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KERNEL SMOOTHING OF THE MEAN PERFORMANCE FOR HOMOGENEOUS CONTINUOUS TIME SEMI-MARKOV PROCESS

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Abstract

The main goal of the present paper is to propose a systematic approach to model performance measurements within the context of continuous-time semi-Markov processes with a finite state space. Specifically, the mean performance is estimated using the kernel method. The uniform strong consistency and the asymptotic normality of the proposed estimator is investigated. Furthermore, a non-parametric kernel estimation of the expected cumulative operational time is addressed. The constructed estimator is proved to be consistent and to converge to a normal random variable as the time of observation becomes large. As an illustration example, a simulation study has been conducted in order to highlight the efficiency as well as the superiority of our method to the standard empirical method.

Keywords: Semi-Markov processes, Kernel estimator, Mean performance, Reliability, Cumulative operational time, Consistency, Asymptotic normality.

1. Introduction and motivation

Stochastic process theory can be seen as an extension of probability theory that allows modeling the evolution of a system through the time. In addition, any serious study of renewal processes would be impossible without using the powerful tool of Markov processes. Markov processes are significant because, in addition to modeling a wide range of interesting phenomena, their lack of memory property allows for the computation of probabilities and expected values that quantify the behavior of the process and the prediction of its potential behavior. Therefore, the starting step of many attempts to model continuous-time processes has been the Markov process. However, the Markov property has its limitations. It imposes restrictions on the distribution of the sojourn time in a state, which is exponentially distributed in the continuous case, though this is not realistic in general. It is adequate to assume that the probability of leaving a state depends on the time already spent there. More precisely, it has become clear that the propensity to move from one state to another often depends strongly on the length of stay in that state. Therefore, any adequate model must incorporate this feature and the semi-Markov process meets the case.

The study of the semi-Markov process is related to the theory of Markov renewal processes (MRP) which can be considered as an extension of the classical renewal theory (see for instance, Pyke [23], [24] and Limnios and Oprisan [13]). More precisely, the semi-Markov processes generalize the renewal processes as well as the Markov jump processes and have numerous applications, especially in reliability (see the pioneer work of Janssen [9] which has made a state of the art for the area of the semi-Markov theory and its applications). Furthermore, there are numerous real-world scenarios where semi-Markov models are relevant, such as in fault-tolerant systems, computer systems and networks, manufacturing systems, healthcare systems, and others.

There is a growing need to assess reliability and performance measure (see, for example, Smith et al. [27]). Meyer [16] created a conceptual framework for performability, defining it as the probability that a system achieves a certain level of accomplishment during a utilization interval $[0, t]$. In other words, for the semi Markov process (Z_t), that describes the evolution of the system through a set of states E, the performance level, or a reward rate, $L(j)$ is associated with each state $j \in \mathbb{E}$, where the reward function $L : E \to \mathbb{R}$ is proposed to be a measure of performance per unit time. The resulting semi-Markov reward process is then able to capture not only the failure and repair of the system components, but the degradable performance as well. Therefore, the development of a performance model is badly needed when we are interested in the level of productivity of a system.

The accumulated reward until time t will be $\phi(t) = \int_0^t L(Z_u)du$ which is an integral functional of the process (Z_t). Integral functionals are very important from theoretical as well as practical point of view. Indeed, in martingale theory, integral functionals are very useful since they are used as compensators, see Koroliuk and Limnios [12]. In statistics, they are used as empirical estimators for stationary distributions in semi-Markov processes [15]. In stochastic applications, they are crucial in some reliability studies, in performance evaluation of computer systems [26] and so on.

Since the work of [2] in which the author has defined combined measures of performance and reliability, many researchers have focused on evaluating performability, particularly in cases where (Z_t) is a Markov process with a finite number of states. Meyer [17] has studied the case of Markov process when the function L is monotonic. Beaudry [3] introduced an algorithm to compute performability until absorption over an infinite interval. Additionally, Donatiello and Iyer [5], have proposed an algorithms for computing performability that do not require the function L to be monotonic.

The semi-Markov case with a finite number of states has been examined by Iyer et al. [7]. The authors demonstrated that the distribution function of $\phi(t)$ satisfies a Markov renewal-type equation and proposed approach to solve it. Fore years later, Ciardo et al. [4] have offered an extension of Beaudry's approach to semi-Markov processes.
Since then, several research papers have been published on the development of estimators and the investigation of their asymptotic properties. In [14], the authors have presented a statistical study of the nonparametric estimation of performability of a finite state space semi-Markov system by using empirical estimator and give consistency and asymptotic normality results for such a system.

A specific scenario occurs when a reward of 1 is assigned to all operational states and 0 to all non-operational states. This case was studied by [21], where the expected reward rate at time t, $\mathbb{E}(\phi(t))$, is known as the instantaneous or point availability $A(t)$. In this case, $\phi(t)$ represents the total time spent in operational states during the interval $[0, t]$.

To the best of our knowledge, there are no existing works on nonparametric kernel estimators of the performance and performability. The main goal of the present paper is to propose a systematic approach to model performance measurements within the context of continuous-time semi-Markov processes with a finite state space. Specifically, the mean performance is estimated using the kernel method. The uniform strong consistency and the asymptotic normality of the proposed estimator is investigated. Furthermore, a non-parametric kernel estimation of the expected cumulative operational time is addressed. The constructed estimator is proved to be consistent and to converge to a normal random variable as the time of observation becomes large. As an illustration example, a simulation study has been conducted in order to highlight the efficiency as well as the superiority of our method to the standard empirical method.

The remainder of the paper is organized as follows. Section 2 presents some definitions and notations of the semi-Markov processes in a countable state space, and these are needed in the work's sequel. An explicit expression of the mean performance of a finite state space semi-Markov system is given in Section 3. The basic elements of statistical estimation are given in Section 4. In Section 5, we introduce and discuss in detail the necessary conditions for establishing the asymptotic properties of the proposed estimator. In Section 6, we illustrate these concepts, measures and estimators through the cumulative operational time as well as a numerical study. Some concluding remarks are given in Section 7.

2. Semi-Markov system and related quantities

Definition 1. (Markov renewal process) Let $E = \{1, \dots, s\}$ be the state space.
A Markov renewal process is a bivariate stochastic process (J_n, T_n) where J_n is the system state at the nth time, and T_n is the nth jump time, we set $T_0 = 0$. The process has to satisfy the following formula:

$$\mathbb{P}\left(J_{\mathrm{n}+1} = j, T_{\mathrm{n}+1} - T_{\mathrm{n}} \leq t \mid J_0, J_1, \dots, J_{\mathrm{n}}, T_0, T_1, \dots, T_{\mathrm{n}}\right) = \mathbb{P}\left(J_{\mathrm{n}+1} = j, T_{\mathrm{n}+1} - T_{\mathrm{n}} \leq t \mid J_{\mathrm{n}}\right), \tag{1}$$

for all $j \in E$, all $t \in \mathbb{R}_+$ and all $n \in \mathbb{N}$.
Moreover, if Equation (1) is independent of n, (J_n, T_n) is considered to be time homogeneous.

Definition 2. (Continuous-time semi-Markov process) Consider a Markov-renewal process $\{(J_n, T_n) : n \in \mathbb{N}\}$ defined on a complete probability space and with state space E. The stochastic process $\{Z_t; t \in \mathbb{R}_+\}$ defined by

$$Z_t = J_{N(t)}, \tag{2}$$

is called a Semi-Markov Process (SMP) where $N(t) := \sup\{n \geq 0 \mid T_n \leq t\}$ is the counting process of the SMP up to time t. Let us also define $X_n = T_n - T_{n-1}, n \geq 1$, the inter-jump times of $Z = (Z_t)_{t \in \mathbb{R}^+}$.

Let us also introduce some functions associated with the process Z:

- The semi-Markov kernel $\mathbf{Q}(t) = \{Q_{ij}(t), i, j \in E\}, t \geq 0$ of Z, is given by

$$Q_{ij}(t) = \mathbb{P}\left(J_{n+1} = j, X_{n+1} \leq t \mid J_n = i\right),$$

  and $p_{ij} := Q_{ij}(\infty) = \mathbb{P}(J_{n+1} = j \mid J_n = i)$ with $\mathbf{p} = (p_{ij})_{i,j \in E}$ is the transition matrix of the process (J_n) which is called the embedded Markov chain (EMC) of Z.

- The conditional sojourn time distribution in state i, given that the next state to be visited is j, denoted F_{ij}, is defined by

$$F_{ij}(t) := \mathbb{P}\left(X_{n+1} \leq t \mid J_n = i, J_{n+1} = j\right).$$

  Meanwhile, the sojourn time distribution in state i, denoted H_i, is defined, for every $t \in \mathbb{R}_+$, by

$$H_i(t) = \mathbb{P}\left(X_{n+1} \leq t \mid J_n = i\right) = \sum_{j \in E} Q_{ij}(t),$$

  and its corresponding survival function is defined by $\overline{H}_i(t) = 1 - H_i(t)$.

- If we consider g to be a locally bounded function and G to be a real right continuous nondecreasing function both defined on $\mathbb{R}_+$, the Stieltjes convolution of the function g with the function G is defined, for every $t \in \mathbb{R}_+$, by

$$g * G(t) = \int_{\mathbb{R}} g(t-x)dG(x) = \int_0^t g(t-x)dG(x).$$

Furthermore, when G and F are cumulative distribution functions, we have

$$G * F(t) = \int_0^t G(t-x)dF(x) = \int_0^t F(t-x)dG(x) = F * G(t).$$

- Now, consider the n-fold convolution of Q by itself, for any $i, j \in E$,

$$Q_{ij}^{(n)}(t) = \begin{cases} \mathbf{1}_{\{i=j, t\geq 0\}} & \text{if } n = 0, \\ Q_{ij}(t) & \text{if } n = 1, \\ \sum_{k\in E} \int_0^t Q_{ik}(ds) Q_{kj}^{(n-1)}(t-s) & \text{if } n \geq 2. \end{cases}$$

- The Markov renewal function denoted $\Psi_{ij}(\cdot)$, is defined, for every $i, j \in E, t \geq 0$, by

$$\begin{aligned} \Psi_{ij}(t) &= E_i[\sum_{n=0}^{\infty} \mathbf{1}_{\{J_n=j, T_n\leq t\}}] \\ &= \sum_{n=0}^{\infty} \mathbb{P}_i\left(J_n = j, T_n \leq t\right) = \sum_{n=0}^{\infty} Q_{ij}^{(n)}(t). \end{aligned}$$

The matrix renewal function is given by

$$\mathbf{\Psi}(t) = \sum_{n=0}^{\infty} \mathbf{Q}^{(n)}(t),$$

where $\mathbf{\Psi}(t) = [\Psi_{ij}(t)]$.

The matrix renewal function $\mathbf{\Psi}(t)$ is the solution of the following Markov renewal equation

$$\mathbf{\Psi}(t) = \mathbf{I}(t) + \mathbf{Q} \star \mathbf{\Psi}(t)^1,$$

where $\mathbf{I}(t) = \mathbf{I}$ when $t \geq 0$ and $\mathbf{I}(t) = 0$ when $t < 0$.

- The transition matrix function $\mathbf{P}(t) = [P_{ij}(t)]$ of the semi-Markov process is defined, for every $i, j \in E, t \geq 0$, by

$$P_{ij}(t) = \mathbb{P}\left(Z_t = j \mid Z_0 = i\right) = \mathbb{P}\left(J_{N(t)} = j \mid J_0 = i\right).$$

It is known, cf. [23], that

$$P_{ij}(t) = \mathbf{1}_{\{i=j\}}\left(1 - \sum_{k\in E} Q_{ik}(t)\right) + \sum_{k\in E} \int_0^t P_{kj}(t-s) Q_{ik}(ds).$$

By solving the above Markov renewal equation, cf. [13], it is seen that, in matrix notation, we have

$$\mathbf{P}(t) = (\mathbf{\Psi} \star (\mathbf{I} - \mathbf{H}))(\mathbf{t}),$$

where $(\mathbf{I} - \mathbf{H})(\mathbf{t}) = diag[1 - H_i(t)]$.

Definition 3. For a semi-Markov process $(Z_t)_{t\in\mathbb{R}^+}$, the limit distribution $\pi = (\pi_1, \ldots, \pi_s)^t$ is defined, when it exists, for every $i, j \in E$, by

$$\pi_j = \lim_{t\to\infty} P_{ij}(t).$$

[1] $\star$ stands for the matrix-Stieltjes convolution of an $n \times r$ matrix function, $\mathbf{A}$, by an $m \times n$ matrix function, $\mathbf{B}$, denoted $\mathbf{B} \star \mathbf{A}$, which can be defined by $(\mathbf{B} \star \mathbf{A})_{ij}(t) = \sum_{k=1}^{n} B_{ik} * A_{kj}(t), \quad i = 1, \ldots, m, j = 1, \ldots, r.$

3. Mean performance

The performance process at time $t \geq 0$, denoted $\Phi(t)$ is the real-valued integral functional of a homogeneous semi-Markov process $(Z_t)_{t\in\mathbb{R}_+}$, cf. [14], defined by

$$\Phi(t) = \int_0^t L(Z_u)\,du = \sum_{j\in E} L(j) \int_0^t \mathbf{1}_{\{z_u=j\}} du, \tag{3}$$

where L is a real-valued function defined on E.

The mean performance at time $t \geq 0$, denoted $\overline{\Phi}(t)$, is defined by

$$\overline{\Phi}(t) := \mathbb{E}[\Phi(t)] = \sum_{i\in E} \alpha_i \overline{\Phi}_i(t) = \sum_{i\in E}\sum_{j\in E} \alpha_i L(j) \int_0^t P_{ij}(u)du, \tag{4}$$

where $\overline{\Phi}_i(t) = \sum_{j\in E} L(j) \int_0^t P_{ij}(u)du$, $\alpha_i = P(J_0 = i)$ and the row vector $\boldsymbol{\alpha} = (\alpha_i : i \in E)$ defines the initial distribution of Z.

4. Elements of statistical estimation

Consider a sample path of the Markov renewal process $(J_n, T_n)_{n\in\mathbb{N}}$

$$\mathcal{Y}(M) := \left(J_0, X_1, \ldots, J_{N(M)-1}, X_{N(M)}, J_{N(M)}, u_M\right), \quad M \in \mathbb{R}_+,$$

where $N(M)$ is defined in Definition 2 and $u_M := M - T_{N(M)}$.

For all $i, j \in$ E, we define:

- $N_i(M) := \sum_{n=1}^{N(M)} 1_{\{J_{n-1}=i\}}$, the number of visits to state i, up to time M.
- $N_{ij}(M) := \sum_{n=1}^{N(M)} \mathbf{1}_{\{J_{n-1}=i, J_n=j\}}$, the number of transitions from i to j, up to time M.

For all $i, j \in E$, $t > 0$, we define the kernel estimator of Q_{ij} and H_i respectively (cf. [1]), by

$$\widehat{Q}_{ij}(t, M) = \frac{1}{N_i(M)} \sum_{l=1}^{N(M)} G\left(\frac{t - X_l}{h_{ij,M}}\right) \mathbf{1}_{\{J_{l-1}=i, J_l=j\}},$$

and

$$\widehat{H}_i(t, M) = \frac{1}{N_i(M)} \sum_{l=1}^{N(M)} G\left(\frac{t - X_l}{h_{i,M}}\right) \mathbf{1}_{\{J_{l-1}=i\}};$$

where $G(t) = \int_{-\infty}^t K(t)dt$, with K is a bounded kernel function.

For fixed states i and j, it should be noted that the smoothing parameter of the previous estimators depends on the sample size, so we should write $h_{ij,N_{ij}(M)} = h_{ij,M}$ (resp. $h_{i,N_i(M)} = h_{i,M}$), however we prefer to use a simpler notation.

The kernel estimator of the Markov renewal function $\Psi_{ij}(t)$, in matrix form, is given by

$$\widehat{\boldsymbol{\Psi}}(t, M) = \sum_{n=0}^{\infty} \widehat{\mathbf{Q}}^{(n)}(t, M). \tag{5}$$

The kernel estimator of the transition matrix function $\mathbf{P}(t)$ at time $t \geq 0$ of the SMP, is given by

$$\widehat{\mathbf{P}}(t, M) = \left(\widehat{\boldsymbol{\Psi}}(\cdot, M) \star \left(\mathbf{I} - \widehat{\mathbf{H}}(., M)\right)\right)(t). \tag{6}$$

Based on Equation (4), an nonparametric kernel estimator of the mean performance $\widehat{\overline{\Phi}}(t, M)$ is given by the following expression:

$$\widehat{\overline{\Phi}}(t, M) = \sum_{i \in E} \sum_{j \in E} \alpha_i L(j) \int_0^t \widehat{P}_{ij}(u, M) du. \tag{7}$$

5. Asymptotic properties

The following assumptions are necessary to derive the asymptotic behaviour of the kernel estimator defined in (7).

5.1. Assumptions

First, we will assume the following two assumptions:

(H.1) The EMC $(J_n)_{n \in \mathbb{N}}$ is an ergodic Markov chain, with stationary distribution ν.

(H.2) The SMP is regular, with finite mean sojourn times m.

Second, the following assumptions are required in order to establish all the asymptotic properties in this paper:

(H.3) i) $Q_{ij}(t)$ and $q_{ij}(t)$ are continuously differentiable with respect to the Lebesgue measure, and let $q_{ij}(t)$ and $q'_{ij}(t)$ be respectively their corresponding Radon-Nikodym derivatives.

ii) The derivative q'_{ij} is bounded.

(H.4) The function G is a distribution function, where its derivative is K.

(H.5) The kernel K is a density function of bounded variation such that $\lim_{x \to \infty} |xK(x)| = 0$ and $\left| \int t^j K^n(t) dt \right| < \infty$ for $j = 0, 1$, and $n = 1, 2$.

(H.6) The smoothing parameter $h_{ij,M}$ satisfies

$$\lim_{M \longrightarrow \infty} h_{ij,M} = 0 \quad \text{and} \quad \lim_{M \longrightarrow \infty} M h_{ij,M} = \infty.$$

5.2. Comments on the assumptions

The structural assumptions **(H.1)** and **(H.2)** are the same as those used classically for the semi-Markov processes framework (see, for instance [1] and [6]). More precisely, the recurrence and the positivity of the EMC $(J_n)_{n \in \mathbb{N}}$ in **(H.1)** ensure that the stationary distribution π_j defined in Definition 3 is strictly positive and unique. Furthermore, since the EMC $(J_n)_{n \in \mathbb{N}}$ is irreducible and aperiodic, the limit in Definition 3 always exists and it is independent of the distribution in the initial state. **(H.2)** means that the counting process $\{N(t) : t \geq 0\}$ has a finite number of jumps in a finite period with probability 1. In addition, under this hypothesis we have $T_n < T_{n+1}$, for any $n \in \mathbb{N}$, and $T_n \to \infty$ as n goes to infinity. Assumption **(H.3)** as imposed on $Q_{ij}(t)$ and $q_{ij}(t)$ is a regularity type hypothesis. Whereas, assumption **(H.3)(i)** is a regularity constraint using to get the strong consistency. the second derivative hypothesis **(H.3)(ii)** establishes more restrictive constraints when going through to state the asymptotic normality of our estimators. **(H.5)-(H.6)** are technical constraints; they are imposed for the sake of the proof's simplicity and brevity.

Before stating our main result, we introduce the following technical lemma which will be necessary to prove our second result.

Lemma 1. For $n = 1, 2$. If **(H.3)-(H.5)** hold, we have

$$\frac{1}{h_{ij,M}} \int_0^{+\infty} K^n\left(\frac{v-x}{h_{ij,M}}\right) dQ_{ij}(x) \leqslant q_{ij}(v) \int_{-\infty}^{+\infty} K^n(z)\,dz + o(h_{ij,M}).$$

Proof of Lemma 1 By using a change of variable followed by Taylor's expansion, we have

$$\begin{aligned}\frac{1}{h_{ij,M}} \int_0^{+\infty} K^n\left(\frac{v-x}{h_{ij,M}}\right) dQ_{ij}(x) &= \int_{-\infty}^{\frac{v}{h_{ij,M}}} K^n(z)\, q_{dr}(v - h_{ij,M}z)dz \\ &= \int_{-\infty}^{\frac{v}{h_{ij,M}}} K^n(z) \left[q_{ij}(v) - h_{ij,M} z q'_{ij}(v^*)\right] dz \\ &\leqslant q_{ij}(v) \int_{-\infty}^{+\infty} K^n(z)\,dz + o(h_{ij,M}),\end{aligned}$$

where $v - h_{ij,M}z \leq v^* \leq v$.

5.3. Main Results

Our first result concerns the uniform strong consistency of the proposed estimator.

Theorem 1. For any fixed $0 \leq t \leq M$ and $i \in E$, under **(H.1)-(H.6)**, the estimator $\widehat{\overline{\Phi}}(t, M)$ of $\overline{\Phi}(t)$ is uniformly strongly consistent, that is

$$\max_{i\in E} \sup_{t\in[0,M]} \mid \widehat{\overline{\Phi}}_i(t, M) - \overline{\Phi}_i(t) \mid \xrightarrow{a.s.} 0 \quad as \quad M \longrightarrow \infty.$$

Proof of Theorem 1 The proof of this theorem is based on (5), (6), (7) and the following inequality:

$$\begin{aligned}\max_{i\in E} \sup_{t\in[0,M]} \mid \widehat{\overline{\Phi}}_i(t, M) - \overline{\Phi}_i(t) \mid &\leq \sum_{j\in E} L(j)t \sum_{n=0}^{\infty} \max_{i\in E} \sup_{t\in[0,M]} \left|\widehat{Q}_{ij}^{(n)}(t, M) - Q_{ij}^{(n)}(t)\right| \\ &+ \sum_{j\in E} L(j)t \sum_{n=0}^{\infty} \max_{i\in E} \sup_{t\in[0,M]} \left|\widehat{Q}_{ij}^{(n)}(t, M) - Q_{ij}^{(n)}(t)\right| * \widehat{H}_j(t, M) \\ &+ \sum_{j\in E} L(j)t \max_{i\in E} \sup_{t\in[0,M]} \left|\widehat{H}_j(t, M) - H_j(t)\right| * \Psi_{ij}(t).\end{aligned}$$

For all $i, j \in E$, $n \in \mathbb{N}^*$ and $M \in \mathbb{R}_+$, based on a straightforward adaptation of the proof of Lemma 1 in [19], we get that the estimator $\widehat{Q}_{ij}^{(n)}(t, M)$ is uniformly strong consistent in $[0, M]$. In addition, the uniform strong consistency of the kernel estimator $\widehat{H}_j(t, M)$ is stated in Theorem 4.1 of [1]. Then,

$$\max_{i\in E} \sup_{t\in[0,M]} \mid \widehat{\overline{\Phi}}_i(t, M) - \overline{\Phi}_i(t) \mid \xrightarrow{a.s.} 0, \quad as \quad M \longrightarrow \infty.$$

Before stating our second result, let us consider the renewal process $\left(T_n^i\right)_{n\geq 0}$ of successive times of visits to state i, then $N_i(t)$ is the counting process of renewals. Let μ_{ii} and μ_{ii}^* denote the mean first passage times of the state i in the MRP and in the corresponding Markov chain $\{J_n; n \geq 0\}$, respectively. Furthermore, μ_{ii} is the mean interarrival times of the eventual delayed renewal process $\left(T_n^i\right), n \geq 0$, i.e., $\mu_{ii} = \mathbb{E}\left[T_2^i - T_1^i\right]$, and $\mu_{ii}^* = \mathbb{E}\left[S_i^* | J_0 = i\right]$ with $S_i^* = \min\{k \geq 1, J_n = i\}$ is the first visit time to the state i.

Theorem 2. For any fixed $0 \leq t \leq M$, if **(H.1)-(H.6)** hold, we have

$$\sqrt{Mh_M}\left[\widehat{\overline{\Phi}}(t,M)-\overline{\Phi}(t)\right] \xrightarrow{D} N\left(0,\sigma^2_{\Phi}(t)\right), \quad as \quad M \longrightarrow \infty,$$

with $h_M = \min_{i,j\in E}\{h_{ij,M}\}$ and the asymptotic variance

$$\sigma^2_{\Phi}(t) \leqslant \sum_{i\in E}\sum_{j\in E}\mu_{ii}\int_0^t \left[\left(R_{ij}-D_i\right)^2 * \left(Q_{ij}(\cdot)\int_{-\infty}^{+\infty}K^2(z)\,dz\right)\right](u)du, \tag{8}$$

where

$$R_{ij} = \sum_{d\in E}\sum_{r\in E}\alpha_d L(r)\left(\Psi_{di} * \Psi_{jr} * \overline{H}_r\right), \tag{9}$$

and

$$D_i = \sum_{d\in E}\sum_{r\in E}\alpha_d L(r)\mathbf{1}_{\{i=r\}}\Psi_{dr}. \tag{10}$$

Proof of Theorem 2 We have,

$$\begin{aligned}\sqrt{Mh_M}\left[\widehat{\overline{\Phi}}(t,M)-\overline{\Phi}(t)\right] &= \sum_{d\in E}\sum_{r\in E}\alpha_d L(r)\sqrt{Mh_M}\left[\int_0^t \hat{P}_{dr}(u,M)du - \int_0^t P_{dr}(u)du\right]\\ &= \sum_{d\in E}\sum_{r\in E}\alpha_d L(r)\sqrt{Mh_M}\left[\int_0^t \left[\left(\hat{\Psi}_{dr}(\cdot,M) * \left(I-\widehat{H}_r(\cdot,M)\right)\right)(u)\right.\right.\\ &\quad \left.\left. - \left(\Psi_{dr} * (I-H_r)\right)(u)\right]du\right].\end{aligned}$$

Note that the last right side can be written as follows:

$$\sum_{d\in E}\sum_{r\in E}\alpha_d L(r)\sqrt{Mh_M}\left[\int_0^t \left(\left(\widehat{\Psi}_{dr}(\cdot,M)-\Psi_{dr}(\cdot)\right) * \left(\widehat{\overline{H}}_r(\cdot,M)-\overline{H}_r(\cdot)\right)\right)(u)du\right.$$
$$\left. + \int_0^t \left(\Psi_{dr}(\cdot) * \left(\widehat{\overline{H}}_r(\cdot,M)-\overline{H}_r(\cdot)\right)\right)(u)du + \int_0^t \left(\left(\widehat{\Psi}_{dr}(\cdot,M)-\Psi_{dr}(\cdot)\right) * \overline{H}_r(\cdot)\right)(u)du\right].$$

According to [1] and by following the same arguments as [18], the first term converges to zero as M tends to infinity.

Consequently, by applying Slutsky's Theorem, we deduce that $\sqrt{Mh_M}\left[\widehat{\overline{\Phi}}(t,M)-\overline{\Phi}(t)\right]$ converges in distribution to the same limit as

$$\sqrt{Mh_M}\sum_{d\in E}\sum_{r\in E}\alpha_d L(r)\left[\int_0^t \left(\Psi_{dr}(\cdot) * \left(\widehat{\overline{H}}_r(\cdot,M)-\overline{H}_r(\cdot)\right)\right)(u)du\right.$$
$$\left. + \int_0^t \left(\left(\widehat{\Psi}_{dr}(\cdot,M)-\Psi_{dr}(\cdot)\right) * \overline{H}_r(\cdot)\right)(u)du\right].$$

By combining Theorem 4.3 (i) of [1] and Theorem 4 (b)[18], along with arguments akin to those employed in [9] p. 214, we deduce that $\sqrt{Mh_M}\left[\widehat{\mathbf{\Psi}}(\cdot,M)-\mathbf{\Psi}(\cdot)\right]_{dr}(t)$ has the same limit in distribution as $\sqrt{Mh_M}[\mathbf{\Psi}(\cdot) \star \Delta\mathbf{Q} \star \mathbf{\Psi}(\cdot)]_{dr}(t)$,
where $\Delta\mathbf{Q} = (\widehat{\mathbf{Q}}-\mathbf{Q})$, for every $t \geq 0, t \leq M$, and for every $d, r \in E$, which is written as follows:

$$\Delta Q_{dr}(\cdot) = \frac{1}{N_d(M)}\sum_{l=1}^{N(M)}\left[G\left(\frac{\cdot - X_l}{h_{dr,M}}\right)\mathbf{1}_{\{J_{l-1}=d,J_l=r\}} - Q_{dr}(\cdot)\mathbf{1}_{\{J_{l-1}=d\}}\right].$$

Furthermore,

$$\Psi_{dr} * \left(\widehat{\overline{H}}_r - \overline{H}_r\right) = -\sum_{k\in E}\Psi_{dr} * \Delta Q_{rk} = -\sum_{k\in E}\sum_{m\in E}\mathbf{1}_{\{m=r\}}\Psi_{dr} * \Delta Q_{mk}.$$

Then, $\sqrt{Mh_M}\left[\widehat{\overline{\Phi}}(t,M)-\overline{\Phi}(t)\right]$ has the same limit as

$$\frac{1}{\sqrt{M}}\sum_{l=1}^{N(M)}\sum_{m\in E}\sum_{k\in E}\frac{M}{N_m(M)}\sqrt{h_M}\left[\int_0^t\left[(R_{mk}-D_m)*\left(G\left(\frac{.-X_l}{h_{mk,M}}\right)\mathbf{1}_{\{J_{l-1}=m,J_l=k\}}\right.\right.\right.$$
$$\left.\left.\left.-Q_{mk}(.)\mathbf{1}_{\{J_{l-1}=m\}}\right)\right](u)du\right],$$

where R_{mk} and D_m are given in (9) and (10).
Apply central limit theorem related to semi-Markov processes (see [25]) to the function $W_f(t)$ such that

$$\begin{aligned}W_f(t)&=\sum_{l=1}^{N(M)}f(J_{l-1},J_l,X_l)\\&=\sum_{l=1}^{N(M)}\sum_{m\in E}\sum_{k\in E}\frac{M}{N_m(M)}\sqrt{h_M}\left[\int_0^t\left[(R_{mk}-D_m)*\left(G\left(\frac{.-X_l}{h_{mk,M}}\right)\mathbf{1}_{\{J_{l-1}=m,J_l=k\}}\right.\right.\right.\\&\left.\left.\left.-Q_{mk}(.)\mathbf{1}_{\{J_{l-1}=m\}}\right)\right](u)du\right],\end{aligned}$$

where, for any fixed $t>0$, we have defined the function $f:E\times E\times\mathbb{R}\to\mathbb{R}$ by

$$f(i,j,x)=\sum_{m\in E}\sum_{k\in E}\frac{M}{N_m(M)}\sqrt{h_M}\left[\int_0^t\left[(R_{mk}-D_m)*\left(G\left(\frac{.-x}{h_{mk,M}}\right)\mathbf{1}_{\{i=m,j=k\}}-Q_{mk}(.)\mathbf{1}_{\{i=m\}}\right)\right](u)du\right].$$

In order to apply the Pyke and Schaufele's CLT, we need to compute the quantities $A_{ij},A_i,B_{ij},B_i,r_i,m_f,\sigma_d^2$ and then $\sigma^2_{\overline{\Phi}}(t)$, using Lemma 1 with assumptions **(H.3)-(H.5)**. We have

$$\begin{aligned}A_i&=\sum_{j\in E}A_{ij}\\&=\sum_{j\in E}\int_0^{+\infty}f\left(i,j,x\right)dQ_{ij}(x)\\&=\sum_{j\in E}\sum_{k\in E}\sum_{m\in E}\int_0^{+\infty}\frac{M}{N_m(M)}\sqrt{h_M}\left[\int_0^t\left[(R_{mk}-D_m)*\left(G\left(\frac{.-x}{h_{mk,M}}\right)\mathbf{1}_{\{i=m,j=k\}}\right.\right.\right.\\&\left.\left.\left.-Q_{mk}(.)\mathbf{1}_{\{i=m\}}\right)\right](u)du\right]dQ_{ij}(x)\\&=\sum_{j\in E}\frac{M}{N_i(M)}\sqrt{h_M}\left[\int_0^t\int_0^u\left(R_{ij}-D_i\right)(u-v)\left(\frac{1}{h_{ij,M}}\int_0^{+\infty}K\left(\frac{v-x}{h_{ij,M}}\right)dQ_{ij}(x)\right)dvdu\right.\\&\left.-\sum_{k\in E}\int_0^t\int_0^u\left(R_{ik}-D_i\right)(u-v)q_{ik}(v)\int_0^{+\infty}dQ_{ij}(x)dvdu\right].\end{aligned}$$

Then

$$A_i\leqslant\sum_{j\in E}\frac{M}{N_i(M)}\sqrt{h_M}\int_0^t\left[\left(R_{ij}-D_i\right)*\left(o\left(h_{ij,M}\right)\right)\right](u)du.$$

For B_i and by using Jensen's inequality and Lemma 1, we have

$$\begin{aligned}
B_i &= \sum_{j\in E} B_{ij} \\
&= \sum_{j\in E} \int_0^{+\infty} [f(i,j,x)]^2 dQ_{ij}(x) \\
&= \sum_{j\in E} \int_0^{+\infty} \left[\frac{M}{N_i(M)} \sqrt{h_M} \left[\int_0^t \left[\left(R_{ij} - D_i\right) * G\left(\frac{. - x}{h_{ij,M}}\right) \right.\right.\right. \\
&\left.\left.\left. - \sum_{k\in E} \left(R_{ik} - D_i\right) * Q_{ik}(.) \right] (u) du \right]\right]^2 dQ_{ij}(x) \\
&\leqslant \sum_{j\in E} \int_0^{+\infty} \left(\frac{M}{N_i(M)} \right)^2 h_M \left[\int_0^t \left[\int_0^u \left(R_{ij} - D_i\right)(u-v) \left(\frac{1}{h_{ij,M}} K\left(\frac{v-x}{h_{ij,M}} \right) \right) dv \right.\right. \\
&\left.\left. - \sum_{k\in E} \int_0^u \left(R_{ik} - D_i\right)(u-v) q_{ik}(v) dv \right]^2 du \right] dQ_{ij}(x).
\end{aligned}$$

Then

$$\begin{aligned}
B_i \leqslant& \sum_{j\in E} \left(\frac{M}{N_i(M)} \right)^2 h_M \int_0^{+\infty} \left[\int_0^t \int_0^u \left[\left(R_{ij} - D_i\right)^2 (u-v) \left(\frac{1}{h_{ij,M}^2} K^2\left(\frac{v-x}{h_{ij,M}} \right) \right) \right.\right. \\
&+ \sum_{k\in E} \left(R_{ik} - D_i\right)^2 (u-v) q_{ik}^2(v) \\
&\left.\left. -2 \sum_{k\in E} \left(R_{ik} - D_i\right)\left(R_{ij} - D_i\right)(u-v) q_{ik}(v) \frac{1}{h_{ij,M}} K\left(\frac{v-x}{h_{ij,M}} \right) \right] dv du \right] dQ_{ij}(x).
\end{aligned}$$

Since $\frac{N_i(M)}{M} \xrightarrow{a.s.} \frac{1}{\mu_{ii}}$ (see [13]), when $M \to +\infty$ and from the assumption **(H.6)**, the second and the third term in the last inequality converge to zero, we get

$$B_i \leqslant \sum_{j\in E} \mu_{ii}^2 \int_0^t \left[\left(R_{ij} - D_i\right)^2 * \left(Q_{ij}(\cdot) \int_{-\infty}^{+\infty} K^2(z)\, dz \right) \right] (u) du.$$

Furthermore,

$$\begin{aligned}
r_d &= \sum_{i\in E} A_i \frac{\mu_{dd}^*}{\mu_{ii}^*} = 0 \qquad \text{as} \qquad M \to \infty, \\
m_f &= \frac{1}{\mu_{dd}} r_d = 0 \qquad \text{as} \qquad M \to \infty, \\
\sigma_\Phi^2(t) &= \frac{1}{\mu_{dd}} \sigma_d^2(t),
\end{aligned}$$

where

$$\begin{aligned}
\sigma_d^2(t) &= \sum_{i\in E} B_i \frac{\mu_{dd}^*}{\mu_{ii}^*} \\
&\leqslant \mu_{dd}^* \sum_{i\in E} \sum_{j\in E} \frac{\mu_{ii}^2}{\mu_{ii}^*} \int_0^t \left[\left(R_{ij} - D_i\right)^2 * \left(Q_{ij}(\cdot) \int_{-\infty}^{+\infty} K^2(z)\, dz \right) \right] (u) du.
\end{aligned}$$

Then, since $\mu_{ii}^* = \dfrac{1}{v_i}$ (see [10]) and $\mu_{ii} = \dfrac{\bar{m}}{v_i}$ (see [13]), where $\bar{m} = \sum_{i\in E} m_i v_i$ is the mean sojourn time of the MRP; we have

$$\sigma_\Phi^2(t) \leqslant \sum_{i\in E} \sum_{j\in E} \mu_{ii} \int_0^t \left[\left(R_{ij} - D_i\right)^2 * \left(Q_{ij}(\cdot) \int_{-\infty}^{+\infty} K^2(z)\, dz \right) \right] (u) du.$$

We obtain from the CLT that $\sqrt{Mh_M}\left[\widehat{\overline{\Phi}}(t,M)-\overline{\Phi}(t)\right]$ converges in distribution, as M tends to infinity, to a zero-mean normal random variable, of variance $\sigma^2_{\Phi}(t)$ given in (8).

6. Applications

The cumulative operational time is considered as one of the most relevant performance measures for the reliability. In this section, we propose a non-parametric kernel estimator of the expected cumulative operational time for the semi-Markov system. Then, we investigate the asymptotic properties of the proposed estimator, namely, the strong consistency and the asymptotic normality. As an illustration example, we apply the previous results to three-state continuous time SMP.

6.1. The cumulative operational time

The state space E is often split into two subsets for reliability research. The first one, let's say U, is made up of up states, whereas the second one, let's say D, is made up of down states. The start of an essential event, such as a component failure related to some reason or a complete repair, might well be associated with the transition into a state. Since we suppose that the system can be fixed, the process alternates between U and D.

The cumulative operational time is defined by

$$W(t)=\int_0^t \mathbf{1}_{\{Z_u\in U\}}du.$$

It represents the total time that the semi-Markov process Z spends in the set of up states U over the interval $[0,t]$.

Making use of the assumptions **(H.1) - (H.2)**, along with the aid of the arguments used in [8], we obtain the following result:

$$\lim_{t\to+\infty}\frac{W(t)}{t}=\frac{\sum_{j\in U}\nu_j m_j}{\sum_{k\in E}\nu_k m_k},$$

where $m_j=\int_0^\infty(1-H_j(t))dt$ is the mean sojourn time in state j.

The quantity we aim to analyze is the expected cumulative operational time of a semi-Markov system, denoted by $\overline{W}(t):=\mathbb{E}[W(t)]$. Which is given by

$$\overline{W}(t)=\sum_{i\in E}\alpha_i\overline{W}_i(t)=\sum_{i\in E}\sum_{j\in U}\alpha_i\int_0^t P_{ij}(u)du, \tag{11}$$

where $\overline{W}_i(t)=\sum_{j\in U}\int_0^t P_{ij}(u)du$.

The expected cumulative operational time serves as a crucial indicator in maintenance studies, facilitating the calculation of average system availability, cf. [21], which is expressed as

$$\overline{A}(t)=\frac{1}{t}\overline{W}(t)=\frac{1}{t}\sum_{i\in E}\sum_{j\in U}\int_0^t P_{ij}(u)du.$$

From the definition of the expected cumulative operational time $\overline{W}(t)$ given in Equation (11), and based on a sample path truncated to the time interval $[0,M]$ of the process, the nonparametric kernel estimator $\widehat{\overline{W}}(t,M)$ is given by

$$\widehat{\overline{W}}(t,M)=\sum_{i\in E}\sum_{j\in U}\alpha_i\int_0^t\widehat{P}_{ij}(u,M)du. \tag{12}$$

The asymptotic properties of the proposed estimator are gathered in the following two corollaries.

Corollary 1. For any fixed $0 \leq t \leq M$, under the same assumptions of Theorem 1, the estimator of the expected operational time, $\widehat{\overline{W}}(t, M)$ is strongly consistent, that is

$$\sup_{t\in[0,M]} \mid \widehat{\overline{W}}(t,M) - \overline{W}(t) \mid \xrightarrow{a.s.} 0 \quad \text{as} \qquad M \to \infty.$$

Proof of Corollary 1
This corollary is a particular case of Theorem 1 and then the proof is omitted.

The following result concerns the asymptotic normality of the proposed estimator.

Corollary 2. For any fixed $0 \leq t \leq M$, we have $\sqrt{Mh_M}\left[\widehat{\overline{W}}(t,M) - \overline{W}(t)\right]$ converges in law, as M tends to infinity, to a zero mean normal random variable with the asymptotic variance

$$\sigma^2_{\overline{W}}(t) \leqslant \sum_{i\in E}\sum_{j\in U}\mu_{ii}\int_0^t \left[\left(Y_{ij} - C_i\right)^2 * \left(Q_{ij}(\cdot)\int_{-\infty}^{+\infty} K^2\left(z\right)dz\right)\right](u)du,$$

where

$$Y_{ij} = \sum_{d\in E}\sum_{r\in U}\alpha_d\left(\Psi_{di} * \Psi_{jr} * \overline{H}_r\right) \quad \text{and} \quad C_i = \sum_{d\in E}\sum_{r\in U}\alpha_d \mathbf{1}_{\{i=r\}}\Psi_{dr}.$$

Proof of Corollary 2
The proof of this result is based on the same arguments as in the proof of Theorem 2.

6.2. Confidence interval

The main purpose of the confidence interval is to supplement the estimate at a point with information about the uncertainty in this estimate. It is considered as a direct application of the Central Limit Theorem. In order to provide a confidence interval for the expected cumulative operational time $\overline{W}(t)$, we need first to propose a consistent estimator of the variance $\sigma^2_{\overline{W}}(t)$. A natural consistent estimator of this variance, denoted by $\widehat{\sigma}^2_{\overline{W}}(t,M)$, is obtained by estimating the parameters involved in this quantity such as $Q_{mk}(t), \overline{H}_j(t)$ and $\Psi_{im}(t)$.
Indeed, from the strong consistency of $\widehat{Q}_{mk}(t,M), \widehat{\overline{H}}_j(t,M)$ and $\widehat{\Psi}_{im}(t,M)$, (see the proof of Theorems 1 and 2 as well as Theorem 4.1 and Theorem 4.2 (v) in [1]), we deduce the strong consistency of $\widehat{\sigma}^2_{\overline{W}}(t,M)$.
Consequently, from Corollary 2, we get

$$\sqrt{Mh_M}\left[\widehat{\overline{W}}(t,M) - \overline{W}(t)\right] \xrightarrow{D} N\left(0, \widehat{\sigma}^2_{\overline{W}}(t,M)\right).$$

Then

$$\frac{\sqrt{Mh_M}}{\widehat{\sigma}_{\overline{W}}(t,M)}\left[\widehat{\overline{W}}(t,M) - \overline{W}(t)\right] \xrightarrow{D} N\left(0,1\right).$$

Hence, for $\alpha \in (0,1)$, an asymptotic $100(1-\alpha)\%$ confidence interval for $\widehat{\overline{W}}(t,M)$ can be straightforwardly computed:

$$I = \left(\widehat{\overline{W}}(t,M) \pm z_{\frac{\alpha}{2}}\frac{\widehat{\sigma}_{\overline{W}}(t,M)}{\sqrt{Mh_M}}\right)$$

where $z_{\frac{\alpha}{2}}$ is the upper $\frac{\alpha}{2}$ quantile of the standard normal distribution.

6.3. Numerical application

To validate our results, we consider a three state system whose state transition diagram is given in Figure 1. States 1 and 2 are up states and state 3 is a down state.
We have two exponential and two Weibull distribution functions as conditional transitions, for all $x \geq 0$, say $H_{12}(x) = 1 - \exp\left(-\lambda_1 x\right), H_{31}(x) = 1 - \exp\left(-\lambda_2 x\right)$,
$H_{23}(x) = 1 - \exp\left[-\left(\frac{x}{\alpha_1}\right)^{\beta_2}\right], H_{21}(x) = 1 - \exp\left[-\left(\frac{x}{a_1}\right)^{\beta_1}\right]$.
The parameters of these distributions are: $\lambda_1 = 0.1, \lambda_2 = 0.2, \alpha_1 = 0.3, \beta_1 = 2, \alpha_2 = 0.1, \beta_2 = 2$.

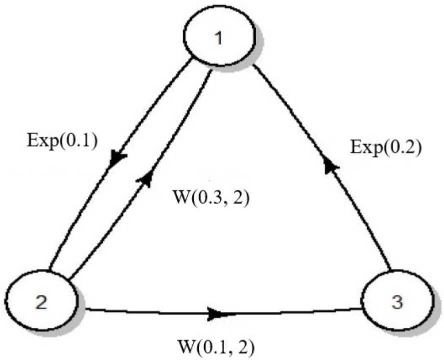

Figure 1: *A three state semi-Markov system.*

The transition probability matrix of the embedded Markov chain (J_n) is:

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0.95 & 0 & 0.05 \\ 1 & 0 & 0 \end{pmatrix}$$

Where the system is defined by the initial distribution $\alpha = (1/3, 1/3, 1/3)$.

To construct the kernel estimator for the mean performance of a continuous-time semi-Markov process. The smoothed function $K(\cdot)$ is chosen to be the quadratic function defined as $K(u) = \frac{3}{4}\left(1 - u^2\right)$ for $|u| \leq 1$ and the cumulative distribution function $G(u) = \int_{-\infty}^{u} \frac{3}{4}\left(1 - z^2\right) \mathbf{1}_{[-1,1]}(z) \mathrm{d}z$. The bandwidth h_M has been obtained by the "PBbw" method, which computes the plug-in bandwidth of the Polansky and Baker method, cf. [22]. We have considered that the observation period is the interval $[0, M]$ with $M = 20000$.

Figure 2 gives a comparison between the kernel estimator of the mean performance for different sample sizes ($M = 2000, M = 10000$ and $M = 20000$). We observe that this estimator converges to the true value of the mean performance as M increases.

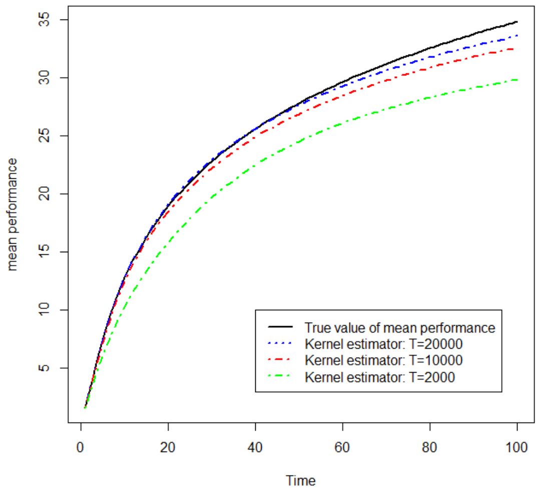

Figure 2: *Comparison between the kernel estimator of the mean performance for different sample sizes and the true value.*

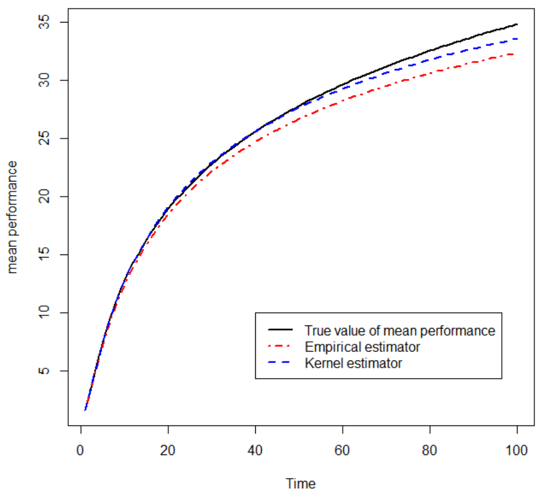

Figure 3: *Comparison between the true values of the mean performance and their estimators (empirical and kernel).*

Figure 3 gives a comparison between the empirical estimator (see [14]) and our kernel estimator of the mean performance. We remark easily that, our method provides better results than the empirical one.

7. Concluding remarks

The application of the nonparametric kernel approach to estimate the mean performance of a continuous-time semi-Markov process is the main element of the work described in this paper. We have proposed a kernel estimator for this quantity then we have provided its asymptotic properties, such as the uniform strong consistency, as well as the asymptotic normality.
Compared to the empirical estimator, the use of this kernel technique approach has a number of benefits. Since the empirical function is always a discontinuous function, the kernel smoothing in particular prevents discontinuities in this function. As a result, the empirical distribution may be considered a poor approximation when knowing that the underlying distribution is continuous. To the best of our knowledge, no limit theorems have been obtained for functionals of homogeneous semi-Markov processes, such as the performance and the related quantities, by using the kernel approach. In particular, we have made an important connection of our results with the reliability theory by focusing on the cumulative operational time of the semi-Markov systems. This crucial indicator is the total time spent by the process in the set of operational states during a specific time interval. It is used to minimize the expected cost of the maintenance process. The uniform strong consistency and asymptotic normality have been stated. In addition, a confidence interval has been constructed. Moreover, a simulation study has been conducted in order to highlight to the efficiency as well as the superiority of our method to the standard empirical method.

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Conflict of interests

The authors declare that there is no conflict of interest.

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