A GENERALIZED POWER SUJATHA DISTRIBUTION WITH PROPERTIES AND APPLICATIONS

Hosenur Rahman Prodhani*, Rama Shanker

•

Department of Statistics, Assam University, Silchar, India
E-mail: *hosenur72@gmail.com, shankerrama2009@gmail.com

Abstract

This paper introduces a generalized power Sujatha distribution as an extension of the two-parameter generalization of Sujatha distribution, initially proposed for analyzing and modeling lifetime data in medical and engineering fields. The existing generalization of Sujatha distribution, being two-parameter, may not always provide a satisfactory fit for certain lifetime data from both theoretical and practical perspectives. The generalized power Sujatha distribution is presented as a comprehensive model, encompassing both the Generalization of Sujatha distribution and the Sujatha distribution as particular cases, specifically for the analysis of data in medical and engineering domains. The paper delves into the statistical properties of the proposed distribution, examining the behavior of its probability density function and cumulative distribution function across varying parameter values. Additionally, the first four raw moments of the distribution are derived and provided. The expressions for the hazard rate function and mean residual life function are obtained, and their behaviors under different parameter values are discussed. Stochastic ordering, a valuable tool for comparing stochastic nature, is also explored. The method of maximum likelihood is discussed for parameter estimation, and a simulation study is conducted to assess the performance of maximum likelihood estimates as sample sizes increase. To validate the applicability of the distribution, two real lifetime data sets from medical and engineering fields are analyzed. The goodness of fit of the generalized power Sujatha distribution is evaluated using the Akaike Information criterion and Kolmogorov-Smirnov statistic. The results demonstrate that the proposed distribution offers a closer fit compared to three-parameter power Quasi Lindley distribution, Three-parameter Sujatha distribution, Generalized gamma distribution, and two-parameter Generalizations of Sujatha distribution, as well as Weibull distribution and one-parameter Sujatha distribution. Given its superior fit over Power Quasi Lindley and Weibull distributions, particularly in the context of modeling and analyzing data from medical and engineering fields, the paper concludes by recommending the generalized power Sujatha distribution as the preferred choice over the considered distributions for such applications.

Keywords: generalization of Sujatha distribution, statistical properties, maximum likelihood estimation, mean residual life function, application.

I. Introduction

Lindly distribution introduced by [1], a one-parameter lifetime distribution obtained through the convex combination of the exponential distribution and gamma distribution. Subsequently, its statistical properties and goodness of fit conducted and examined by [2]. Their findings revealed that the Lindley distribution outperforms the exponential distribution in terms of fit. Another one-parameter lifetime distribution, the Sujatha distribution (SD), was proposed by [3] using a similar convex combination approach. SD is a convex combination of exponential (η) distribution, gamma $(2,\eta)$ distribution and gamma $(3,\eta)$ distribution, which provides better fit on some dataset as compared to exponential and Lindley distribution. The probability density function (pdf) and the cumulative distribution function (cdf) of SD are given by

$$f(x;\eta)=\frac{\eta^3}{\eta^2+\eta+2}\left(1+x+x^2\right)e^{-\eta x};\; x>0,\eta>0 \tag{1}$$

$$F(x;\eta)=1-\left[1+\frac{\eta x(\eta x+\eta+2)}{\eta^2+\eta+2}\right]e^{-\eta x};\; x>0,\eta>0 \tag{2}$$

and comprehensively explored the statistical and mathematical properties of the Sujatha distribution (SD). This included a detailed discussion on moments and their associated measures, hazard function, mean residual life function, Bonferroni and Lorenz curves, stochastic ordering, mean deviation about the mean and the median, as well as stress-strength reliability and delved into the estimation of parameters using the maximum likelihood method and provided insights into the practical applications of SD in modelling lifetime data. Despite the improved fit of the Sujatha distribution (SD) compared to the exponential and Lindley distributions, limitations arise in situations where these one-parameter distributions fail to provide a good fit. Recognizing this, [4] addressed the issue by introducing a generalized version, termed the Generalization of Sujatha distribution (GSD). This extension involves the incorporation of an additional parameter into the probability density function (pdf) of the SD. The introduction of this extra parameter enhances the flexibility of the GSD, making it better suited to accommodate a broader range of data patterns as compared to its one-parameter counterpart, the SD. The GSD is characterized by its pdf and cdf

$$f(x;\eta,\omega)=\frac{\eta^3}{\eta^2+\eta+2\omega}\left(1+x+\omega x^2\right)e^{-\eta x};x>0,\eta>0,\omega>0 \tag{3}$$

$$F(x;\eta,\omega)=1-\left[1+\frac{\eta x(\omega\eta x+\eta+2\omega)}{\eta^2+\eta+2\omega}\right]e^{-\eta x};\; x>0,\eta>0\;,\omega>0 \tag{4}$$

In their study, [4] applied the GSD to four lifetime datasets and demonstrated that the GSD provides a superior fit compared to the SD, Lindley distribution, Aradhana distribution, and exponential distribution. It is important to highlight that the GSD reverts to the SD under specific conditions $\omega=1$. Additionally, [5] introduced the power Sujatha distribution (PSD) by employing power transformation on the SD. The PSD is defined by its pdf and cdf.

$$f(x;\eta,\tau)=\frac{\tau\eta^3}{\eta^2+\eta+2}x^{\tau-1}\left(1+x^\tau+x^{2\tau}\right)e^{-\eta x^\tau};x>0,\eta>0,\tau>0 \tag{5}$$

$$F(x;\eta,\tau)=1-\left[1+\frac{\eta x^{\tau}\left(\eta x^{\tau}+\eta+2\right)}{\eta^{2}+\eta+2}\right]e^{-\eta x^{\tau}};x>0,\eta>0,\tau>0 \tag{6}$$

Using power transformation in quasi Lindley distribution (QLD) discussed by [6], a Power quasi Lindley distribution (PQLD) has been proposed by [7] and defined by its pdf and cdf as

$$f\left(x;\eta,\omega,\tau\right)=\frac{\tau\eta}{\omega+1}x^{\tau-1}\left(\omega+\eta x^{\tau}\right)e^{-\eta x^{\tau}};x>0,\eta>0,\omega>0,\tau>0 \tag{7}$$

$$F(x;\eta,\omega,\tau)=1-\left[1+\frac{\eta x^{\tau}}{\omega+1}\right]e^{-\eta x^{\tau}};x>0,\eta>0,\omega>0,\tau>0 \tag{8}$$

It can be readily confirmed that the power Lindley distribution (PLD), as presented by [8], and the Lindley distribution proposed by [1] are specific cases of the PQLD for $\omega=\eta$ and $\omega=\eta$, $\tau=1$.

A generalized gamma distribution (GGD) proposed by [9] defined by its pdf and cdf as

$$f\left(x;\eta,\omega,\tau\right)=\frac{\tau\eta^{\omega}}{\Gamma(\omega)}x^{\tau\omega-1}e^{-\eta x^{\tau}};x>0,\eta>0,\omega>0,\tau>0 \tag{9}$$

$$F(x;\eta,\omega,\tau)=1-\frac{\Gamma\left(\omega,\eta x^{\tau}\right)}{\Gamma\left(\omega\right)};x>0,\eta>0,\omega>0,\tau>0 \tag{10}$$

Recently, a three-parameter Sujatha distribution (ThPSD) proposed by [10], and later its various statistical properties and applications has been studied by [11]. Its pdf and cdf define as

$$f\left(x;\eta,\omega,\tau\right)=\frac{\eta^{2}}{2\left(\eta^{2}+\tau+\omega\right)}\left(2\eta+2\tau x+\eta\omega x^{2}\right)e^{-\eta x};x>0,\eta>0,\omega>0,\tau>0 \tag{11}$$

$$F\left(x;\eta,\omega,\tau\right)=1-\left[1+\frac{\eta^{2}\omega x^{2}+2\eta\left(\tau+\omega\right)x}{2\left(\eta^{2}+\tau+\omega\right)}\right]e^{-\eta x};x>0,\eta>0,\omega>0,\tau>0 \tag{12}$$

The main motivation for proposing a Generalized Power Sujatha distribution is due to the fact that PQLD provides much closer fit than PLD, and the PSD provides much better fit than the PLD, it is expected and hoped that GPSD would provide better fit than PQLD. We discussed various important statistical properties of GPSD, estimation of parameters using maximum likelihood methods and applications to two lifetime data.

II. Generalized Power Sujatha distribution

Considering the power transformation $X = Y^{\frac{1}{\tau}}$ in the pdf (3), the pdf of a Generalized Power Sujatha distribution (GPSD) can be obtained as

$$f(x;\eta,\omega,\tau)=\frac{\tau\eta^3}{\eta^2+\eta+2\omega}\left(1+x^\tau+\omega x^{2\tau}\right)x^{\tau-1}e^{-\eta x^\tau};x>0,\eta>0,\omega>0,\tau>0 \tag{13}$$

$$= p_1 f_1(x;\eta,\tau)+p_2 f_2(x;\eta,\tau)+(1-p_1-p_2)\,f_3(x;\eta,\tau)$$

where $p_1=\dfrac{\eta^2}{\eta^2+\eta+2\omega}$, $p_2=\dfrac{\eta}{\eta^2+\eta+2\omega}$

$$f_1(x;\eta,\tau)=\tau\eta x^{\tau-1}e^{-\eta x^\tau},\ f_2(x;\eta,\tau)=\frac{\tau\eta^2}{\Gamma(2)}x^{2\tau-1}e^{-\eta x^\tau},\ f_3(x;\eta,\tau)=\frac{\tau\eta^3}{\Gamma(3)}x^{3\tau-1}e^{-\eta x^\tau}$$

This means that GPSD is a convex combination of Weibull (η,τ), Generalized Gamma $(2,\eta,\tau)$ and Generalized Gamma $(3,\eta,\tau)$ distributions. The corresponding cdf of GPSD can be obtained as

$$F(x;\eta,\omega,\tau)=1-\left[1+\frac{\eta x^\tau\left(\omega\eta x^\tau+\eta+2\omega\right)}{\eta^2+\eta+2\omega}\right]e^{-\eta x^\tau};x>0,\omega>0,\eta>0,\tau>0 \tag{14}$$

The behaviour of the pdf and the cdf of GPSD are presented in figures 1 and 2 respectively.

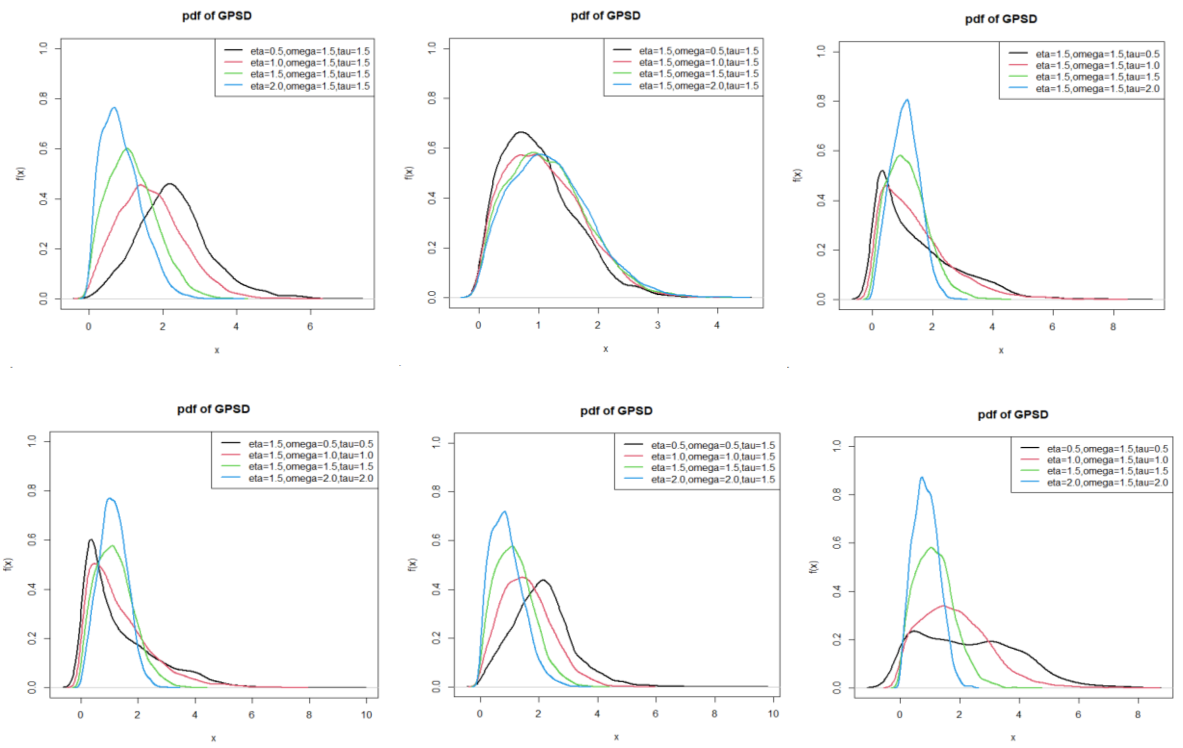

Figure 1: *pdf of GPSD for different values of the parameters*

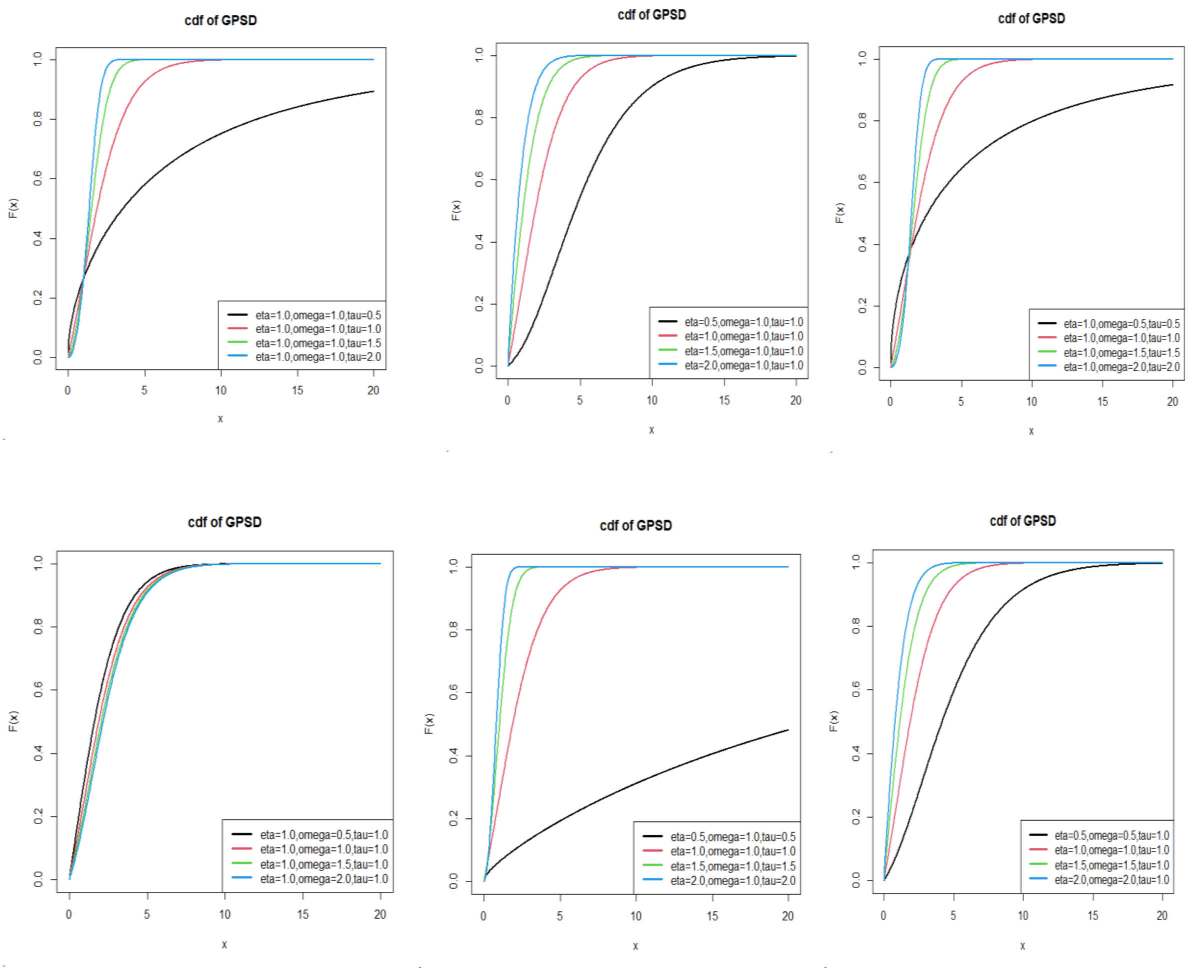

Figure 2: *cdf of GPSD for different values of the parameters*

III. Statistical Properties of GPSD

I. Survival Function

The Survival function of GPSD can be expressed as

$$S(x;\eta,\omega,\tau)=\left[\frac{\left(\eta^2+\eta+2\omega\right)+\eta x^\tau\left(\omega\eta x^\tau+\eta+2\omega\right)}{\eta^2+\eta+2\omega}\right]e^{-\eta x^\tau};x>0,\omega>0,\eta>0,\tau>0 \qquad (15)$$

II. Reverse Hazard Function

The reverse hazard function of GPSD can be obtained as

$$r(x;\eta,\omega,\tau)=\frac{\tau\eta^3\left(1+x^\tau+\omega x^{2\tau}\right)x^{\tau-1}e^{-\eta x^\tau}}{\left(\eta^2+\eta+2\omega\right)-\left[\left(\eta^2+\eta+2\omega\right)+\eta x^\tau\left(\omega\eta x^\tau+\eta+2\omega\right)\right]e^{-\eta x^\tau}};x>0,\omega>0,\eta>0,\tau>0 \qquad (16)$$

III. Hazard Function

The hazard function of GPSD can be obtained as

$$h(x;\eta,\omega,\tau)=\frac{\tau\eta^{3}\left(1+x^{\tau}+\omega x^{2\tau}\right)x^{\tau-1}}{\left(\eta^{2}+\eta+2\omega\right)+\eta x^{\tau}\left(\omega\eta x^{\tau}+\eta+2\omega\right)} \tag{17}$$

The behaviours of the hazard function of GPSD are explained in the following figure 3.

IV. Cumulative Hazard Function

The cumulative hazard function of GPSD can be obtained as

$$H(x;\eta,\omega,\tau)=-\log\left[\frac{\left\{\left(\eta^{2}+\eta+2\omega\right)+\eta x^{\tau}\left(\omega\eta x^{\tau}+\eta+2\omega\right)\right\}e^{-\eta x^{\tau}}}{\left(\eta^{2}+\eta+2\omega\right)}\right] \tag{18}$$

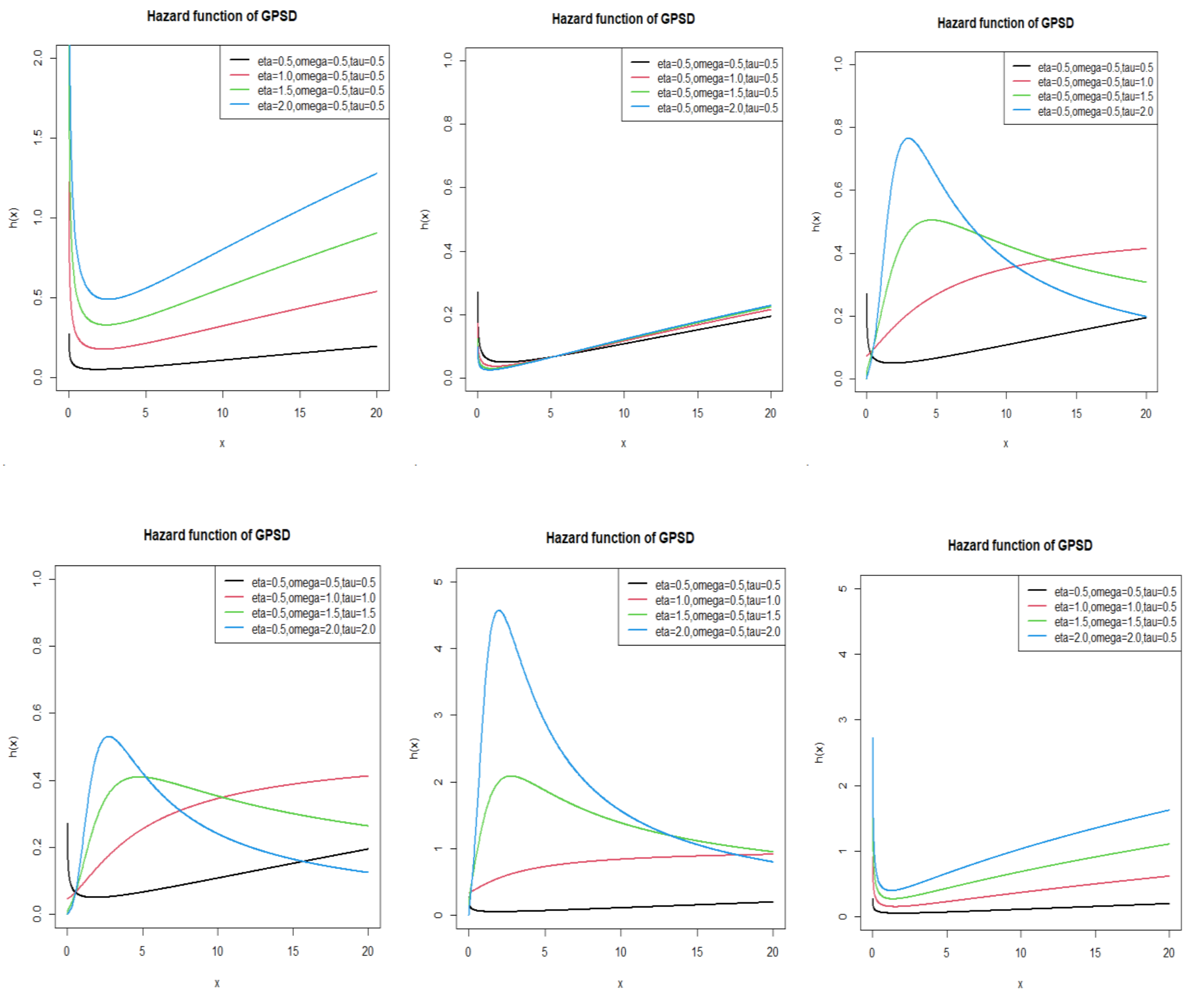

Figure 3: *Hazard function of GPSD for different values of the parameters*

V. Stochastic Ordering

In probability theory and statistics, a stochastic order quantifies the concept of one random variable being bigger than another. A random variable X is said to be smaller than a random variable Y in the:

i. Stochastic order $\left(X \leq_{st} Y\right)$ if $F_X(x) \geq F_Y(y)$ for all x
ii. Hazard rate order $\left(X \leq_{hr} Y\right)$ if $h_X(x) \geq h_Y(y)$ for all x
iii. Mean residual life order $\left(X \leq_{mrl} Y\right)$ if $m_X(x) \geq m_Y(y)$ for all x
iv. Likelihood ratio order $\left(X \leq_{lr} Y\right)$ if $\dfrac{f_X(x)}{f_Y(y)}$ decrease in x

The following results due to [12] are well known for establishing stochastic ordering of distributions

$$\begin{array}{c} X <_{lr} Y \Rightarrow X <_{hr} Y \Rightarrow X <_{mrl} Y \\ \Downarrow \\ X <_{st} Y \end{array}$$

Theorem: Let $X \sim$ GPSD $\left(\eta_1, \omega_1, \tau_1\right)$ and $Y \sim$ GPSD$\left(\eta_2, \omega_2, \tau_2\right)$. If $\omega_1 = \omega_2, \tau_1 = \tau_2$ and $\eta_1 > \eta_2$ or $\omega_1 = \omega_2, \eta_1 = \eta_2$ and $\tau_1 < \tau_2$ or $\eta_1 = \eta_2, \tau_1 = \tau_2$ and $\omega_1 < \omega_2$ then $X <_{lr} Y$ hence $X <_{hr} Y$, $X <_{mrl} Y$ and $X <_{st} Y$.

Proof: We have $$\frac{f_X(x)}{f_Y(x)} = \frac{\tau_1 \eta_1^3 \left(\eta_2^2 + \eta_2 + 2\omega_2\right)}{\tau_2 \eta_2^3 \left(\eta_1^2 + \eta_1 + 2\omega_1\right)} \left(\frac{1 + x^{\tau_1} + \omega_1 x^{2\tau_1}}{1 + x^{\tau_2} + \omega_2 x^{2\tau_2}}\right) x^{\tau_1 - \tau_2} e^{-\left(\eta_1 x^{\tau_1} - \eta_2 x^{\tau_2}\right)}$$

$$\log\left(\frac{f_X(x)}{f_Y(x)}\right) = \log\left[\frac{\tau_1 \eta_1^3 \left(\eta_2^2 + \eta_2 + 2\omega_2\right)}{\tau_2 \eta_2^3 \left(\eta_1^2 + \eta_1 + 2\omega_1\right)}\right] + \log\left(\frac{1 + x^{\tau_1} + \omega_1 x^{2\tau_1}}{1 + x^{\tau_2} + \omega_2 x^{2\tau_2}}\right) + \left(\tau_1 - \tau_2\right)\log x - \left(\eta_1 x^{\tau_1} - \eta_2 x^{\tau_2}\right)$$

$$\frac{d}{dx}\left[\log\left(\frac{f_X(x)}{f_Y(x)}\right)\right] = \frac{\tau_1 x^{\tau_1 - 1} + 2\omega_1 \tau_1 x^{2\tau_1 - 1}}{1 + x^{\tau_1} + \omega_1 x^{2\tau_1}} - \frac{\tau_2 x^{\tau_2 - 1} + 2\omega_2 \tau_2 x^{2\tau_2 - 1}}{1 + x^{\tau_2} + \omega_2 x^{2\tau_2}} + \frac{\tau_1 - \tau_2}{x} - \left(\eta_1 \tau_1 x^{\tau_1 - 1} - \eta_2 \tau_2 x^{\tau_2 - 1}\right)$$

Thus, for $\omega_1 = \omega_2, \tau_1 = \tau_2$ and $\eta_1 > \eta_2$ or $\omega_1 = \omega_2, \eta_1 = \eta_2$ and $\tau_1 < \tau_2$ or $\eta_1 = \eta_2, \tau_1 = \tau_2$ and $\omega_1 < \omega_2$, $\dfrac{d}{dx}\left[\log\left(\dfrac{f_X(x)}{f_Y(x)}\right)\right] < 0$. This means that $X <_{lr} Y$ hence $X <_{hr} Y$, $X <_{mrl} Y$ and $X <_{st} Y$.

VI. Mean Residual Life Function

The mean residual life function of GPSD can be obtained as

$$m(x,\eta,\omega,\tau)=\frac{\left(\eta^2+\eta+2\right)\Gamma\left(\frac{1}{\tau},\eta x^\tau\right)+\left(\eta+2\omega\right)\Gamma\left(\frac{1}{\tau}+1,\eta x^\tau\right)+\omega\Gamma\left(\frac{1}{\tau}+2,\eta x^\tau\right)}{\tau\eta^{\frac{1}{\tau}}\left[\left(\eta^2+\eta+2\omega\right)+\eta x^\tau\left(\omega\eta x^\tau+\eta+2\omega\right)\right]e^{-\eta x^\tau}} \tag{19}$$

The behaviours of the mean residual life function of GPSD are explained in the following figure 4.

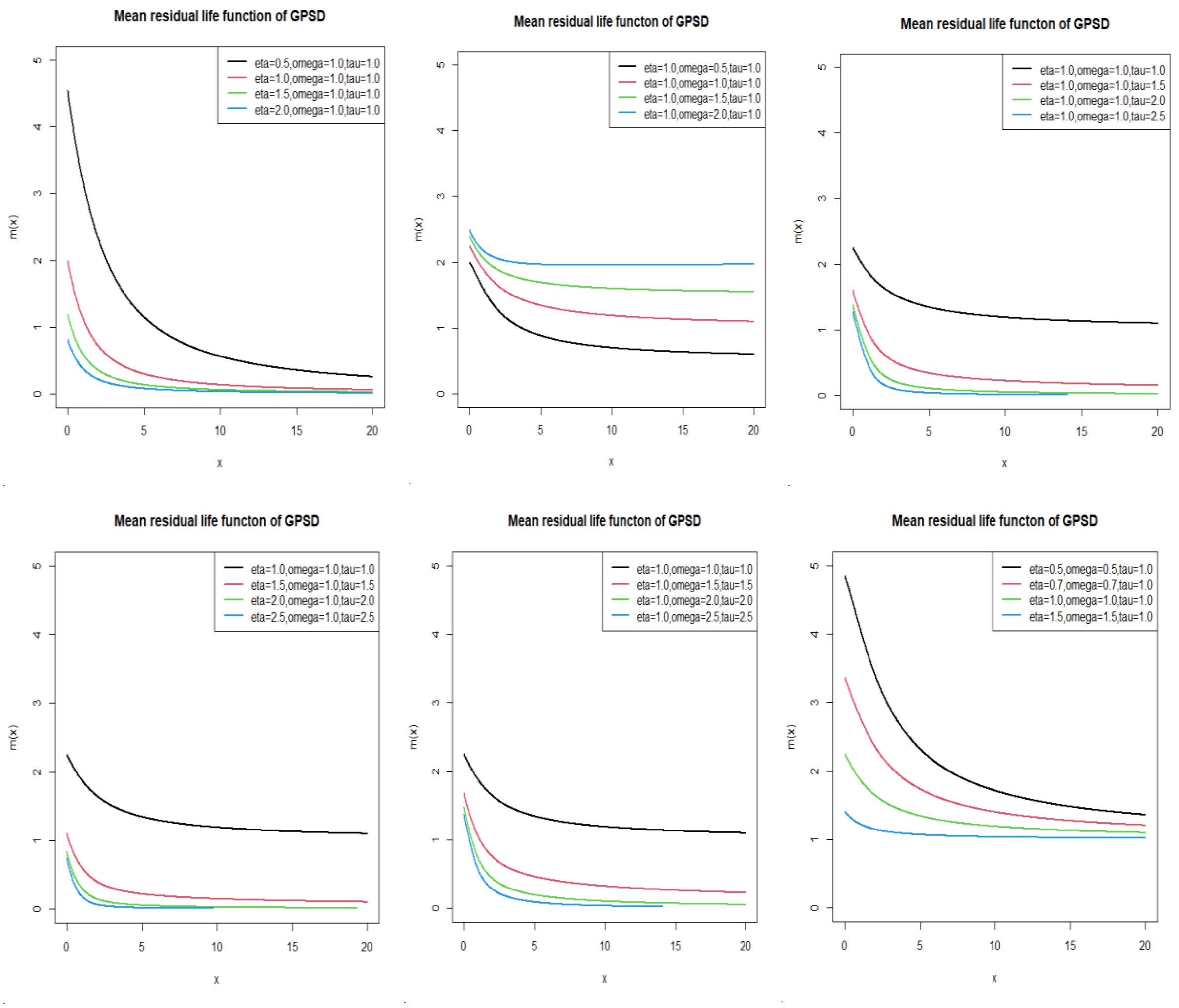

Figure 4: *Mean residual life function of GPSD for different values of the parameters*

IV. Moments Based Measures

The rth moments about origin of GPSD can be obtained as

$$\mu_r{}' = \int_0^\infty x^r \, f(x,\eta,\omega,\tau)dt = \frac{r\Gamma\left(\frac{r}{\tau}\right)\left[\eta^2\tau^2 + \eta\tau(r+\tau) + \omega(r+\tau)(r+2\tau)\right]}{\tau^3\eta^{\frac{r}{\tau}}\left(\eta^2+\eta+2\omega\right)} \tag{20}$$

Putting $r = 1,2,3,4$ in (20) we get

$$\mu_1{}' = \frac{\Gamma\left(\frac{1}{\tau}\right)\left[\eta^2\tau^2 + \eta\tau(1+\tau) + \omega(1+\tau)(1+2\tau)\right]}{\tau^3\eta^{\frac{1}{\tau}}\left(\eta^2+\eta+2\omega\right)}$$

$$\mu_2{}' = \frac{2\Gamma\left(\frac{2}{\tau}\right)\left[\eta^2\tau^2 + \eta\tau(2+\tau) + \omega(2+\tau)(2+2\tau)\right]}{\tau^3\eta^{\frac{2}{\tau}}\left(\eta^2+\eta+2\omega\right)}$$

$$\mu_3{}' = \frac{3\Gamma\left(\frac{3}{\tau}\right)\left[\eta^2\tau^2 + \eta\tau(3+\tau) + \omega(3+\tau)(3+2\tau)\right]}{\tau^3\eta^{\frac{3}{\tau}}\left(\eta^2+\eta+2\omega\right)}$$

$$\mu_4{}' = \frac{4\Gamma\left(\frac{4}{\tau}\right)\left[\eta^2\tau^2 + \eta\tau(4+\tau) + \omega(4+\tau)(4+2\tau)\right]}{\tau^3\eta^{\frac{4}{\tau}}\left(\eta^2+\eta+2\omega\right)}$$

V. Maximum Likelihood Estimation of the Parameters

Let's as $\left(x_1, x_2, ..., x_n\right)$ be a random sample of size n taken from GPSD$\left(\eta,\omega,\tau\right)$. The likelihood function is defined as

$$L = \left(\frac{\tau\eta^3}{\left(\eta^2+\eta+2\omega\right)}\right)^n \prod_{i=1}^n \left(1+x_i^\tau + \omega x_i 2\tau\right) x^{\tau-1} e^{-\eta x_i^\tau}$$

The log-likelihood function is obtained as

$$\log L = n\left[3\log\eta + \log\tau - \log\left(\eta^2+\eta+2\omega\right)\right] + \sum_{i=1}^n \log\left(1+x_i^\tau + \omega x_i^{\,2\tau}\right) + (\tau-1)\sum_{i=1}^n \log x_i - \eta\sum_{i=1}^n x_i^\tau$$

Now, the log-likelihood equations are given by

$$\frac{\partial \log L}{\partial \eta}=\frac{3n}{\eta}-\frac{n(2\eta+1)}{\eta^2+\eta+2\omega}-\sum_{i=1}^{n} x_i^{\tau}=0$$

$$\frac{\partial \log L}{\partial \omega}=-\frac{2n}{\eta^2+\eta+2\omega}+\sum_{i=1}^{n}\frac{x_i^{2\tau}}{1+x_i^{\tau}+\omega x_i^{2\tau}}=0$$

$$\frac{\partial \log L}{\partial \tau}=\frac{n}{\tau}-+\sum_{i=1}^{n}\frac{x_i^{\tau}\log x_i+\omega x_i^{2\tau}\log x_i}{1+x_i^{\tau}+\omega x_i^{2\tau}}+\sum_{i=1}^{n}\log x_i-\eta\sum_{i=1}^{n}x_i^{\tau}\log x_i=0$$

Solving these three log-likelihood equations directly may not be straightforward. Utilizing maximization techniques within R software is necessary to iteratively solve the likelihood function until sufficiently close values of the parameters are achieved. This methodology is crucial for addressing the intricacies inherent in solving log-likelihood equations and ensuring the accuracy and reliability of the parameter estimates in statistical analyses.

Fisher's scoring method is widely used in statistical software packages like R for optimizing likelihood-based models, especially in cases where direct analytical solutions are challenging. It is a powerful tool for estimating parameters in a variety of statistical models, contributing to the robustness and efficiency of the estimation process.

$$\frac{\partial^2 \log L}{\partial \eta^2}=-\frac{3n}{\eta^2}-\frac{2n\left(\eta^2+\eta+2\omega\right)-n(2\eta+1)^2}{\left(\eta^2+\eta+2\omega\right)^2}$$

$$\frac{\partial^2 \log L}{\partial \omega^2}=\frac{4n}{\left(\eta^2+\eta+2\omega\right)}-\sum_{i=1}^{n}\frac{x_i^{4\tau}}{\left(1+x_i^{\tau}+\omega x_i^{2\tau}\right)^2}=0$$

$$\frac{\partial^2 \log L}{\partial \tau^2}=-\frac{n}{\tau^2}-+\sum_{i=1}^{n}\frac{\left(\log x_i\right)^2\left(x_i^{\tau}+\omega x_i^{2\tau}\right)-\left(x_i^{\tau}\log x_i+\omega x_i^{2\tau}\log x_i\right)^2}{1+x_i^{\tau}+\omega x_i^{2\tau}}+\eta\sum_{i=1}^{n}x_i^{\tau}\left(\log x_i\right)^2$$

$$\frac{\partial^2 \log L}{\partial \eta\,\partial \omega}=\frac{2n(2\eta+1)}{\left(\eta^2+\eta+2\omega\right)^2}=\frac{\partial^2 \log L}{\partial \omega\,\partial \eta}$$

$$\frac{\partial^2 \log L}{\partial \eta\,\partial \tau}=\sum_{i=1}^{n}x_i^{\tau}\log x_i=\frac{\partial^2 \log L}{\partial \tau\,\partial \eta}$$

$$\frac{\partial^2 \log L}{\partial \tau\,\partial \omega}=\sum_{i=1}^{n}\frac{x_i^{2\tau}\log x_i\left(1+x_i^{\tau}+\omega x_i^{2\tau}\right)-\left(x_i^{\tau}\log x_i+\omega x_i^{2\tau}\log x_i\right)x_i^{2\tau}}{\left(1+x_i^{\tau}+\omega x_i^{2\tau}\right)^2}=\frac{\partial^2 \log L}{\partial \omega\,\partial \tau}$$

For finding the MLEs $(\hat{\eta},\hat{\omega},\hat{\tau})$ of parameters (η,ω,τ) of GPSD, following equations can be solved

$$\begin{bmatrix} \frac{\partial^2 \log L}{\partial \eta^2} & \frac{\partial^2 \log L}{\partial \eta \partial \omega} & \frac{\partial^2 \log L}{\partial \eta \partial \tau} \\ \frac{\partial^2 \log L}{\partial \omega \partial \eta} & \frac{\partial^2 \log L}{\partial \omega^2} & \frac{\partial^2 \log L}{\partial \omega \partial \tau} \\ \frac{\partial^2 \log L}{\partial \tau \partial \eta} & \frac{\partial^2 \log L}{\partial \tau \partial \omega} & \frac{\partial^2 \log L}{\partial \tau^2} \end{bmatrix}_{\substack{\hat{\eta}=\eta_0 \\ \hat{\omega}=\omega_0 \\ \hat{\tau}=\tau_0}} \begin{bmatrix} \hat{\eta}-\eta_0 \\ \hat{\omega}-\omega_0 \\ \hat{\tau}-\tau_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial \log L}{\partial \eta} \\ \frac{\partial \log L}{\partial \omega} \\ \frac{\partial \log L}{\partial \tau} \end{bmatrix} \qquad (21)$$

where η_0, ω_0 and τ_0 are the initial values of η, ω and τ . These equations are solved iteratively till close estimates of parameters are obtained.

VI. Simulation Studies

In this section, a simulation study was conducted to assess the performance of maximum likelihood estimators for the GPSD. The investigation involved the examination of mean estimates, biases (B), mean square errors (MSEs), and variances of the Maximum Likelihood Estimates (MLEs). These metrics were computed using the following formulas:

$$Mean = \frac{1}{n}\sum_{i=1}^{n}\hat{H}_i a\,,\ B = \frac{1}{n}\sum_{i=1}^{n}\left(\hat{H}_i - H\right),\ MSE = \frac{1}{n}\sum_{i=1}^{n}\left(\hat{H}_i - H\right)^2,\ Variance = MSE - B^2$$

where $H = \eta, \omega, \tau$ and $\hat{H}_i = \hat{\eta}_i, \hat{\omega}_i, \hat{\tau}_i$

the simulation results for different parameter values of GPSD are presented in tables 1 and 2 respectively. The steps for simulation study are as follows:

a. Data is generated using the acceptance-rejection method in simulation studies, a commonly employed approach to generate random samples from a target distribution when the inverse transform method of simulation is impractical or inefficient. The acceptance-rejection method for generating random samples from GPSD involves the following steps:

   i. Generate a random variable Y distributed as Gamma (η, ω)
   ii. Generate U distributed as Uniform $(0,1)$
   iii. If $U \le \frac{f(y)}{M\, g(y)}$, then set $X = Y$ (“accept the sample”); otherwise (“reject the sample”) and if reject then repeat the process: step (i-iii) until getting the required samples. Where M is a constant.

b. The sample sizes are taken as $n = 25, 50, 100, 200, 300$
c. The parameter values are set as values $\eta = 1.0$, $\omega = 0.1$, $\tau = 1.0$. and $\eta = 1.5$, $\omega = 1.5$, $\tau = 1.5$
d. Each sample size is replicated 10000 times

The findings from Tables 1 and 2 demonstrate that with an increase in sample size, biases, MSEs, and variances of the MLEs for the parameters decrease correspondingly. This result is in line with the first-order asymptotic theory.

Table-1: *The Mean values, Biases, MSEs and Variances of GPSD for parameter values* $\eta = 1.0$, $\omega = 0.1$, $\tau = 1.0$

Parameters	Sample size (n)	Mean	Bias	MSE	Variance
η	25	1.036569	0.03656	0.00220	0.00087
	50	1.035792	0.03579	0.00204	0.00076
	100	1.02854	0.02854	0.00140	0.00059
	200	1.02554	0.02554	0.00113	0.00048
	300	1.02470	0.02470	0.00106	0.00045
ω	25	0.12499	0.02499	0.00084	0.00021
	50	0.12213	0.02213	0.00065	0.00016
	100	0.12105	0.02105	0.00058	0.00014
	200	0.11927	0.01927	0.00050	0.00013
	300	0.11691	0.01691	0.00041	0.00012
τ	25	1.13596	0.13596	0.01984	0.00135
	50	1.13276	0.13276	0.01862	0.00099
	100	1.13212	0.13212	0.01828	0.00082
	200	1.13025	0.13025	0.01757	0.00060
	300	1.50287	0.00287	0.00030	0.00030

Table-2: *The Mean values, Biases, MSEs and Variances of GPSD for parameter values* $\eta = 1.5$, $\omega = 1.5$, $\tau = 1.5$

Parameters	Sample size (n)	Mean	Bias	MSE	Variance
η	25	1.50935	0.50935	0.00106	0.00097
	50	1.50575	0.00575	0.00082	0.00078
	100	1.50525	0.00525	0.00074	0.00071
	200	1.50299	0.00299	0.00062	0.00061
	300	1.50217	0.00217	0.00055	0.00055
ω	25	1.50832	0.00832	0.00087	0.00080
	50	1.50765	0.00765	0.00063	0.00057
	100	1.50558	0.00558	0.00055	0.00052
	200	1.50406	0.00406	0.00042	0.00041
	300	1.50355	0.00355	0.00038	0.00037
τ	25	1.50703	0.00703	0.00041	0.00036
	50	1.50388	0.00388	0.00037	0.00036
	100	1.50355	0.00355	0.00033	0.00031
	200	1.50304	0.00304	0.00031	0.00030
	300	1.50287	0.00287	0.00030	0.00030

VII. Applications Of GPSD

To assess the goodness of fit of the Generalized Power Sujatha distribution (GPSD) compared to other three-parameter, two-parameter, and one-parameter lifetime distributions, the following real lifetime datasets have been examined

Data set-1: The COVID-19 data set has the following source,

http://covid.gov.pk/stats/pakistan. It contains the daily recovered cases of COVID-19 inPakistan from 24 March to 28 April 2020 (36 days). The considered values are given:

2, 2, 3, 4, 26, 24, 25, 19, 4, 40, 87, 172, 38, 105, 155, 35, 264, 69, 283, 68, 199, 120, 67, 36,102, 96, 90, 181, 190, 228, 111, 163, 204, 192, 627, 263.

Data set-2: The following set of complete left skewed data, discussed by [13], reports the failure times of 20 components. The values are:

0.481, 1.196, 1.438, 1.797, 1.811, 1.831, 1.885, 2.104, 2.133, 2.144, 2.282, 2.322, 2.334,2.341, 2.428, 2.447, 2.511, 2.593, 2.715, 3.218.

Table 3: *MLEs, $-2\log L$, AIC, K-S and p-value of the considered distributions for the data set-1*

Distribution	MLE			$-2\log L$	AIC	K-S	p-value
	$\hat{\eta}$	$\hat{\omega}$	$\hat{\tau}$				
GPSD	0.1553	0.1000	0.6031	**416.84**	**422.84**	**0.10**	**0.88**
PQLD	0.1000	0.1648	0.6383	417.00	423.00	0.16	0.34
TPSD	0.1000	12.4100	0.6929	815.27	821.27	0.86	0.00
GGD	0.1000	1.7828	0.6282	417.41	423.41	0.19	0.16
GSD	0.0251	7.2040	…	470.51	474.51	0.20	0.11
WD	0.1000	…	0.5483	429.56	433.56	0.68	0.00
SD	0.0250	…	…	467.13	469.13	0.27	0.01

Table 4: *MLEs, $-2\log L$, AIC, K-S and p-value of the considered distributions for the data set-2*

Distribution	MLE			$-2\log L$	AIC	K-S	p-value
	$\hat{\eta}$	$\hat{\omega}$	$\hat{\tau}$				
GPSD	0.2587	0.3236	2.8755	**33.40**	**39.40**	**0.22**	**0.25**
PQLD	0.1000	0.3464	3.5053	33.49	39.49	0.57	0.00
TPSD	1.4280	14347.6730	0.1000	50.23	56.23	0.410	0.00
GGD	0.1000	1.4004	3.2768	35.61	41.61	0.60	0.00
GSD	1.4280	9294.9234	…	50.23	54.23	0.38	0.00
WD	0.1000	…	3.0368	38.15	42.15	0.57	0.00
SD	1.0538	…	…	60.20	62.20	0.51	0.00

From Table-3 and Table -4 we observed that GPSD has the lower $-2\log L$, AIC, and K-S values and higher p-values as compared to PQLD, TPSD, GGD, GSD, WD and SD. Hence, we may conclude that GPSD give the better fit than PQLD, TPSD, GGD, GSD, WD and SD. Further, it is also clear from the fitted plot of distributions in figure 5 that GPSD provide much closer fit over PQLD, TPSD, GGD, GSD, WD and SD.

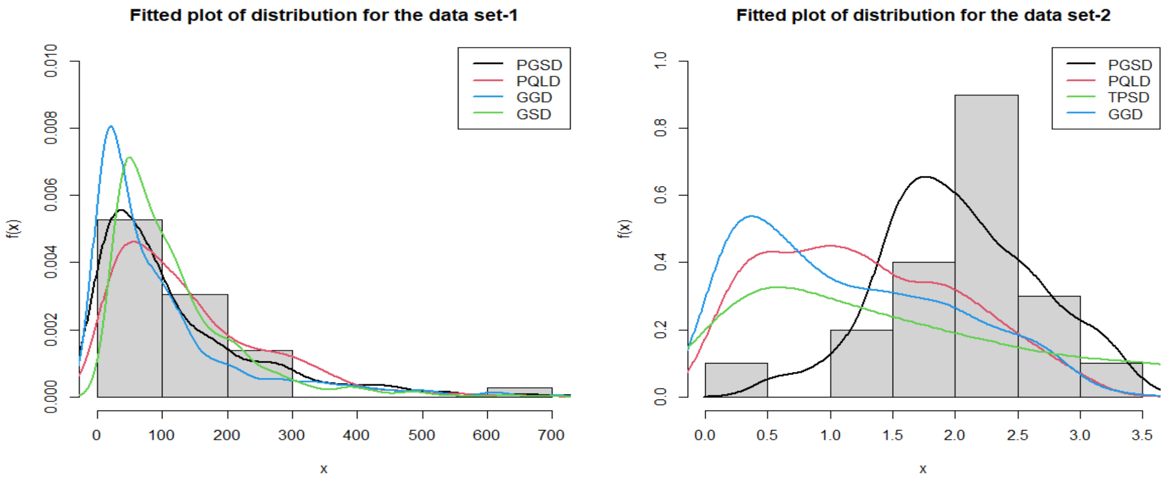

Figure 5: *Graph of the fitted plot of distributions for the data set-1 and data set-2*

VIII. Conclusion

This paper introduces the Generalized Power Sujatha distribution (GPSD) and explores its statistical properties, including moments, survival function, hazard function, reversed hazard function, mean residual life function, and stochastic ordering. The study employs maximum likelihood estimation to estimate the distribution's parameters and evaluates their performance through a simulation study. Additionally, the paper applies GPSD to two real lifetime datasets, comparing its fit with other distributions such as PQLD, TPSD, GGD, GSD, WD and SD. Results indicate that GPSD offers a superior fit compared to PQLD, TPSD, GGD, GSD, WD and SD.

References

[1] Lindley, D.V. (1958). Fiducial distributions and Bayes' theorem, Journal of the Royal Statistical Society, Series B, 20(1):102 – 107.

[2] Ghitany, M.E., Atieh, B. and Nadarajah, S. (2008). Lindley distribution and its application, Mathematics and Computer in Simulation, 78:493-506.

[3] Shanker, R. (2016): Sujatha Distribution and Its Applications**,** *Statistics in Transition-New Series,* 17(3): 391- 410.

[4] Shanker, R., Shukla K. K. and Fesshaye, H. (2017). A Generalization of Sujatha distribution and its applications with real lifetime data, *Journal of Institute of Science and Technology*. 22(1):66-83.

[5] Shanker, R. and Shukla, K.K (2019). A two-parameter power Sujatha distribution with properties and application, International Journal of Mathematics and Statistics, 20(3):11-22.

[6] Shanker, R. and Mishra, A. (2013). A quasi-Lindley distribution, African Journal of Mathematics and Computer Science Research, 6(4): 64-71.

[7] Alkarni S. Power quasi-Lindley distribution (2015). Properties and application. Asian Journal of Mathematics and Computer Research. 10(2):179-195.

[8] Ghitany, M.E., Al-Mutairi, D.K., Balakrishnan N. and Al-Enezi, L.J. (2013). Power Lindley distribution and associated inference, Computational Statistics & Data Analysis, 64, 20-33.

[9] Stacy, E.W. (1962). A generalized gamma distribution, Annals Mathematical Statistics, 33, 1187-1192.

[10] Nwikpe, B. J. and Iwoke, I. A. (2020). A three parameter Sujatha distribution with application, *European Journal of Statistics and Probability*, 8(3):22-34.

[11] Prodhani, H. R. and Shanker, R. (2023). On some statistical properties and applications of three-parameter Sujatha distribution, *Reliability: Theory & Applications*, 18(3 (74)): 514-527.

[12] Shaked, M. and Shanthikumar, J. (1994). Stochastic Orders and Their Applications. Academic Press, New York.

[13] Murthy, D.N.P, Xie, M. & Jiang, R. (2004). Weibull models, John Wily & Sons Inc., Hoboken.