COMPARATIVE BAYESIAN ANALYSIS OF THE INVERSE TOPP-LEONE DISTRIBUTION

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Abstract

This paper focuses on the Bayesian estimation of the shape parameter for the Inverse Topp-Leone (ITL) distribution. To achieve this, we employ both the extended Jeffrey's prior and the gamma prior, facilitating the derivation of posterior distributions for the shape parameter. The Bayesian estimators are calculated under various loss functions, including the squared error loss function (SELF), entropy loss function (ELF), precautionary loss function (PLF), and Linex loss function (LLF), each chosen to address different practical scenarios and estimator biases. In addition to the Bayesian approach, we also explore maximum likelihood estimation (MLE) to provide a comparative benchmark. The performance of these estimators is assessed and compared based on mean squared error (MSE) across multiple sample sizes, allowing for a detailed evaluation of estimator robustness and accuracy. A real-world dataset is then analyzed to further demonstrate the relative efficiency of each estimator under the different loss functions, providing practical insights into the applicability of each estimation approach for the ITL distribution. This analysis offers a comprehensive perspective on the versatility and precision of Bayesian and classical estimation methods for the ITL model.

Keywords: Bayesian analysis, Jeffery's prior, Gamma prior, Maximum likelihood estimation, Loss functions

I. Introduction

Topp-Leone distribution belongs to the distribution family which has support $[0,1]$. It indicates the j-shape form of density function along with bathtub shape of its hazard function. This distribution is used for the analysis of failure data. The probability density function of Topp-Leone is given by

$$g(x,\theta)=2\theta x^{\theta-1}(1-x)(2-x)^{\theta-1};\ \ 0\le x\le 1,\theta>0 \tag{1}$$

Since the Topp-Leone distribution is newly formulated distribution proposed by Topp and Leone [20]. This distribution has been studied by several authors such as Nadarajah [12], Ghitney et al [6, 7], Genc [8], Al-Zahrani [5], MirMostafaee [11], Vicari et al. [21]. Recently Hassan et al. [9] explored the inverse of Topp-Leone distribution. Researchers have extensively explored and generalized a variety of probability distributions. For instance, Rather and Ozel [17, 18] investigated the weighted Power Lindley distribution, and also examined the length-biased Power Lindley distribution, providing insights into its properties and applications. In addition, Rather et al. [19] introduced the exponentiated Ailamujia distribution and discussed its real-life applications. Qayoom and Rather [13] conducted a comprehensive study of the Weighted Transmuted Mukherjee-Islam distribution, analysing its statistical properties. Qayoom and Rather [14] also explored a new generalization of the Transmuted Mukherjee-Islam distribution. More recently, Qayoom et al. [15] presented an extension of the Lindley distribution, examining its practical utility in real-world scenarios.

Let X follows the probability distribution function of Topp-Leone distribution, then the transformation $Y=\frac{1}{X}-1$ is said to follow inverse of Topp-Leone distribution having probability density function (p.d.f) as

$$f(y,\theta)=2\theta y\frac{(1+2y)^{\theta-1}}{(1+y)^{2\theta+1}};y>0,\theta>0 \tag{2}$$

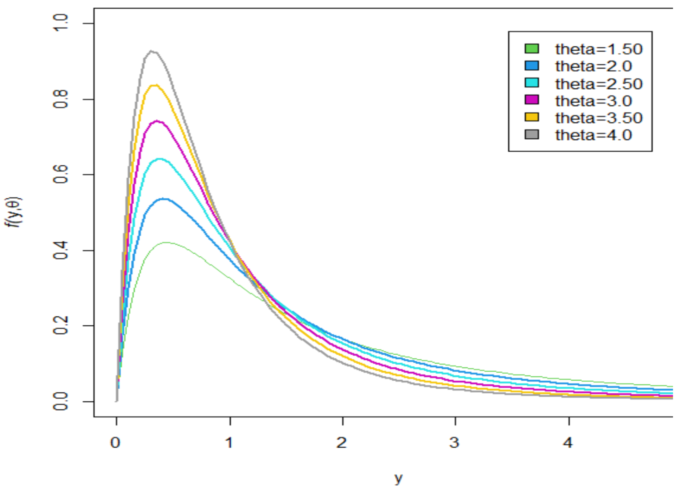

Figure 1: *p.d.f plot of ITLD under different values of parameters*

Figure 1, illustrates some possible shapes of p.d.f for varying parameters. The corresponding cumulative distribution function (c.d.f) of (2) is given by

$$F(y,\theta)=1-\left[\frac{(1+2y)^{\theta}}{(1+y)^{2\theta}}\right];y\geq 0,\theta>0 \tag{3}$$

II. Maximum Likelihood Estimation

Let $y_1,y_2,\ldots,y_n$ be random samples from inverse Topp-Leone distribution given by (2), then the likelihood function becomes

$$l=\prod_{i=1}^{n}f(y_i,\theta) \tag{4}$$

$$l=\prod_{i=1}^{n}(2\theta)y_i(1+y_i)^{-(2\theta+1)}(1+2y_i)^{\theta-1} \tag{5}$$

The log-likelihood function is

$$\log l = n\log 2 + n\log\theta + \sum_{i=1}^{n}\log y_i - (2\theta+1)\sum_{i=1}^{n}\log(y_i+1) + (\theta-1)\sum_{i=1}^{n}\log(2y_i+1) \tag{6}$$

Differentiate w.r.t θ we get

$$\frac{\partial \log l}{\partial\theta} = \frac{n}{\theta} - 2\sum_{i=1}^{n}\log(y_i+1) + \sum_{i=1}^{n}\log(2y_i+1) \tag{7}$$

Solving $\frac{\partial \log l}{\partial\theta} = 0$, we get

$$\hat{\theta} = \frac{n}{2\sum_{i=1}^{n}\log(y_i+1) - \sum_{i=1}^{n}\log(2y_i+1)} \tag{8}$$

III. Bayesian Estimation of Inverse Topp-Leone Distribution

Bayesian estimation is a highly effective approach for estimating the parameters of distribution models. This method incorporates prior knowledge to determine the posterior distribution of a lifetime distribution's parameters. From a Bayesian perspective, selecting an appropriate prior is flexible, as no single prior can be universally preferred; the choice depends on the available information about the parameter. When little prior knowledge about the parameter is available, a non-informative prior is typically chosen to minimize bias. However, when sufficient prior information is available, using an informative prior enhances the accuracy of the estimation. The goal of the current study is to derive a Bayesian estimation of the parameter for the inverse Topp-Leone distribution, specifically employing an extended Jeffreys prior and a gamma prior. In recent years, Bayesian estimation methods have gained considerable attention. For instance, Ahmad et al. [2] investigated Bayesian parameter estimation for the two-parameter exponentiated gamma distribution, while Mudasir et al. [10] focused on the weighted Erlang distribution. Raqab and Madi [16] explored Bayesian estimation for the exponentiated Rayleigh distribution. Recently, Ahmad et al. [3] examined Bayesian parameter estimation for the inverse Ailamujia distribution using various loss functions. In this study, both extended Jeffreys and gamma priors are considered. The extended Jeffreys prior is a non-informative prior, useful when interpretive information about the parameters is limited, while the gamma prior provides a more informative approach when substantial parameter knowledge is available. This Bayesian framework aims to enhance parameter estimation accuracy for the inverse Topp-Leone distribution, with relevance across various applied fields.

IV. Posterior Distribution of Inverse Topp-Leone Distribution

Suppose $y = (y_1, y_2, .., y_n)$ denotes the n recorded values of (2). Then its likelihood function is given by

$$L(y|\theta) = \prod_{i=1}^{n} 2\theta \frac{y_i}{(y_{i+1})^2}\left[\frac{2}{y_i+1} - \frac{1}{(y_i+1)^2}\right]^{\theta-1} \tag{9}$$

$$L(y|\theta) = 2^n\theta^n \prod_{i=1}^{n}\frac{y_i}{(y_i+1)^2} e^{-(\theta+1)\sum_{i=1}^{n}-\log\left[\frac{2}{y_1+1} - \frac{1}{(y_i+1)^2}\right]} \tag{10}$$

We assume the prior distribution of θ to be extended Jeffery's prior proposed by Alkutubi [4], is given by

$$g(\theta) \propto [I(\theta)]^c, c \in R^+$$

Where $[I(\theta)] = -nE\left[\frac{\partial^2 \log f(y,\theta)}{\partial^2 \theta}\right]$ is the Fisher information matrix, for the distribution (2),then

$$g(\theta) = K\left[\frac{n}{\theta^2}\right]^c \tag{11}$$

The posterior distribution of θ under the assumption of extended Jeffrey's prior i.e $g(\theta) \propto \frac{1}{\theta^{2c}}$ is given by

$$h(\theta|y) \propto L(y|\theta)g(\theta)$$

$$\Rightarrow h(\theta|y) \propto \prod_{i=1}^{n} 2^n \frac{y_i}{(y_i+1)^2} e^{\sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1} - \frac{1}{(y_i+1)^2}\right]} \theta^{n-2c} e^{-\theta \sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1} - \frac{1}{(y_i+1)^2}\right]} \tag{12}$$

$$\Rightarrow h(\theta|y) \propto K\theta^{n-2c} e^{-\theta \sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1} - \frac{1}{(y_i+1)^2}\right]} \tag{13}$$

Where K is independent of θ

$$K^{-1} = \int_0^\infty \theta^{n-2c} e^{-\theta \sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1} - \frac{1}{(y_i+1)^2}\right]} d\theta$$

$$K^{-1} = \frac{\Gamma(n-2c+1)}{\left\{\sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1} - \frac{1}{(y_i+1)^2}\right]\right\}^{n-2c+1}}$$

Therefore

$$K = \frac{\left\{\sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1} - \frac{1}{(y_i+1)^2}\right]\right\}^{n-2c+1}}{\Gamma(n-2c+1)} \tag{14}$$

Hence the posterior distribution of θ is given by

$$h(\theta|y) = \frac{T^{n-2c+1}}{\Gamma(n-2c+1)} \theta^{n-2c} e^{-\theta T} \tag{15}$$

Where,

$$T = \sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1} - \frac{1}{(y_i+1)^2}\right]$$

I. Estimation under square error loss function

The squared error loss function is defined as $l(\hat{\theta}, \theta) = c_1(\hat{\theta} - \theta)^2$ for some constant c_1. The risk function is given by

$$R(\hat{\theta}, \theta) = E[l(\hat{\theta}, \theta)]$$

$$R(\hat{\theta}, \theta) = \int_0^\infty l(\hat{\theta}, \theta) h(\theta|y) d\theta$$

$$R(\hat{\theta}, \theta) = \int_0^\infty c_1(\hat{\theta} - \theta)^2 \frac{T^{n-2c+1}}{\Gamma(n-2c+1)} \theta^{n-2c} e^{-\theta T} d\theta \tag{16}$$

$$R(\hat{\theta}, \theta) = \frac{T^{n-2c+1}}{\Gamma(n-2c+1)} c_1 \int_0^\infty (\hat{\theta} - \theta)^2 \theta^{n-2c} e^{-\theta T} d\theta \tag{17}$$

$$R(\hat{\theta}, \theta) = \frac{T^{n-2c+1}}{\Gamma(n-2c+1)} c_1 \left[\hat{\theta}^2 \int_0^\infty \theta^{n-2c} e^{-\theta T} d\theta + \int_0^\infty \theta^{n-2c+2} e^{-\theta T} d\theta - 2\hat{\theta} \int_0^\infty \theta^{n-2c+1} e^{-\theta T} d\theta\right]$$

After solving the integral, we get

$$R(\hat{\theta},\theta)=\frac{T^{n-2c+1}}{\Gamma(n-2c+1)}c_1\begin{bmatrix}\hat{\theta}\frac{\Gamma(n-2c+1)}{T^{n-2c+1}}+\frac{(n-2c+1)(n-2c+1)\Gamma(n-2c+1)}{T^{n-2c+2}}\\ -2\hat{\theta}\frac{(n-2c+1)\Gamma(n-2c+1)}{T^{n-2c+1}}\end{bmatrix}$$

$$R(\hat{\theta},\theta)=c_1\left[\hat{\theta}^2+\frac{(n-2c+1)(n-2c+2)}{T^2}-2\hat{\theta}\frac{(n-2c+1)}{T}\right]$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial\hat{\theta}}=0$, we get

$$\hat{\theta}_s=\frac{(n-2c+1)}{T} \tag{18}$$

Where,

$$T=\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]$$

II. Estimation under entropy loss function

The entropy loss function is defined as $l(\delta)=b[\delta-\log(\delta)-1]; b>0$, $\delta=\frac{\hat{\theta}}{\theta}$. The risk function is given by

$$R(\hat{\theta},\theta)=E[l(\delta)]$$

$$R(\hat{\theta},\theta)=\int_0^\infty l(\delta)h(\theta|y)d\theta$$

$$R(\hat{\theta},\theta)=\int_0^\infty b[\delta-\log(\delta)-1]h(\theta|y)d\theta \tag{19}$$

$$R(\hat{\theta},\theta)=b\frac{T^{n-2c+1}}{\Gamma(n-2c+1)}\int_0^\infty\left[\frac{\hat{\theta}}{\theta}-\log\left(\frac{\hat{\theta}}{\theta}\right)-1\right]\theta^{n-2c}e^{-\theta T}d\theta \tag{20}$$

$$R(\hat{\theta},\theta)=\frac{T^{n-2c+1}}{\Gamma(n-2c+1)}b\left[\hat{\theta}\int_0^\infty\theta^{n-2c-1}e^{-\theta T}d\theta-\log(\hat{\theta})\int_0^\infty\theta^{n-2c}e^{-\theta T}d\theta+\int_0^\infty\log(\theta)\theta^{n-2c}e^{-\theta T}d\theta-\int_0^\infty\theta^{n-2c}e^{-\theta T}d\theta\right]$$

After solving the integral, we get

$$R(\hat{\theta},\theta)=b\frac{T^{n-2c+1}}{\Gamma(n-2c+1)}\left[\hat{\theta}\frac{\Gamma(n-2c)}{T^{n-2c}}-\log(\hat{\theta})\frac{\Gamma(n-2c+1)}{T^{n-2c+1}}+\frac{\Gamma'(n-2c+1)}{T^{n-2c+1}}-\frac{\Gamma(n-2c+1)}{T^{n-2c+1}}\right]$$

$$R(\hat{\theta},\theta)=b\left[\hat{\theta}\frac{T}{n-2c}-\log(\hat{\theta})+\psi'(n-2c+1)-1\right] \tag{21}$$

Where $\psi'(.)$ denotes the digamma function

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial\hat{\theta}}=0$, we get

$$\hat{\theta}_e=\frac{(n-2c)}{T} \tag{22}$$

Where,

$$T = \sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1} - \frac{1}{(y_i+1)^2}\right]$$

III. Estimation under precautionary loss function

The entropy loss function is defined as $l(\hat{\theta},\theta) = \frac{(\hat{\theta}-\theta)^2}{\hat{\theta}}$. The risk function is given by

$$R(\hat{\theta},\theta) = E[l(\hat{\theta},\theta)]$$

$$R(\hat{\theta},\theta) = \int_0^\infty l(\hat{\theta},\theta)h(\theta|y)d\theta$$

$$R(\hat{\theta},\theta) = \frac{T^{n-2c+1}}{\Gamma(n-2c+1)} \int_0^\infty \frac{(\hat{\theta}-\theta)^2}{\hat{\theta}} \theta^{n-2c} e^{-\theta T} d\theta \tag{23}$$

$$R(\hat{\theta},\theta) = \frac{T^{n-2c+1}}{\Gamma(n-2c+1)}\left[\hat{\theta}\int_0^\infty \theta^{n-2c} e^{-\theta T} d\theta + \frac{1}{\hat{\theta}}\int_0^\infty \theta^{n-2c+2} e^{-\theta T} d\theta - 2\int_0^\infty \theta^{n-2c+1} e^{-\theta T} d\theta\right]$$

After solving the integral, we get

$$R(\hat{\theta},\theta) = \left[\hat{\theta} + \frac{1}{\hat{\theta}}\frac{(n-2c+1)(n-2c+2)}{T^2} - \frac{2(n-2c+1)}{T}\right] \tag{24}$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial \hat{\theta}} = 0$, we get

$$\hat{\theta}_p = \frac{[(n-2c+1)(n-2c+2)]^{\frac{1}{2}}}{T} \tag{25}$$

Where,

$$T = \sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1} - \frac{1}{(y_i+1)^2}\right]$$

IV. Estimation under linex loss function

The linex loss function is defined as $l(\hat{\theta},\theta) = \exp\{b_1(\hat{\theta}-\theta)\} - b_1(\hat{\theta}-\theta) - 1$.The risk function is given by

$$R(\hat{\theta},\theta) = E[l(\hat{\theta},\theta)]$$

$$R(\hat{\theta},\theta) = \int_0^\infty l(\hat{\theta},\theta)h(\theta|y)d\theta$$

$$R(\hat{\theta},\theta) = \frac{T^{n-2c+1}}{\Gamma(n-2c+1)} \int_0^\infty \left[\exp\{b_1(\hat{\theta}-\theta)\} - b_1(\hat{\theta}-\theta) - 1\right]\theta^{n-2c} e^{-\theta T} d\theta \tag{26}$$

$$R(\hat{\theta},\theta) = \frac{T^{n-2c+1}}{\Gamma(n-2c+1)}\left[e^{b_1\hat{\theta}}\int_0^\infty \theta^{n-2c} e^{-\theta(b_1+T)} d\theta - b_1\hat{\theta}\int_0^\infty \theta^{n-2c} e^{-\theta T} d\theta + b_1\int_0^\infty \theta^{n-2c+1} e^{-\theta T} d\theta - \int_0^\infty \theta^{n-2c} e^{-\theta T} d\theta\right]$$

After solving the integral, we get

$$R(\hat{\theta},\theta) = \left[e^{b_1\theta}\left(\frac{T}{b_1+T}\right) - b_1\hat{\theta} + \frac{b_1(n-2c+1)}{T} - 1\right] \tag{27}$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial \hat{\theta}}=0$, we get

$$\hat{\theta}_l=\frac{(n-2c+1)}{b_1}\log\left(\frac{b_1+T}{T}\right) \tag{28}$$

Where,

$$T=\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]$$

V. Posterior Distribution of Inverse Topp-Leone Distribution under Gamma Prior

Suppose $y=(y_1,y_2,..,y_n)$ denotes the n recorded values of (2). Then its likelihood function is given by

$$L(y|\theta)=\prod_{i=1}^{n}2\theta\frac{y_i}{(y_{i+1})^2}\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]^{\theta-1} \tag{29}$$

$$L(y|\theta)=2^n\theta^n\prod_{i=1}^{n}\frac{y_i}{(y_i+1)^2}e^{-(\theta+1)\sum_{i=1}^{n}-\log\left[\frac{2}{y_1+1}-\frac{1}{(y_i+1)^2}\right]} \tag{30}$$

We assume the prior distribution of θ to be gamma prior

The posterior distribution of θ under the assumption of gamma prior i.e., $g(\theta)\propto\frac{a^b}{\Gamma(b)}\theta^{b-1}e^{-a\theta}$ is given by

$$h(\theta|y)\propto L(y|\theta)g(\theta)$$

$$\Rightarrow h(\theta|y)\propto 2^n\theta^n\prod_{i=1}^{n}\frac{y_i}{(y_i+1)^2}e^{-(\theta+1)\sum_{i=1}^{n}-\log\left[\frac{2}{y_1+1}-\frac{1}{(y_i+1)^2}\right]}\frac{a^b}{\Gamma(b)}\theta^{b-1}e^{-a\theta}$$

$$\Rightarrow h(\theta|y)\propto\prod_{i=1}^{n}2^n\frac{y_i}{(y_i+1)^2}e^{\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]}\frac{a^b}{\Gamma(b)}\theta^{n+b-1}e^{-\theta\left(a+\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]\right)}$$

$$\Rightarrow h(\theta|y)\propto K\theta^{n+b-1}e^{-\theta\left(a+\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]\right)}$$

Where K is independent of θ

$$K^{-1}=\int_0^\infty\theta^{n+b-1}e^{-\theta\left(a+\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]\right)}d\theta \tag{31}$$

$$K^{-1}=\frac{\Gamma(n+b)}{\left\{a+\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]\right\}^{n+b}} \tag{32}$$

Therefore,

$$K=\frac{\left\{a+\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]\right\}^{n+b}}{\Gamma(n+b)} \tag{33}$$

Hence the posterior distribution of θ is given by

$$h(\theta|y)=\frac{(a+T)^{n+b}}{\Gamma(n+b)}\theta^{n+b-1}e^{-(a+T)\theta} \tag{34}$$

Where,

$$T=\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]$$

I. Estimation under square error loss function

The squared error loss function is defined as $l(\hat{\theta},\theta)=c_1(\hat{\theta}-\theta)^2$ for some constant c_1. The risk function is given by

$$R(\hat{\theta},\theta)=E[l(\hat{\theta},\theta)]$$

$$R(\hat{\theta},\theta)=\int_0^\infty l(\hat{\theta},\theta)h(\theta|y)d\theta$$

$$R(\hat{\theta},\theta)=c_1\frac{(a+T)^{n+b}}{\Gamma(n+b)}\int_0^\infty(\hat{\theta}-\theta)^2\theta^{n+b-1}e^{-(a+T)\theta}d\theta \tag{35}$$

$$R(\hat{\theta},\theta)=c_1\frac{(a+T)^{n+b}}{\Gamma(n+b)}\left\{\hat{\theta}^2\int_0^\infty\theta^{n+b-1}e^{-(a+T)\theta}d\theta+\int_0^\infty\theta^{n+b+1}e^{-(a+T)\theta}d\theta-2\hat{\theta}\int_0^\infty\theta^{n+b}e^{-(a+T)\theta}d\theta\right\}$$

After solving the integral, we get

$$R(\hat{\theta},\theta)=c_1\left\{\hat{\theta}^2+\frac{(n+b)(n+b+1)}{(a+T)^2}-2\hat{\theta}\frac{(n+b)}{a+T}\right\} \tag{36}$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial\hat{\theta}}=0$, we get

$$\hat{\theta}_s=\frac{n+b}{a+T} \tag{37}$$

Where,

$$T=\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]$$

II. Estimation under entropy loss function

The entropy loss function is defined as $l(\delta)=b[\delta-\log(\delta)-1]; b>0, \delta=\frac{\hat{\theta}}{\theta}$. The risk function is given by

$$R(\hat{\theta},\theta)=E[l(\delta)]$$

$$R(\hat{\theta},\theta)=\int_0^\infty l(\delta)h(\theta|y)d\theta$$

$$R(\hat{\theta},\theta)=\int_0^\infty b[\delta-\log(\delta)-1]h(\theta|y)d\theta \tag{38}$$

$$R(\hat{\theta},\theta)=b\frac{(a+T)^{n+b}}{\Gamma(n+b)}\int_0^\infty\left[\frac{\hat{\theta}}{\theta}-\log\left(\frac{\hat{\theta}}{\theta}\right)-1\right]\theta^{n+b-1}e^{-(a+T)\theta}d\theta \tag{39}$$

After solving the integral, we get

$$R(\hat{\theta},\theta)=b\left\{\hat{\theta}\frac{(a+T)}{(n+b-1)}-\log\hat{\theta}+\psi'(n+b)-1\right\} \tag{40}$$

Where $\psi'(.)$ denotes the digamma function

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial\hat{\theta}}=0$, we get

$$\hat{\theta}_e=\frac{n+b-1}{a+T} \tag{41}$$

Where,

$$T=\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]$$

III. Estimation under precautionary loss function

The entropy loss function is defined as $l(\hat{\theta},\theta)=\frac{(\hat{\theta}-\theta)^2}{\hat{\theta}}$ .The risk function is given by

$$R(\hat{\theta},\theta)=E[l(\hat{\theta},\theta)]$$

$$R(\hat{\theta},\theta)=\int_0^\infty l(\hat{\theta},\theta)h(\theta|y)d\theta$$

$$R(\hat{\theta},\theta)=\frac{(a+T)^{n+b}}{\Gamma(n+b)}\int_0^\infty\frac{(\hat{\theta}-\theta)^2}{\hat{\theta}}\theta^{n+b-1}e^{-(a+T)\theta}d\theta \tag{42}$$

$$R(\hat{\theta},\theta)=\frac{(a+T)^{n+b}}{\Gamma(n+b)}\left\{\hat{\theta}\int_0^\infty\theta^{n+b-1}e^{-(a+T)\theta}d\theta+\frac{1}{\hat{\theta}}\int_0^\infty\theta^{n+b+1}e^{-(a+T)\theta}d\theta-2\int_0^\infty\theta^{n+b}e^{-(a+T)\theta}d\theta\right\}$$

After solving the integral, we get

$$R(\hat{\theta},\theta)=\left\{\hat{\theta}+\frac{1}{\hat{\theta}}\frac{(n+b)(n+b+1)}{(a+T)^2}-2\frac{(n+b)}{(a+T)}\right\} \tag{43}$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial\hat{\theta}}=0$, we get

$$\hat{\theta}_p=\frac{[(n+b)(n+b+1)]^{\frac{1}{2}}}{a+T} \tag{44}$$

Where,

$$T=\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]$$

IV. Estimation under linex loss function

The linex loss function is defined as $l(\hat{\theta},\theta)=\exp\{b_1(\hat{\theta}-\theta)\}-b_1(\hat{\theta}-\theta)-1$.The risk function is given by

$$R(\hat{\theta},\theta)=E[l(\hat{\theta},\theta)]$$

$$R(\hat{\theta},\theta)=\int_0^\infty l(\hat{\theta},\theta)h(\theta|y)d\theta$$

$$R(\hat{\theta},\theta)=\frac{(a+T)^{n+b}}{\Gamma(n+b)}\int_0^\infty\left[\exp\{b_1(\hat{\theta}-\theta)\}-b_1(\hat{\theta}-\theta)-1\right]\theta^{n+b-1}e^{-(a+T)\theta}d\theta \qquad (45)$$

After solving the integral, we get

$$\hat{\theta}_l=\frac{1}{b_1}\log\left(\frac{a+b_1+T}{a+T}\right)^{n+b} \qquad (46)$$

Where,

$$T=\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]$$

VI. Simulation Analysis

This section is dedicated to the simulation analysis, we generate $N=1500$ random samples of size n= 50, 100 and 150 to represent a small, medium and large data set from inverse Topp-Leone distribution for specific values of $\theta=0.5$ and 1. The shape parameter is estimated with maximum likelihood estimation and Bayesian using extended Jeffery's prior and gamma prior. For extended Jeffrey's prior we chose $c=0.5$ and 1 and the value of loss function $b_1=0.06$ and 0.09. In case of gamma prior we chose $a=0.5,1.0$ and $b=0.5,1.0$ with loss function $b_1=0.06$ and 0.09. R software is used for simulation analysis in order to examine and compare the efficiency of the estimates for different sample sizes with different values of loss functions. The results are presented in table 1 and 2.

Table 1: *Mean Square Error for $\hat{\theta}$ Using Jeffery's Prior*

n	$\hat{\theta}$	C	$\hat{\theta}_s$	$\hat{\theta}_e$	$\hat{\theta}_p$	$\hat{\theta}_l$	
						$b_1=0.06$	$b_1=0.09$
50	0.5	0.5	0.01019658	0.01019497	0.01019739	0.01019656	0.01019654
		1.0	0.01006666	0.01006506	0.01006746	0.01006662	0.0100666
	1	0.5	0.1605382	0.1605446	0.160535	0.1605383	0.1605383
		1.0	0.1591694	0.1591759	0.1591662	0.1591696	0.1591697
100	0.5	0.5	0.01003337	0.01003257	0.01003377	0.01003335	0.01003335
		1.0	0.01002572	0.01002492	0.01002612	0.0100257	0.0100257
	1	0.5	0.1603843	0.1603875	0.1603827	0.1603844	0.1603844
		1.0	0.1602739	0.1602771	0.1602723	0.160274	0.160274
150	0.5	0.5	0.01001125	0.01001071	0.01001151	0.01001123	0.01001123
		1.0	0.01004535	0.01004483	0.01004563	0.01004535	0.01004533
	1	0.5	0.1600195	0.1600216	0.1600184	0.1600195	0.1600196
		1.0	0.1600716	0.1600738	0.1600706	0.1600717	0.1600717

$\hat{\theta}_s=$ Square error loss function, $\hat{\theta}_e=$ Estimation under Entropy,

$\hat{\theta}_p=$ Estimation under Precautionary, $\hat{\theta}_l=$ Estimation under LINEX

In table 1, Bayes estimation with squared error loss function under extended Jeffery's prior the lesser values in most cases. Moreover, when sample size increase from 50 to 150, the mean square error decreases quite significantly.

Table 2: *Mean Square Error for $\hat{\theta}$ Using gamma Prior*

n	$\hat{\theta}$	a	b	$\hat{\theta}_s$	$\hat{\theta}_e$	$\hat{\theta}_p$	$\hat{\theta}_l$	
							$b_1 = 0.06$	$b_1 = 0.09$
50	0.5	0.5	0.5	0.02337317	0.02337461	0.02337391	0.0233732	0.0233732
		0.5	1.0	0.02335441	0.02335582	0.02335512	0.02335444	0.02335444
	1	0.5	0.5	0.2494094	0.2494161	0.2494127	0.2494095	0.2494096
		0.5	1.0	0.4267061	0.4267121	0.4267091	0.4267061	0.4267062
100	0.5	0.5	0.5	6.735e-06	6.7374e-06	6.7365e-06	6.7356e-06	6.7356e-06
		0.5	1.0	7.1562e-06	7.1517e-06	7.1540e-06	7.1561e-06	7.1561e-06
	1	0.5	0.5	0.2500997	0.250103	0.2501014	0.2500998	0.2500998
		0.5	1.0	0.2490105	0.2490138	0.2490121	0.2490105	0.2490104
150	0.5	0.5	0.5	4.5992e-06	4.6011e-06	4.6002e-06	4.59938e-06	4.59938e-06
		0.5	1.0	4.6406e-06	4.6426e-06	4.6416e-06	4.64060e-06	4.64060e-06
	1	0.5	0.5	0.2502639	0.2502661	0.250265	0.2502639	0.2502639
		0.5	1.0	0.2497457	0.2497479	0.2497468	0.2497457	0.2497457

In table 2, Bayes estimation with squared error loss function under gamma prior the lesser values in most cases. Moreover, when sample size increase from 50 to 150, the mean square error decreases quite significantly.

VII. Application

In this section we provide a real life data sets through which the efficiency of the estimators and posterior risks of different loss function has been obtained.

The data of 40 patients suffering from blood cancer(leukaemia) from one ministry of health hospitals in Saudi Arabia (see Abouammah et al. [1]).

By using different loss functions, the Bayesian estimates and posterior risks of the posterior distribution through both priors are as follows where posterior risks are in parenthesis.

Table 3: *Bayes Estimation and Posterior Risks Using Jeffery's Prior*

$\hat{\theta}$	C	$\hat{\theta}_s$	$\hat{\theta}_e$	$\hat{\theta}_p$	$\hat{\theta}_l$	
					$b_1 = 0.06$	$b_1 = 0.09$
1.0	0.5	0.5836 (0.0085)	0.5690 (3.622)	0.5908 (8.616)	0.5833 (0.0350)	0.5832 (0.0525)
	1.0	0.5690 (0.0083)	0.5544 (3.597)	0.5763 (8.403)	0.5688 (0.0341)	0.5686 (0.0512)
	1.5	0.5544 (0.0080)	0.5398 (3.571)	0.5617 (8.191)	0.5542 (0.0332)	0.5541 (0.0499)
2.0	0.5	0.5836 (0.0085)	0.5690 (3.622)	0.5908 (8.616)	0.5833 (0.0350)	0.5832 (0.0525)
	1.0	0.5690 (0.0083)	0.5544 (3.597)	0.5763 (8.403)	0.5688 (0.0341)	0.5686 (0.0512)
	1.5	0.5544 (0.0080)	0.5398 (3.571)	0.5617 (8.191)	0.5542 (0.0332)	0.5541 (0.0499)

$\hat{\theta}_s$ =Square error loss function, $\hat{\theta}_e$ =Estimation under Entropy, $\hat{\theta}_p$ =Estimation under Precautionary, and $\hat{\theta}_l$ = Estimation under LINEX

Table 4: *Bayes Estimation and Posterior Risks Using Gamma Prior*

$\hat{\theta}$	a	b	$\hat{\theta}_s$	$\hat{\theta}_e$	$\hat{\theta}_p$	$\hat{\theta}_l$	
						$b_1 = 0.06$	$b_1 = 0.09$
1.0	0.5	0.5	0.5866 (0.0084)	0.5721 (4.247)	0.5793 (0.5699)	0.5864 (0.0351)	0.5862 (0.0527)
	0.5	1.0	0.5939 (0.0086)	0.5794 (4.247)	0.5866 (0.5771)	0.5936 (0.0356)	0.5935 (0.0534)
	1.0	0.5	0.5824 (0.0083)	0.5680 (4.254)	0.5752 (0.5637)	0.5821 (0.0349)	0.5820 (0.0524)
2.0	0.5	0.5	0.5866 (0.0084)	0.5721 (4.247)	0.5793 (0.5699)	0.5864 (0.0351)	0.5862 (0.0527)
	0.5	1.0	0.5939 (0.0086)	0.5794 (4.247)	0.5866 (0.5771)	0.5936 (0.0356)	0.5935 (0.0534)
	1.0	0.5	0.5824 (0.0083)	0.5680 (4.254)	0.5752 (0.5637)	0.5821 (0.0349)	0.5820 (0.0524)

Among other loss functions, it is evident from table 3 and table 4. That the square error loss function shows smaller Bayes posterior risk under the both assumptions (extended Jeffery's prior and gamma prior). According to decision rule of less Bayes posterior risk, we accomplish that square error loss function is more useful than others.

VIII. Conclusion

In this paper, we have initially obtained the Bayes posterior distribution and estimation of parameter of the inverse Topp-Leone distribution under both informative and non-informative priors. We have discussed different loss functions among them square error loss function provides less Bayes posterior risk. Eventually through simulation analysis and application, the performance of the estimators has been achieved.

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