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A NEW CLASS OF COS-G FAMILY OF DISTRIBUTIONS WITH APPLICATIONS

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Abstract

This paper introduces a novel family of probability distributions, termed the Cos-G family, which is derived from a trigonometric transformation approach. We present the general structural properties of this family and focus on one of its unique members. This newly proposed distribution, formulated from the inverse Weibull distribution, exhibits flexible hazard rate shapes, including reverse-J, increasing, and inverted bathtub forms. We investigate its fundamental statistical properties and employ the maximum likelihood estimation method to estimate its parameters. The performance of the estimation technique is assessed through a Monte Carlo simulation, revealing that biases and mean square errors decrease as sample size increases, ensuring reliable parameter estimation even for small samples. To illustrate its practical applicability, we fit the suggested model to three real-world datasets and compare its performance against existing models using various goodness-of-fit measures and model selection criteria. The results confirm the superiority of the proposed model in capturing complex data structures.

Keywords: Cos-G distribution, Inverse Weibull, Moment, Estimation, Goodness of fit.

1. Introduction

Real-world events are frequently studied using statistical distributions. Both novel developments for their application and the theory of statistical distributions are thoroughly researched. To explain a variety of real-world phenomena, several families of distributions have been developed. In fact, this fresh advancement in distribution theory is an ongoing practice. The majority of probability distributions suggested in the literature have a lot of parameters, which gives the model more adaptability. Some authors claim that it is challenging to acquire these estimates using numerical resources [1]. For modeling actual data, it is better to develop models with a limited number of parameters and a high level of flexibility. A team of scientists made the decision to use trigonometric functions to seek novel distributions in order to achieve this objective. Trigonometric models have gained popularity among scholars in recent years due to their adaptability and ability to be understood mathematically. Souza et al. [2] suggested a new class of trigonometric cosine distribution with a bathtub-shaped or increasing failure rate function called the Cos-G Class of distribution with base parameters ($\omega > 0$) among the various trigonometric G-family. The cumulative distribution function (CDF) for the Cos-G class of distribution are

$$F(x;\omega) = -\int_{0}^{\frac{\pi}{2}K(x;\omega)} sin(t)dt = 1 - \cos\left[\frac{\pi}{2}K(x;\omega)\right]; x \in \Re.$$

Souza et al. [3] utilized a similar methodology to propose the Sin-G family of distributions and include the Sin-Inverse Weibull distribution in the Sin-G class. Similarly, Souza et al. [4]

introduced a new Tan-G class with an increasing failure rate function or bathtub-shaped failure rate function, and focused on examining the Tan-BXII distribution as a member. A CDF exists for both the Sin-G and Tan-G classes of distributions.

$$F(x;\omega) = \int_0^{\frac{\pi}{2}K(x;\omega)} \cos(t)dt = \sin\left[\frac{\pi}{2}K(x;\omega)\right]; x \in \Re.$$

$$F(x;\omega) = \int_0^{\frac{\pi}{4}K(x;\omega)} sec^2(t)dt = \tan\left[\frac{\pi}{4}K(x;\omega)\right]; x \in \Re.$$

where $K(x;\omega)$ is the CDF of any parent distribution and ω is the vector of parameters of the parent distribution. The new sin-G family was created by [5], who also studied the sin-inverse Weibull model in specific. The CDF of the novel sin-G family of distribution are

$$F(x;\omega) = \int_0^{\frac{\pi}{4}K(x;\omega)(K(x;\omega)+1)} cos(t)dt = \sin\left[\frac{\pi}{4}K(x;\omega)(K(x;\omega)+1)\right]; x \in \Re.$$

Also, Chesneau and Jamal [6] have defined the sine Kumaraswamy-G family of distributions as having two extra parameters to this family. Muhammad et al. [7] have defined the exponentiated sine-G family and analyzed the particular distribution as an exponentiated sine-Weibull distribution. Another trigonometric function-related probability model introduced by [8] is called arctan generalized exponential distribution. Using the sine-G family of distribution [9] have developed a new two-parameter model called sine Burr XII distribution. A new family of distributions related to the Sine function was developed by [10] and used with medical data. As a result, we have observed that the simple functions have a trigonometric distribution and are tractable formally see [3]. Additionally, without the use of any extra parameters, the sine transformation can significantly increase $G(x)$ flexibility [6]. We are drawn to the cosine metamorphosis family because of these appealing qualities. In this research, we created a new family of trigonometric models using the cosine function, which we named the new class of cos-G family (NCC-G) of distributions.

This study is divided into several sections. In Section 2, we introduce the methodology of model development and key functions of the family of distributions. Section 3 presents some general properties of the NCC-G family, while Section 4 discusses methods of estimation. In Section 5, we introduce a specific member of the NCC-G family and present a detailed study, and in the application Section6, we provide the application of this model using three real datasets. Finally, Section 7 contains the conclusion.

2. The NCC-G family of Distribution (NCC-G FD)

In this study, a new family of distributions called NCC-G is suggested using the T-X approach as defined by [11]. Consider a baseline CDF, represented by $G(x;\xi)$, and a vector of associated parameters, denoted by $\xi > 0$. The ratio of $G(x;\xi)$ and $1 + G(x;\xi)$ can be treated as a function of the new family of distributions. For further information, refer to [12]. Mathematically it can be expressed as $\frac{G(x;\xi)}{1+G(x;\xi)} \to 0$ as $G(x;\xi) \to 0$; $\frac{G(x;\xi)}{1+G(x;\xi)} \to \frac{1}{2}$ as $G(x;\xi) \to 1$ The CDF $F(x;\xi)$ of the NCC-G family of distributions is defined as

$$F(x;\xi) = -\int_0^{\pi\frac{G(x;\xi)}{1+G(x;\xi)}} \sin(t) = 1 - \cos\left[\pi\frac{G(x;\xi)}{1+G(x;\xi)}\right]; x \in \Re. \quad (1)$$

Differentiating the Equation (1), the PDF $f(x;\xi)$ of the family can be written as

$$f(x;\xi) = \pi \sin\left[\pi\frac{G(x;\xi)}{1+G(x;\xi)}\right]\frac{g(x;\xi)}{(1+G(x;\xi))^2}; x \in \Re. \quad (2)$$

2.1. Survival Function

The survival function of NCC-G FD is presented as

$$R(x;\xi) = 1 - F(x;\xi) = \cos\left[\pi \frac{G(x;\xi)}{1+G(x;\xi)}\right]; x \in \Re.$$

2.2. Hazard Function

The Hazard function of NCC-G FD can be expressed as

$$H(x;\xi) = \frac{f(x;\xi)}{R(x;\xi)} = \pi \sin\left[\pi \frac{G(x;\xi)}{1+G(x;\xi)}\right] \frac{g(x;\xi)}{(1+G(x;\xi))^2} \left[\cos\left(\pi \frac{G(x;\xi)}{1+G(x;\xi)}\right)\right]^{-1}; x \in \Re.$$

2.3. The Quantile Function

The quantile function is useful in statistical analysis and modeling, as it provides a way to estimate percentiles and other summary statistics of a probability distribution. Suppose $Q(p)$ is the smallest value of X for which the probability that $X \leqslant$ to that value is at least p. The quantile function $Q(p;\xi)$ of CDF $F(x;\xi)$ of NCC-G FD can be obtained as

$$Q(p;\xi) = G^{-1}\left[\frac{\cos^{-1}(1-p)}{\pi - \cos^{-1}(1-p)}\right]; \quad p \in (0,1). \tag{3}$$

Using equation (3) we can calculate the median, upper and lower quartile, quartile deviation (QD), coefficient of QD, skewness, and kurtosis, which are presented in Table 1.

Table 1: *Various measures based on quantiles of NCC-G FD*

Statistics	Expressions
Median	$G^{-1}\left[\frac{\cos^{-1}(0.5)}{\pi-\cos^{-1}(0.5)}\right]$
Lower Quartile	$G^{-1}\left[\frac{\cos^{-1}(0.75)}{\pi-\cos^{-1}(0.75)}\right]$
Upper Quartile	$G^{-1}\left[\frac{\cos^{-1}(0.25)}{\pi-\cos^{-1}(0.25)}\right]$
QD	$\frac{1}{2}\left[G^{-1}\left(\frac{\cos^{-1}(0.25)}{\pi-\cos^{-1}(0.25)}\right) - G^{-1}\left(\frac{\cos^{-1}(0.75)}{\pi-\cos^{-1}(0.75)}\right)\right]$
Coefficient of QD	$\frac{\left[G^{-1}\left(\frac{\cos^{-1}(0.25)}{\pi-\cos^{-1}(0.25)}\right) - G^{-1}\left(\frac{\cos^{-1}(0.75)}{\pi-\cos^{-1}(0.75)}\right)\right]}{\left[G^{-1}\left(\frac{\cos^{-1}(0.25)}{\pi-\cos^{-1}(0.25)}\right) + G^{-1}\left(\frac{\cos^{-1}(0.75)}{\pi-\cos^{-1}(0.75)}\right)\right]}$
Skewness ([13])	$\frac{Q\left(\frac{3}{4};\xi\right)-2Q\left(\frac{1}{2};\xi\right)+Q\left(\frac{1}{4};\xi\right)}{Q\left(\frac{3}{4};\xi\right)-Q\left(\frac{1}{4};\xi\right)}$
Kurtosis ([14])	$\frac{Q\left(\frac{7}{8};\xi\right)-Q\left(\frac{5}{8};\xi\right)-Q\left(\frac{1}{8};\xi\right)+Q\left(\frac{3}{8};\xi\right)}{Q\left(\frac{3}{4};\xi\right)-Q\left(\frac{1}{4};\xi\right)}$

3. Some Properties of NCC-G FD

3.1. Useful Expansion of NCC-G FD

Exponentiated distributions can be used to generate useful linear expansions. The CDF of the exponentiated-G (Exp-G) distribution for more information see [3, 15, 16], exponentiated distributions have well-known properties for a wide range of baseline CDF $G(x;\varphi)$ with parameter $z > 0$ is given by

$$G_z(x;\varphi) = [G(x;\varphi)]^z; x \in \Re, \text{ where } x \in \Re. \tag{4}$$

The PDF corresponding to (4) can be presented as

$$g_z(x;\varphi) = zg(x;\varphi)\left[G(x;\varphi)\right]^{(z-1)}, x \in \Re.$$

We can express the density function of the NCC-G FD in linear form using the series expansions shown below.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!} = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \frac{y^9}{9!} - \cdots ; -\infty < x < \infty.$$

$$(1+y)^p = \sum_{n=0}^{\infty} \binom{p}{n} y^n = 1 + \frac{p}{1!}y + \frac{p(p-1)}{2!}y^2 + \frac{p(p-1)(p-2)}{3!}y^3 + \cdots ; |y| < 1.$$

The PDF of NCC-G FD is

$$f(x,\xi) = g(x,\xi) \sum_{i=0}^{\infty} \frac{\pi^{2i+2}(-1)^i}{(2i+1)!} \left(1 + G(x,\xi)\right)^{2i-1} \left(G(x,\xi)\right)^{2i+1}. \quad (5)$$

Further expanding Equation (5) using generalized binomial series expansion. The expression for $f(x;\xi)$ becomes

$$f(x,\xi) = g(x,\xi) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Delta_{ij} \left(G(x,\xi)\right)^{2i+j+1}, \quad (6)$$

where $\Delta_{ij} = \frac{\pi^{2i+2}(-1)^i}{(2i+1)!} \binom{2i-1}{j}$.

3.2. Moments

The r^{th} order moment (μ_r') about the origin for the NCC-G FD is

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx \quad (7)$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Delta_{ij} \int_{-\infty}^{\infty} x^r \left(G(x,\xi)\right)^{2i+j+1} g(x,\xi) dx. \quad (8)$$

Further moments can also be calculated using the quantile function for more detail see [17] as Let $G(x;\xi) = p \Rightarrow g(x;\xi)dx = dp; 0 \leqslant p \leqslant 1.$

$$\mu_r' = E(X^r) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Delta_{ij} \int_{0}^{1} p^{2i+j+1} Q_G^r(p) dp, 0 < p < 1.$$

where $G(x;\xi) = p$ and $Q_G(p)$ is the function of quantile.

3.3. Moment Generating Function (MGF)

The MGF ($M_X(t)$) for the NCC-G FD is

$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mu_r' = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^k}{k!} \Delta_{ij} \int_{-\infty}^{\infty} x^k g(x,\xi) \left(G(x,\xi)\right)^{2i+j+1} dx$$

Let $G(x;\xi) = p \Rightarrow g(x;\xi)dx = dp; 0 \leqslant p \leqslant 1.$

$$M_X(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^k}{k!} \Delta_{ij} \int_{-\infty}^{\infty} p^{2i+j+1} Q_G^l(p) dp, 0 < p < 1,$$

where $G(x;\xi) = p$ and $Q_G(p)$ is the quantile function of the baseline distribution.

3.4. Incomplete Moments

The Incomplete moments of the NCC-G FD can be defined as $M_r(y) = \int_0^y x^r f(x)dx$. Therefore incomplete moments for NCC-G FD are given by

$$M_r(y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\int_{-\infty}^{y} \Delta_{ij} x^r g(x;\xi) \left(G(x,\xi)\right)^{2i+j+1} dx. \tag{9}$$

Alternately, $M_r(y)$ may be defined in terms of quantile function as

$$M_r(y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij} \int_0^{G(y)} p^{2i+j+1} Q_G^r(p)dp; 0 < p < 1.$$

3.5. Mean Residual Life

The mean residual life of the NCC-G FD can be defined as $\bar{M}(y) = \frac{1}{F(y)}\left[\mu - \int_{-\infty}^{y} xf(x)dx\right] - y$. Therefore, the mean residual life for NCC-G FD is given by

$$\bar{M}(y) = \frac{1}{F(y)}\left[\mu - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}\int_{-\infty}^{y} xg(x;\xi)\left\{G(x;\xi)\right\}^{2i+j+1} dx\right] - y.$$

Alternatively, $\bar{M}(y)$ may be calculated in term of quantile function as

$$\bar{M}(y) = \frac{1}{F(y)}\left[\mu - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}\int_0^{G(y)} p^{2i+j+1} Q_G(p)dp\right] - y.$$

3.6. Inequality Measure

Lorenz and Bonferroni curves are utilized in various fields such as insurance, econometrics, and demography, among others, to analyze measures of inequality such as income and poverty.
i) **Lorenz Curve**
Lorenz curve is defined as $L_{F(y)} = \frac{1}{\mu}\int_{-\infty}^{y} xf(x)dx$, where μ is the mean of x, hence Lorenz curve for NCC-G FD is given by

$$L_{F(y)} = \frac{1}{\mu}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}\int_{-\infty}^{y} xg(x;\xi)\left(G(x,\xi)\right)^{2i+j+1} dx. \tag{10}$$

Alternatively, in terms of quantile function as

$$L_{F(y)} = \frac{1}{\mu}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}\int_{-\infty}^{G(y)} p^{2i+j+1} Q_G(p)dp.$$

ii) **the Bonferroni Curve**
The Bonferroni curve is given by $B_{F(y)} = \frac{L_{F(y)}}{F(y)}$. From Equation (10), the Bonferroni curve for the NCC-G FD is obtained as

$$B_{F(y)} = \frac{1}{\mu F(y)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}\int_{-\infty}^{y} xg(x;\xi)\left(G(x,\xi)\right)^{2i+j+1} dx.$$

3.7. Entropy

Entropy is a concept used to describe the degree of variation or uncertainty associated with a random variable. Its applicability is widespread and can be observed in various disciplines such as probability theory, medicine, insurance, engineering, life sciences, etc. in general.

i) **Renyi's Entropy**

Entropy, which serves as a measure of the amount of variation or uncertainty associated with a random variable, finds its applications across several disciplines, including engineering, econometrics, and financial mathematics. Renyi [18] proposed the concept of entropy as a metric for quantifying variability and uncertainty, and it can be computed as follows: $R_\rho(X) = \frac{1}{1-\rho}\log\int\limits_{-\infty}^{\infty}\{f(x)\}^\rho\,dx$; $\rho > 0$ and $\rho \neq 1$. Applying Taylor™s series expansion $[f(x,\xi)]^\rho$ can be obtained in the form

$$[f(x;\xi)]^\rho = \pi^\rho\,(g(x;\xi))^\rho\left[\sin\left(\pi\frac{G(x;\xi)}{1+G(x;\xi)}\right)\right]^\rho(1+G(x;\xi))^{-2\rho}.$$

By considering the function of Taylor series $\left[\sin\left(\pi\frac{G(x;\xi)}{1+G(x;\xi)}\right)\right]^\rho$ at the point $s = 1/4$, we can write

$$[\sin(\pi s)]^\rho = \sum_{k=0}^{\infty}\sum_{r=0}^{k}a_k\binom{k}{r}(-1)^{k-r}\left(\frac{1}{4}\right)^{k-r}s^r,$$

where $a_k = \frac{1}{k!}\left[\{\sin(\pi s)\}^\rho\right]^{(k)}|_{s=\frac{1}{4}}$. We have selected $s = \frac{1}{4}$ because $\{\sin(\pi s)\}^\rho$ is infinitely differentiable at this point and $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \cos\left(\frac{\pi}{4}\right)$, which allows to have a more tractable expression for a_k at a given ρ

$$[f(x;\xi)]^\rho = \pi^\rho\,(g(x;\xi))^\rho\sum_{k=0}^{\infty}\sum_{r=0}^{k}a_k\binom{k}{r}(-1)^{k-r}\left(\frac{1}{4}\right)^{k-r}(G(x;\xi))^r\,(1+G(x;\xi))^{-(2\rho+r)} \tag{11}$$

Further expanding Equation (11) using generalized binomial series expansion. The expression for $[f(x;\xi)]^\rho$ becomes

$$[f(x;\xi)]^\rho = \pi^\rho\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{m=0}^{\infty}(-1)^{m+k-r}a_k\binom{k}{r}\left(\frac{1}{4}\right)^{k-r}\binom{(2\rho+r)+m-1}{m}(G(x;\xi))^{r+m}\,(g(x;\xi))^\rho \tag{12}$$

Substituting $[f(x,\xi)]^\rho$ into the expression for $R_\rho(X)$, the Renyi™s entropy for NCC-G family of distribution is given by

$$R_\rho(X) = \frac{1}{1-\rho}\log\left[\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{m=0}^{\infty}\psi_{mkr}\int\limits_{-\infty}^{\infty}(g(x;\xi))^\rho\,(G(x;\xi))^{r+m}\,dx\right],$$

where $\psi_{mkr} = (-1)^{m+k-r}\,\pi^\rho a_k\binom{k}{r}\left(\frac{1}{4}\right)^{k-r}\binom{(2\rho+r)+m-1}{m}$.

ii) **q-Entropy**

The q-entropy is given by

$$H(\rho) = \frac{1}{1-\rho}\log\left[1-\int\limits_{-\infty}^{\infty}\{f(x)\}^\rho\,dx\right];\rho > 0 \text{ and } \rho \neq 1.$$

Substituting $[f(x,\xi)]^\rho$ from Equation (12) into the expression for $H(\rho)$, the q-Entropy for NCC-G FD is given by

$$H(\rho) = \frac{1}{1-\rho}\log\left[1-\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{m=0}^{\infty}\psi_{mkr}\int\limits_{-\infty}^{\infty}(g(x;\xi))^\rho\,(G(x,\xi))^{r+m}\,dx\right];\rho > 0 \text{ and } \rho \neq 1.$$

iii) **Shannon's Entropy**
The Shannon's entropy for a random variable X with pdf $f(x)$ is a special case of the Renyi's entropy when $\rho \uparrow 1$. Shannon entropies are defined as $\eta_X = E(-\log f(x))$. For the NCC-G family of distribution is given by

$$\eta_X = E\left[-\log\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta_{ij}g(x,\xi)\left(G(x,\xi)\right)^{2i+j+1})\right\}\right].$$

4. Estimation Method

4.1. Maximum Likelihood Estimation (MLE)

The parameters of the NCC-G FD are estimated in this section using the method of maximum likelihood. Given random sample $x_1, ..., x_n$ of size n with parameters vector ξ from the NCC-G FD, let $u = \xi^T$ be $(p \times 1)$ parameter vectors, then the log density and total log-likelihood function respectively, are given by

$$l(x;\xi) = \log\pi + \log\left[\sin\left\{\pi\frac{G(x;\xi)}{1+G(x;\xi)}\right\}\right] - 2\log\left(1+G(x;\xi)\right) + \log g(x;\xi)$$

and

$$l(\underline{x},\xi) = n\log\pi + \sum_{i=1}^{n}\log\left[\sin\left\{\pi\frac{G(x_i;\xi)}{1+G(x_i;\xi)}\right\}\right] - 2\sum_{i=1}^{n}\log\left(1+G(x_i;\xi)\right) + \sum_{i=1}^{n}\log g(x_i;\xi). \quad (13)$$

Differentiating Equation (13) gives the score function's components of $V(u) = \left(\frac{\partial l}{\partial\xi}\right)^T$ as follows,

$$\frac{\partial l}{\partial\xi} = \pi\sum_{i=1}^{n}\cot\left\{\pi\frac{G(x_i;\xi)}{1+G(x_i;\xi)}\right\}\frac{G_k'(x_i;\xi)}{\left(1+G(x_i;\xi)\right)^2} - 2\sum_{i=1}^{n}\frac{G_k'(x_i;\xi)}{\left(1+G(x_i;\xi)\right)} + \sum_{i=1}^{n}\frac{g_k'(x_i;\xi)}{g(x_i;\xi)},$$

where $g_k'(x_i;\xi) = \frac{dg(x_i;\xi)}{d\xi}$, $g_k''(x_i;\xi) = \frac{d^2g(x_i;\xi)}{d^2\xi}$, $G_k'(x_i;\xi) = \frac{dG(x_i;\xi)}{d\xi}$ and $G_k''(x_i;\xi) = \frac{d^2G(x_i;\xi)}{d^2\xi}$.

5. Special member of NCC-G FD

Generalization of several distributions can be made using the NCC-G FD. The special distribution, a new class of cosine inverse Weibull distribution, is introduced in this section.

5.1. New class Cos inverse Weibull (NCC-IW) istribution

The CDF and PDF of the Inverse Weibull (IW) distribution are respectively given by

$$G(x) = 1 - exp(-\alpha x^{-\beta};\ x > 0, \alpha, \beta > 0$$

and

$$g(x) = \alpha\beta x^{-\beta-1}exp(-\alpha x^{-\beta}$$

Hence using the CDF and PDF of IW, the CDF and PDF of the NCC-IW distribution are given by

$$F(x;\alpha,\beta) = 1 - \cos\left[\pi\frac{\exp(-\alpha x^{-\beta})}{1+\exp(-\alpha x^{-\beta})}\right]; x > 0 \quad (14)$$

$$f(x;\alpha,\beta) = \pi\alpha\beta x^{-(\beta+1)}\sin\left[\pi\frac{\exp(-\alpha x^{-\beta})}{1+\exp(-\alpha x^{-\beta})}\right]\frac{\exp(-\alpha x^{-\beta})}{\left(1+\exp(-\alpha x^{-\beta})\right)^2}; x > 0 \quad (15)$$

The reliability and hazard functions, respectively, are given by

$$R(x;\alpha,\beta) = \cos\left[\pi \frac{\exp(-\alpha x^{-\beta})}{1+\exp(-\alpha x^{-\beta})}\right]; x > 0.$$

and

$$H(x;\alpha,\beta) = \pi\alpha\beta x^{-(\delta+1)} \frac{\exp(-\alpha x^{-\beta})}{(1+\exp(-\alpha x^{-\beta}))^2} \sin\left[\pi \frac{\exp(-\alpha x^{-\beta})}{1+\exp(-\alpha x^{-\beta})}\right] \left[\cos\left(\pi \frac{\exp(-\alpha x^{-\beta})}{1+\exp(-\alpha x^{-\beta})}\right)\right]^{-1}.$$

The quantile function for the NCC-IW distribution is presented below

$$Q_X(p) = \left[-\frac{1}{\alpha}\log\left(\frac{\cos^{-1}(1-p)}{\pi - \cos^{-1}(1-p)}\right)\right]^{-\frac{1}{\beta}}, p \in (0,1)$$

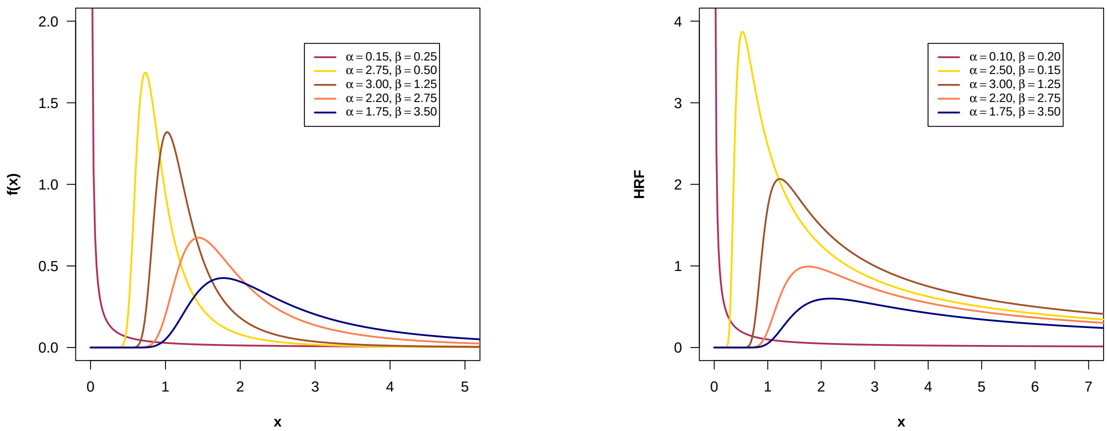

Figure 1: *Shapes of PDF and HRF of NCC-IW distribution*

5.2. Linear Expansion

Using Equation (6), Equation (15) can be expressed in linear form as

$$f(x;\xi) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega_{ij} x^{-(\beta+1)} \exp\left\{-(2i+j+2)\alpha x^{-\beta}\right\}, \tag{16}$$

where $\Omega_{ij} = \frac{\pi^{2i+2}\alpha\beta(-1)^i}{(2i+1)!}\binom{2i-1}{j}$.

5.3. Moments

Using the PDF defined in Equation (16), the r^{th} order non-central moment ($\mu_r^{'}$) for the NCC-IW distribution can be presented as

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega_{ij}^{*} \frac{\Gamma\left(\frac{\beta-r}{\beta}\right)}{[\alpha\{(2i+j)+2\}]^{\frac{\beta-r}{\beta}}}; \quad \forall \beta > r, \tag{17}$$

where $\Omega_{ij}^{*} = \frac{\pi^{2i+2}\alpha(-1)^i}{(2i+1)!}\binom{2i-1}{j}$ and $\Gamma(.)$ is the gamma function.

5.4. Skewness and Kurtosis

Using the Equation (17) we can obtain the first four $(r = 1, 2, 3, 4)$ non-central moments as:

$$Mean = \mu_1^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}^{*}\left[\alpha\{(2i+j)+2\}\right]^{-\frac{\beta-1}{\beta}}\Gamma\left(\frac{\beta-1}{\beta}\right); \quad \forall\, \beta > 1,$$

$$\mu_2^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}^{*}\frac{\Gamma\left(\frac{\beta-2}{\beta}\right)}{\left[\alpha\{(2i+j)+2\}\right]^{\frac{\beta-2}{\beta}}}; \quad \forall\, \beta > 2,$$

$$\mu_3^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}^{*}\frac{\Gamma\left(\frac{\beta-3}{\beta}\right)}{\left[\alpha\{(2i+j)+2\}\right]^{\frac{\beta-3}{\beta}}}; \quad \forall\, \beta > 3,$$

and

$$\mu_4^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}^{*}\frac{\Gamma\left(\frac{\beta-4}{\beta}\right)}{\left[\alpha\{(2i+j)+2\}\right]^{\frac{\beta-4}{\beta}}}; \quad \forall\, \beta > 4.$$

Similarly, we can calculate the central moments using the above non-central moments as

$$\mu_1 = \mu_1^{'},$$

$$\mu_2 = \mu_2^{'} - \mu_1^{'2},$$

$$\mu_3 = \mu_3^{'} - 3\mu_1^{'}\mu_2^{'} + 2\mu_1^{'3},$$

and

$$\mu_4 = \mu_4^{'} - 4\mu_3^{'}\mu_2^{'} + 6\mu_2^{'}\mu_1^{'2} - 2\mu_1^{'4}$$

Therefore skewness and kurtosis for the NCC-IW distribution are $\beta_1 = \frac{\mu_3^2}{\mu_2^3}$ and $\beta_2 = \frac{\mu_4}{\mu_2^2}$ respectively.

5.5. MGF

The MGF $(M_X(t))$ for the NCC-IW distribution is

$$M_X(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{t^k\Omega_{ij}^{*}}{k!}\frac{\Gamma\left(\frac{\beta-r}{\beta}\right)}{\left[\alpha\{(2i+j)+2\}\right]^{\frac{\beta-r}{\beta}}}; \quad \forall\, \beta > r. \tag{18}$$

5.6. Incomplete moments

The incomplete moments for NCC-IW distribution are given by

$$\begin{aligned} M_r(y) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}\int_0^y x^{r-(\delta+1)}\exp\left\{-(2i+j+2)\alpha x^{-\beta}\right\}dx \\ &= \frac{1}{\beta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}\frac{\gamma\left(\frac{\beta-r}{\beta}, (2i+j+2)\alpha y^{-\beta}\right)}{\left\{(2i+j+2)\alpha\right\}^{\frac{\beta-r}{\beta}}}. \end{aligned}$$

where $\gamma(.)$ incomplete gamma function.

5.7. Mean Residual Life

The mean residual life for the NCC-IW distribution is given by

$$\bar{M}(y) = \frac{1}{F(y)}\left[\mu - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}\int_{0}^{y} x^{-\beta}\exp\left\{-(2i+j+2)\alpha x^{-\beta}\right\}\right] - y$$
$$= \frac{1}{F(y)}\left[\mu - \frac{1}{\beta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}\frac{\gamma\left(\frac{\beta-1}{\beta},(2i+j+2)\alpha y^{-\beta}\right)}{\{(2i+j+2)\alpha\}^{\frac{\beta-1}{\beta}}}\right] - y.$$

5.8. Entropy

i) **Renyi™s Entropy** The Renyi™s entropy for NCC-IW distribution is given by

$$R_{\rho}(X) = \frac{1}{1-\rho}\log\left[\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{m=0}^{\infty}\psi_{krm}(\alpha\beta)^{\rho}\int_{0}^{\infty} x^{-\rho(\beta+1)}\exp(-(r+m+\rho)\alpha x^{-\beta})dx\right]$$
$$= \frac{1}{1-\rho}\log\left[\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{m=0}^{\infty}\psi_{krm}\frac{(\alpha\beta)^{\rho}}{\beta}\frac{\Gamma\left(\left\{\frac{(\rho-1)(\beta+1)}{\beta}+1\right\}\right)}{\{(r+m+\rho)\alpha\}^{\frac{(\rho-1)(\beta+1)}{\beta}+1}}\right]$$

where $\psi_{mkr} = (-1)^{m+k-r}\pi^{\rho}a_k\binom{k}{r}\left(\frac{1}{4}\right)^{k-r}\binom{(2\rho+r)+m-1}{m}$.

ii) **q-Entropy**

The q-Entropy for NCC-IW distribution is given by

$$H(\rho) = \frac{1}{1-\rho}\log\left[1 - \psi_{krm}(\alpha\beta)^{\rho}\int_{0}^{\infty} x^{-\rho(\beta+1)}\exp(-(r+m+\rho)\alpha x^{-\beta})dx\right]$$
$$= \frac{1}{1-\rho}\log\left[1 - \psi_{krm}\frac{(\alpha\beta)^{\rho}}{\beta}\frac{\Gamma\left(\left\{\frac{(\rho-1)(\beta+1)}{\beta}+1\right\}\right)}{\{(r+m+\rho)\alpha\}^{\frac{(\rho-1)(\beta+1)}{\beta}+1}}\right];$$

where $\rho > 0$, $\rho \neq 1$ and $\psi_{krm} = (-1)^{m+k-r}\pi^{\rho}a_k\binom{k}{r}\left(\frac{1}{4}\right)^{k-r}\binom{(2\rho+r)+m-1}{m}$.

iii) **Shannon's Entropy**

The Shannon entropy for the NCC-IW distribution is given by

$$\eta_X = E\left[-\log\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}x^{-(\beta+1)}\exp\left\{-(2i+j+2)\alpha x^{-\beta}\right\}\right\}\right].$$

5.9. Inequality Measure

i) **Lorentz Curve:** The Lorenz curve for NCC-IW distribution is given by

$$L_{F(y)} = \frac{\alpha\beta}{\mu}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}\int_{-\infty}^{y} x^{-\beta}\exp(-\alpha(2i+j+2)x^{-\beta})dx$$
$$= \frac{\alpha}{\mu}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Omega_{ij}\frac{\gamma\left(\frac{\beta-1}{\beta},(2i+j+2)\alpha y^{-\beta}\right)}{\{(2i+j+2)\alpha\}^{\frac{\beta-1}{\beta}}}.$$

ii) **Boneferroni Curve**

The Bonferroni curve for the NCC-IW distribution is given by

$$
\begin{aligned}
B_{F(y)} &= \frac{1}{\mu F(y)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} T_{ij} \int_{-\infty}^{y} x^{-\beta} \exp(-\alpha(2i+j+1)x^{-\beta}) dx \\
&= \frac{1}{\delta \mu F(y)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Omega_{ij} \frac{\gamma\left(\frac{\beta-1}{\beta}, (2i+j+2)\alpha y^{-\beta}\right)}{\{(2i+j+2)\alpha\}^{\frac{\beta-1}{\beta}}}.
\end{aligned}
$$

5.10. Parameter estimation of NCC-IW distribution

Our current focus is on determining the parameters of the NCC-IW model through the MLE method. The objective is to compute the MLEs for the parameters α and β. To achieve this, we will examine the log-likelihood of a vector $X = (x_1, ..., x_n)^T$ of size n composed of independent random variables from the NCC-IW distribution.

$$
\begin{aligned}
l(x;\alpha,\beta) = n\log(\pi\alpha\beta) - (\beta+1)\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log sin\left[\pi\frac{\exp(-\alpha x_i^{-\beta})}{1+\exp(-\alpha x_i^{-\beta})}\right] \\
-2\sum_{i=1}^{n}\log\left(1+\exp(-\alpha x_i^{-\beta})\right) - \alpha\sum_{i=1}^{n} x_i^{-\beta} \qquad (19)
\end{aligned}
$$

Partially differentiating the Equation (19) with respect to β and α yields the components of the score function $V(u) = \left(\frac{\partial l}{\partial \beta}, \frac{\partial l}{\partial \alpha}\right)^T$ as follows

$$
\begin{aligned}
\frac{\partial l}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\log x_i + \pi\alpha\sum_{i=1}^{n}\frac{x_i^{-\beta}\log(x_i)\exp(-\alpha x_i^{-\beta})}{\left(1+\exp(-\alpha x_i^{-\beta})\right)^2}\cot\left[\pi\frac{\exp(-\alpha x_i^{-\beta})}{1+\exp(-\alpha x_i^{-\beta})}\right] + \\
2\alpha\sum_{i=1}^{n}\frac{x_i^{-\beta}\log(x_i)\exp(-\alpha x_i^{-\beta})}{\left(1+\exp(-\alpha x_i^{-\beta})\right)} + \alpha\sum_{i=1}^{n} x_i^{-\beta}\log(x_i).
\end{aligned}
$$

and

$$
\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - \pi\sum_{i=1}^{n}\frac{x_i^{-\beta}exp(-\alpha x_i^{-\beta})}{\left(1+\exp(-\alpha x_i^{-\beta})\right)^2}\cot\left[\pi\frac{\exp(-\alpha x_i^{-\beta})}{1+\exp(-\alpha x_i^{-\beta})}\right] + 2\sum_{i=1}^{n}\frac{x_i^{-\beta}exp(-\alpha x_i^{-\beta})}{\left(1+\exp(-\alpha x_i^{-\beta})\right)} - \sum_{i=1}^{n} x_i^{-\beta}.
$$

The MLEs of β and α are obtained by maximizing $l(x;\alpha,\beta)$ in β and α, which can be done by solving simultaneously the equation: $\frac{\partial l}{\partial \beta} = 0$ and $\frac{\partial l}{\partial \alpha} = 0$.

5.11. Simulation Study

We used the maxLik R package developed by [19] to create samples from the quantile function defined in Equation (14) for various parameter combinations of the NCC-IW distribution. The MLEs were calculated for each sample using the `maxLik()` function with the BFGS algorithm. This allowed us to test parameter estimation problems, such as the sharpness or flatness of the likelihood function and provided estimates for the size and direction (underestimate or overestimate) of the MLEs bias. We repeated the procedure 1000 times, with 25 samples of sizes ranging from 10 to 250. We then calculated the bias and mean square error (MSE) for each simulation. In addition, we provided lower confidence limit (LCL) and upper confidence limit (UCL) estimated values with a 5% level of significance. The results of the experiment are summarized in Tables 2 and 3, which show the bias and MSEs for each parameter, along with the LB and UB for the MLEs. As the table shows, the MLE method consistently estimates the parameters of the proposed model. Moreover, as the sample size increases, the MLEs gradually approach the actual values of α and β. In Figures 2 and 3, we have displayed a clear picture of MSEs with 95% confidence bound (dark region) for α and β.

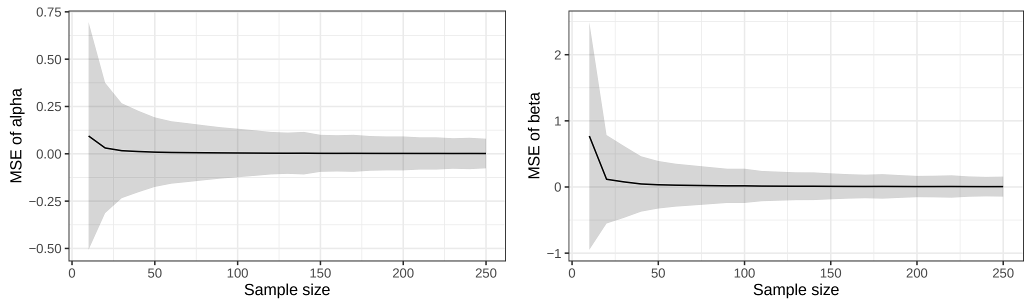

Figure 2: *MSE plots of α and β with 95% CI for initial values $\alpha = 0.5$ and $\beta = 1.25$.*

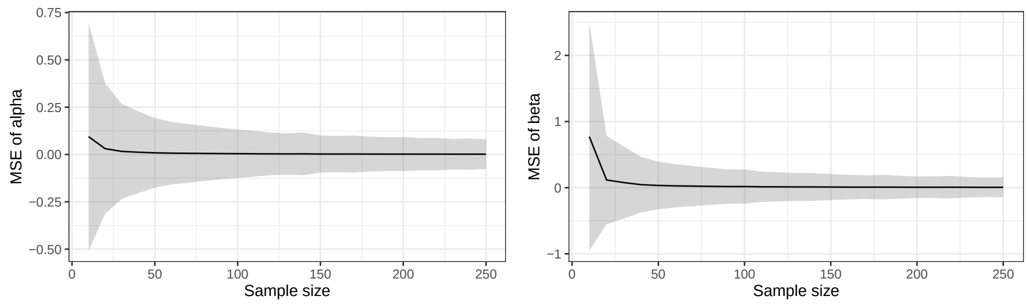

Figure 3: *MSE plots of α and β with 95% CI for initial values $\alpha = 0.75$ and $\beta = 1.5$.*

6. Application

Employing three real data sets, we demonstrate the applicability of the NCC-IW distribution in this section. The data sets employed for the application of the suggested distribution are given below:

6.1. Model Analysis

To analyze the data sets under study, we calculate several widely used goodness-of-fit statistics. The fitted models are then compared using various measures, including the log-likelihood value (-2logL), Akaike information criterion (AIC), Hannan-Quinn information criterion (HQIC), Anderson-Darling (AD), Kolmogorov-Smirnov (KS) with p-values, and Cramer-von Mises (CVM). For additional information see [1]. All the essential computations are carried out in R-software. For the comparison of fitting capability we have selected some models such as inverse Weibull (IW), arctan generalized exponential (ArcTGE) [8], arctan Lomax (ArcTLx) [20], arcsine exponential (ASE) [21], Tan Burr XII (TBXII) [4], New Cosine Weibull (NCW) [22], Exponentiated Cos Weibull (EcosW) [7], arcsine exponentiated Weibull (ASEW) [23], Cos Weibull (CosW) [2] and Sine inverse Weibull (Sin-IW) [3].

Data set I:

The dataset from [24] contains information on the relief times of 20 patients who were administered an analgesic. An analgesic is a type of medication that is commonly used to reduce pain, and the relief time refers to the duration for which the patients experience relief from their pain after taking the medication. The data are "1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, and 2.0".

Table 2: *Bias, MSEs, and LCL and UCL for MLEs with initial values $\alpha = 0.5$ and $\beta = 1.25$.*

n	$bias_\alpha$	$bias_\beta$	mse_α	mse_β	LCL_α	UCL_α	LCL_β	UCL_β
10	0.0904	0.1500	0.0444	0.4191	0.3381	1.0299	0.7585	2.5927
20	0.0445	0.0633	0.0143	0.0823	0.3761	0.7924	0.8920	1.9588
30	0.0248	0.0283	0.0079	0.0400	0.3893	0.7076	0.9577	1.7161
40	0.0194	0.0249	0.0056	0.0295	0.3918	0.6853	0.9747	1.6231
50	0.0152	0.0182	0.0040	0.0219	0.4117	0.6556	1.0243	1.5883
60	0.0136	0.0174	0.0034	0.0191	0.4155	0.6376	1.0242	1.5585
70	0.0126	0.0100	0.0027	0.0146	0.4284	0.6194	1.0481	1.512
80	0.0083	0.0138	0.0023	0.0140	0.4260	0.6108	1.0596	1.5091
90	0.0069	0.0141	0.0020	0.0119	0.4239	0.6032	1.0692	1.4905
100	0.0070	0.0074	0.0021	0.0105	0.4316	0.6074	1.0601	1.467
110	0.0072	0.0060	0.0017	0.0095	0.4308	0.5913	1.0866	1.4777
120	0.0037	0.0130	0.0014	0.0090	0.4359	0.5831	1.0944	1.4520
130	0.0060	0.0139	0.0014	0.0083	0.4393	0.5823	1.0997	1.4567
140	0.0062	0.0107	0.0013	0.0074	0.4405	0.5821	1.1054	1.4331
150	0.0058	0.006	0.0013	0.0068	0.4424	0.5814	1.1116	1.4269
160	0.0054	0.0102	0.0012	0.0062	0.4428	0.5776	1.1226	1.4222
170	0.0060	0.007	0.0011	0.0061	0.4505	0.5729	1.1146	1.4056
180	0.0032	0.0044	0.0010	0.0054	0.4455	0.5704	1.1236	1.4032
190	0.0041	0.0059	0.0010	0.0056	0.4487	0.5716	1.1212	1.4118
200	0.0031	0.0026	9.00E-04	0.0049	0.4472	0.5687	1.1261	1.3952
210	0.0044	0.0058	9.00E-04	0.0048	0.4493	0.5619	1.1206	1.3927
220	0.0033	0.0076	8.00E-04	0.0047	0.4526	0.5645	1.1295	1.4010
230	0.0046	-6.00E-04	8.00E-04	0.0047	0.4529	0.5606	1.1258	1.3934
240	0.0036	0.0042	7.00E-04	0.0041	0.4531	0.5606	1.1335	1.3816
250	0.0028	0.0014	7.00E-04	0.0040	0.4505	0.5561	1.1315	1.3801

Table 3: *Bias, MSEs and LCL and UCL for MLEs with initial values $\alpha = 0.75$ and $\beta = 1.5$.*

n	$bias_\alpha$	$bias_\beta$	mse_α	mse_β	LCL_α	UCL_α	LCL_β	UCL_β
10	0.1424	0.2518	0.0942	0.7713	0.5183	1.5550	0.9717	3.5167
20	0.0692	0.0882	0.0309	0.1166	0.5616	1.1901	1.0949	2.3295
30	0.0363	0.0789	0.0164	0.0774	0.581	1.0422	1.1612	2.1797
40	0.0294	0.0383	0.0121	0.0458	0.6083	1.0183	1.1821	2.0153
50	0.0227	0.0311	0.0088	0.0337	0.6022	0.9684	1.2209	1.9084
60	0.0193	0.0287	0.0071	0.0276	0.6224	0.9468	1.2437	1.8913
70	0.0162	0.025	0.0063	0.0240	0.6247	0.9314	1.2604	1.8680
80	0.0139	0.0238	0.0055	0.0206	0.6397	0.9192	1.2795	1.8384
90	0.0130	0.0131	0.0048	0.0175	0.6507	0.9126	1.2956	1.8022
100	0.0107	0.0176	0.0043	0.0175	0.6418	0.8926	1.2919	1.8124
110	0.0085	0.0112	0.0038	0.0137	0.6507	0.8873	1.3022	1.7490
120	0.0085	0.0098	0.0033	0.0126	0.652	0.8749	1.3022	1.7395
130	0.0071	0.0152	0.0031	0.0116	0.6577	0.8734	1.3151	1.7406
140	0.0091	0.0123	0.0033	0.0115	0.6556	0.8793	1.3232	1.7467
150	0.0065	0.0139	0.0025	0.0101	0.6637	0.8666	1.3344	1.7227
160	0.0075	0.0051	0.0024	0.0090	0.6693	0.8575	1.3315	1.7045
170	0.0074	0.0110	0.0025	0.0083	0.6709	0.8606	1.3356	1.7040
180	0.0080	0.0105	0.0022	0.0090	0.6712	0.8471	1.3412	1.7089
190	0.0053	0.0116	0.0021	0.0078	0.6685	0.8571	1.3481	1.6881
200	0.0052	0.0046	0.0021	0.0068	0.6694	0.8492	1.3471	1.6665
210	0.0040	0.0056	0.0019	0.0070	0.6731	0.8479	1.3615	1.6799
220	0.0055	0.0051	0.0019	0.0075	0.6758	0.8441	1.3438	1.6867
230	0.0055	0.0049	0.0017	0.0062	0.6799	0.8439	1.3583	1.6725
240	0.0049	0.0086	0.0018	0.0057	0.6770	0.8414	1.3603	1.6553
250	0.0055	0.0092	0.0016	0.0060	0.6822	0.8406	1.3621	1.6657

Table 4: *MLEs with SE (in parentheses) (dataset I)*

Model	Parameter(SE)		
NCC-IW(α,β)	3.2906(0.5941)	3.9558(1.0140)	–
IW(λ,θ)	4.0175(0.7060)	6.0224(2.0083)	–
ArcTGE(α,λ,θ)	0.0000(1.5645)	19.3864(6.0429)	1.8579(0.2245)
ArcTLx(α,β,θ)	147.2664(44.0127)	0.2871(0.2782)	12.3869(9.6739)
ASE(θ)	127.8946(4.8432)	–	–
ASEW(λ,θ,v)	1.0488(0.1284)	104.561(19.0921)	3.1656(0.1303)
NCW(λ,θ)	0.2505(0.0810)	2.2930(0.3402)	–
TBXII(λ,ν,θ)	1.3946(0.1597)	10.3624(5.0171)	0.3937(0.2735)
ECosW(β,λ,θ)	0.2386(0.0486)	0.2789(0.0712)	2.7222(0.1824)
CosW(β,δ)	2.2183(0.3323)	0.5655(0.0471)	–
SinIW(δ,θ)	5.3385(1.4594)	2.8386(0.4882)	–

In Tables 4, 6, and 8, we have presented the estimated values of the parameters and their corresponding standard error (SE in parentheses) of the models under study using the MLE method. Similarly, in Tables 5, 7, and 9, we have presented the model selection and goodness of fit statistics such as log-likelihood, AIC, HQIC, KS, AD, and CVM for all three data sets. It has been observed that the suggested model NCC-IW has the least statistics as compared to IW, ArcTGE, ArcTLx, ASE, ASEW, NCW, TBXII, ECosW, CosW, and Sin-IW. Hence NCC-G is more flexible (even four trigonometric distributions having three parameters) and provides a good fit. Also, we have displayed the graphical illustrations of the fitted models in Figures 5, 7, and 9. These figures also verified that the NCC-G model can perform well as compared to candidate models.

Table 5: *Some selection criteria and goodness-of-fit statistics (dataset-I)*

Model	-2logL	AIC	HQIC	KS	p(KS)	CVM	p(CVM)	AD	p(AD)
NCC-IW	31.1170	35.1170	35.5057	0.1148	0.9548	0.0306	0.9770	0.1772	0.9956
IW	30.8174	34.8174	35.2062	0.1020	0.9854	0.0266	0.9880	0.1545	0.9984
ArcGE	33.4131	39.4131	39.9962	0.1516	0.7473	0.0767	0.7169	0.4214	0.8256
ArcLmx	35.6262	41.6262	42.2094	0.1240	0.9182	0.0662	0.7806	0.5268	0.7175
ASE	154.7472	156.7472	156.9416	0.8863	0.0000	5.1247	0.0000	31.4397	0.0000
ASEW	31.1885	37.1885	37.7716	0.1170	0.9470	0.0363	0.9551	0.2096	0.9877
NCW	48.6870	52.6870	53.0757	0.1467	0.7829	0.1078	0.5521	0.7800	0.4940
Tan-BXII	31.0804	37.0804	37.6636	0.0919	0.9959	0.0231	0.9944	0.1377	0.9994
CosW	40.6035	46.6035	47.1867	0.1922	0.4508	0.1840	0.3022	1.0593	0.3267
NCosW	37.4854	41.4854	41.8742	0.1770	0.5576	0.1279	0.4681	0.7563	0.5118
Sin-IW	31.1572	35.1572	35.5460	0.1069	0.9763	0.0292	0.9813	0.1808	0.9949

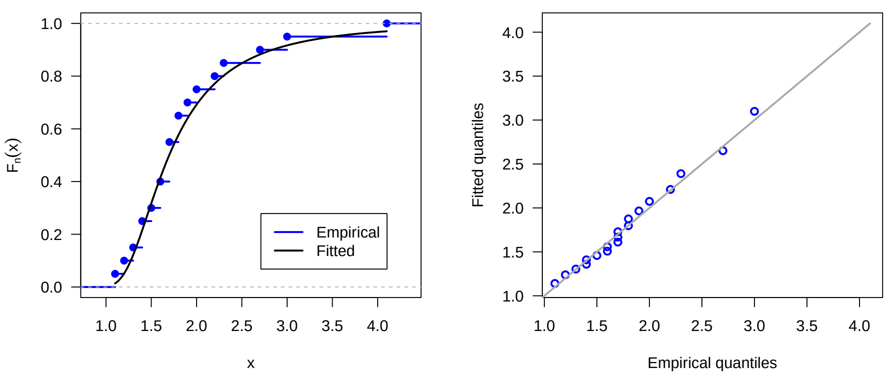

Figure 4: *KS and Q-Q plots (dataset-I).*

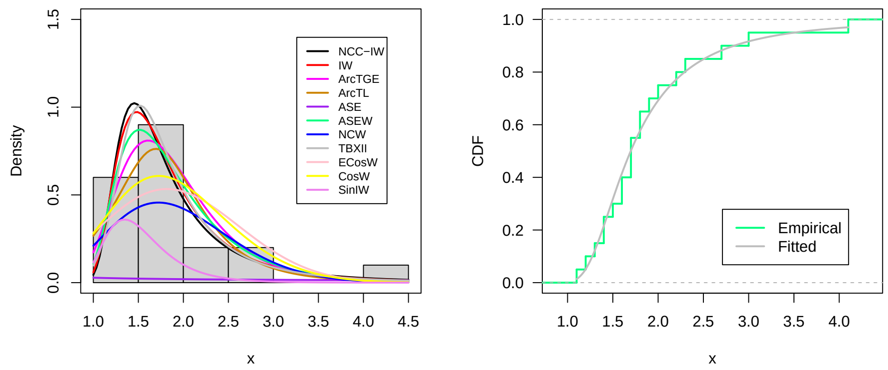

Figure 5: *Estimated PDF (left) and empirical vs estimated CDF (right) (dataset-I).*

Data set-II

The following data set was obtained by [25] consisting of 128 observations on the time intervals, measured in seconds, between the arrivals of vehicles at a specific location on a road. 0.2, 0.5, 0.8, 0.8, 0.8, 1.0, 1.1, 1.2, 1.2, 1.2, 1.2, 1.2, 1.3, 1.4, 1.5, 1.5, 1.6, 1.6, 1.6, 1.7, 1.8, 1.8, 1.8, 1.8, 1.8, 1.9, 1.9, 1.9, 1.9, 1.9, 1.9, 1.9, 2.0, 2.1, 2.1, 2.2, 2.3, 2.3, 2.4, 2.4, 2.5, 2.5, 2.5, 2.6, 2.6, 2.7, 2.8, 2.8, 2.9, 3.0, 3.0, 3.1, 3.2, 3.4, 3.7, 3.9, 3.9, 3.9, 4.6, 4.7, 5.0, 5.1, 5.6, 5.7, 6.0, 6.0, 6.1, 6.6, 6.9, 6.9, 7.3, 7.6, 7.9, 8.0, 8.3, 8.8, 8.8, 9.3, 9.4, 9.5, 10.1, 11.0, 11.3, 11.9, 11.9, 12.3, 12.9, 12.9, 13.0, 13.8, 14.5, 14.9, 15.3, 15.4, 15.9, 16.2, 17.6, 20.1, 20.3, 20.6, 21.4, 22.8, 23.7, 24.7, 29.7, 30.6, 31.0, 33.7, 34.1, 34.7, 36.8, 40.1, 40.2, 41.3, 42.0, 44.8, 49.8, 51.7, 55.7, 56.5, 58.1, 70.5, 72.6, 87.1, 88.6, 91.7, 119.8, 125.3´

Table 6: *MLEs with SE (in parentheses) (dataset-II)*

model	Parameter(SE)		
NCC-IW(α,β)	0.6748(0.0476)	2.0442(0.1524)	–
IW(λ,θ)	0.8183(0.0540)	2.6651(0.2527)	–
ArcTGE(α,λ,θ)	1.00E-04(0.3004)	0.6685(0.0737)	0.0475(0.0063)
ArcTLx(α,β,θ)	1.00E-04(1.7292)	0.0833(0.0310)	1.6100(0.3969)
ASE(θ)	11.6716(1.5853)	–	–
ASEW(λ,θ,v)	0.2832(0.0181)	36.6683(2.9617)	2.8476(0.1367)
NCW(λ,θ)	0.3294(0.0389)	0.5770(0.0359)	–
TBXII(λ,ν,θ)	1.5673(0.3041)	2.7668(0.7182)	0.2870(0.0962)
ECosW(β,λ,θ)	0.1949(0.0172)	0.4319(0.0000)	0.7179(1.00E-04)
CosW(β,δ)	0.5566(0.0350)	0.1655(0.0220)	–
SinIW(δ,θ)	3.0753(0.2472)	0.5944(0.0385)	–

Table 7: *Some selection criteria and goodness-of-fit statistics (dataset-II)*

model	-2logL	AIC	HQIC	KS	p(KS)	CVM	p(CVM)	AD	p(AD)
NCC-IW	924.4949	928.4949	930.8125	0.0616	0.717	0.1390	0.4253	1.0806	0.3175
IW	921.1562	925.1562	927.4738	0.0604	0.7381	0.1323	0.4488	0.974	0.3711
ArcGE	948.3638	954.3638	957.8402	0.1481	0.0073	0.8221	0.0064	4.4936	0.0050
ArcLmx	929.2980	935.2980	938.7744	0.0976	0.1745	0.2488	0.1899	1.7906	0.1202
ASE	956.6371	958.6371	959.7959	0.1501	0.0063	0.6261	0.0191	4.0879	0.0079
ASEW	912.5793	918.5793	922.0557	0.0886	0.2680	0.1858	0.2969	1.1082	0.3051
NCW	998.8923	1002.892	1005.210	0.1207	0.0480	0.3327	0.1096	2.8599	0.0323
Tan-BXII	917.7461	923.7461	927.2225	0.0787	0.4057	0.2086	0.2516	1.2355	0.2544
ECosW	937.7763	943.7763	947.2527	0.1170	0.0603	0.4531	0.0523	2.8386	0.0332
CosW	927.9831	931.9831	934.3006	0.1162	0.0632	0.3467	0.1003	2.1999	0.0716
Sin-IW	916.8895	920.8895	923.2071	0.0868	0.2903	0.1819	0.3056	1.1174	0.3010

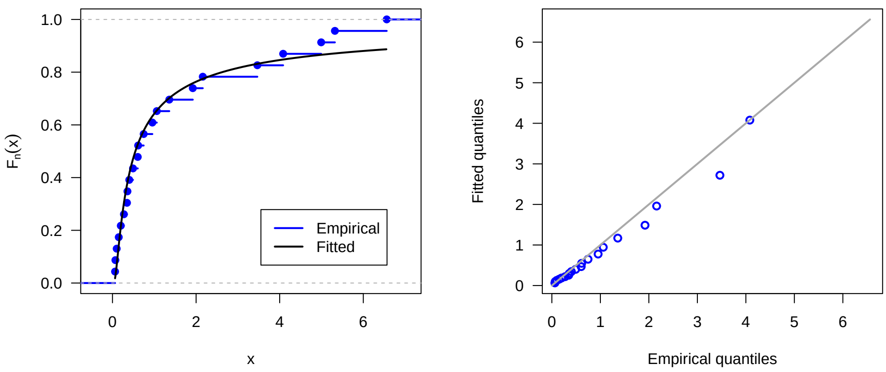

Figure 6: *KS and Q-Q plots (dataset-II).*

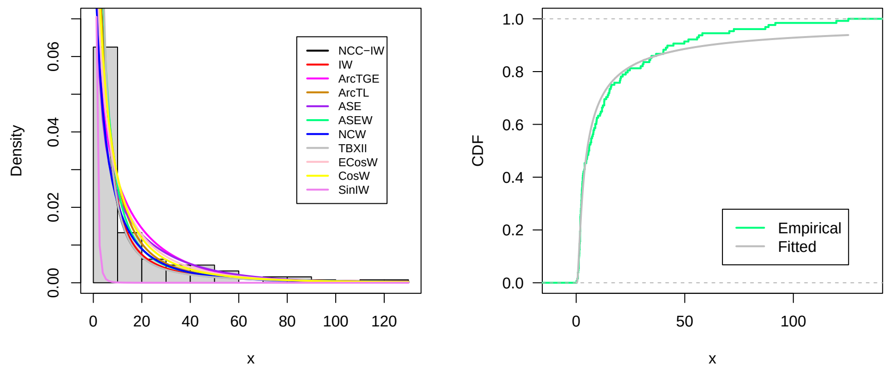

Figure 7: *Estimated PDF (left) and empirical vs estimated CDF (right) (dataset-II).*

Data set-III

We have used the real data reported by [26] and it represents the failure time of 30 items "0.602, 0.603, 0.603, 0.615, 0.652, 0.663, 0.688, 0.705, 0.761, 0.770, 0.868, 0.884, 0.898, 0.901, 0.911, 0.918, 0.935, 0.953, 0.983, 1.009, 1.040, 1.097, 1.097, 1.148, 1.296, 1.343, 1.422, 1.540, 1.555, 1.653"

Table 8: *MLEs with SE (in parentheses) (dataset-III).*

Model	Parameter(SE)		
NCC-IW(α, β)	3.178(0.4814)	0.4482(0.1032)	–
IW(λ, θ)	3.8881(0.555)	0.4275(0.1108)	–
ArcTGE(α, λ, θ)	4.2E-07(1.3892)	31.5836(5.4017)	4.168(0.1055)
ArcTLx(α, β, θ)	143.7539(4.3514)	0.3264(0.2077)	18.9209(10.5266)
ASE(θ)	194.3346(5.9316)	–	–
ASEW(λ, θ, v)	1.2673(0.1458)	41.0829(6.893)	5.4445(0.3188)
NCW(λ, θ)	1.172(0.1733)	2.662(0.3434)	–
TBXII(λ, ν, θ)	0.7768(0.1457)	7.3693(3.3169)	0.6354(0.5186)
ECosW(β, λ, θ)	0.1726(0.0315)	2.368(0.0046)	3.2934(5.00E-04)
CosW(β, δ)	2.5699(0.3352)	1.0907(0.0648)	–
SinIW(δ, θ)	0.8297(0.1375)	2.7635(0.3757)	–

Table 9: *Some selection criteria and goodness-of-fit statistics (dataset-III).*

Model	-2logL	AIC	HQIC	KS	p(KS)	CVM	p(CVM)	AD	p(AD)
NCC-IW	8.7135	12.7135	13.6100	0.1608	0.4197	0.0992	0.5920	0.6056	0.6413
IW	7.6494	11.6494	12.5459	0.1432	0.5700	0.0793	0.6995	0.5146	0.7306
ArcGE	7.2658	13.2658	14.6105	0.0901	0.9679	0.0479	0.8924	0.3760	0.8712
ArcLmx	12.3954	18.3954	19.7402	0.1262	0.7260	0.0542	0.8542	0.5154	0.7297
ASE	224.6097	226.6097	227.0579	0.9413	0.0000	8.6137	0.0000	62.4475	0.0000
ASEW	6.4640	12.4640	13.8087	0.1141	0.8293	0.0561	0.8419	0.4091	0.8385
NCW	26.5153	30.5153	31.4118	0.1328	0.6649	0.0797	0.6969	0.7164	0.5440
Tan-BXII	8.9087	14.9087	16.2535	0.1074	0.8792	0.0536	0.8575	0.4021	0.8456
ECosW	12.0969	18.0969	19.4416	0.1238	0.7477	0.1053	0.5622	0.7074	0.5514
CosW	9.8726	13.8726	14.7692	0.1004	0.9230	0.0741	0.7304	0.5449	0.7001
Sin-IW	7.1457	11.1457	12.0422	0.1111	0.8526	0.0567	0.8383	0.4210	0.8265

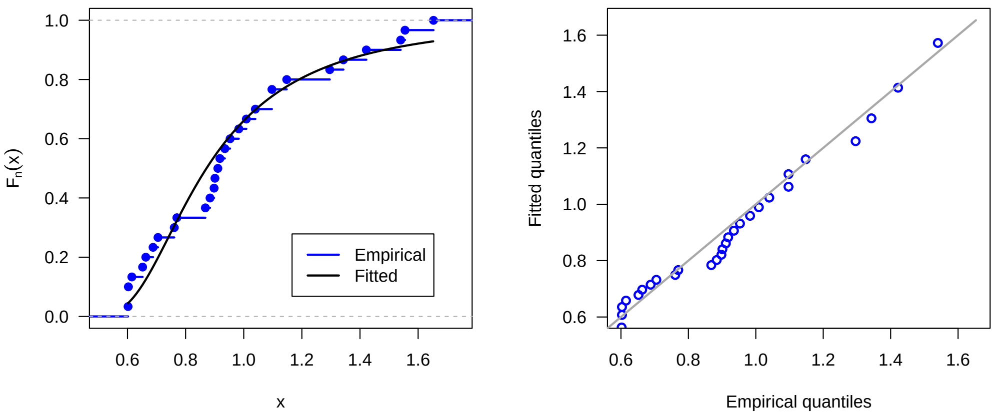

Figure 8: *KS and Q-Q plots (dataset-III).*

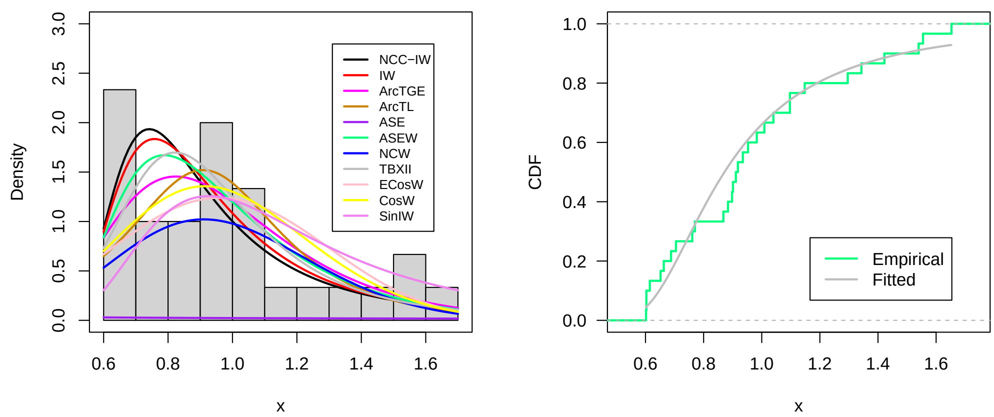

Figure 9: *Estimated PDF (left) and empirical vs estimated CDF (right) (dataset-III).*

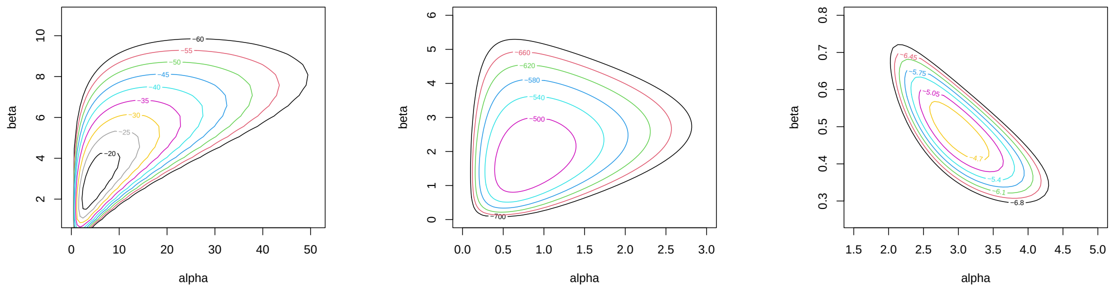

Figure 10: *Countour plots for three data sets under study respectively.*

7. Conclusion

A new family of distributions, known as the Cos-G family, has been developed by transforming the cosine function based on the ratio of CDF $G(x)$ and $1 + G(x)$ of a baseline distribution. The general properties of this distribution family have been described. To create a member of this family with a hazard function that is increasing, decreasing, or inverted bathtub-shaped, the Inverse Weibull distribution was used as a baseline distribution. The resulting distribution, called NCC-IW, was analyzed for its statistical properties, and its parameters were estimated using the MLE method. The estimation procedure was evaluated through a Monte Carlo simulation, which showed that the biases and mean square errors decrease as the sample size increases, even for small samples. The NCC-IW distribution was then applied to three real data sets, and using model selection criteria and goodness-of-fit test statistics, it was shown to outperform other existing models with more parameters. This suggests that the Cos-G family and its member distribution have wide applications in fields such as medical science, reliability engineering, and survival analysis, and can lead to the development of new models in the future.

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