APPLICATION OF FUZZY DYNAMIC GROUP MULTI-CRITERIA DECISION MAKING BASED ON Z-NUMBERS

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Abstract

Dynamic group multi-criteria decision making is essential for making informed, balanced, and adaptive decisions in complex and evolving environments. By integrating multiple methodologies and considering the dynamic nature of criteria and group interactions, dynamic group multi-criteria decision making provides a robust framework for decision-making across various fields and applications. Dynamic group fuzzy multi-criteria decision making under Z-information is a sophisticated approach that incorporates the dynamic aspects of decision making, the involvement of multiple stakeholders, and the use of fuzzy logic to handle uncertainties and imprecise information. Z-information refers to a type of uncertain information that combines fuzzy numbers and Z-numbers, where Z-numbers account for both the reliability of the information and its fuzziness. By integrating fuzzy logic and Z-numbers, it effectively handles dual uncertainties of fuzziness and reliability, while dynamically adapting to changes in criteria and stakeholder preferences. In this article, a dynamic multi-criteria decision-making model is proposed to solve strategic vendor selection problems that need to be evaluated in different time periods and involve uncertainty. Z-information is used to express uncertainty and in the proposed model, the decision-making group is asked to evaluate the alternatives in different time periods, and the evaluations made for these different periods are combined.

Keywords: Fuzzy logic, Z-numbers, multi-criteria decision making, vendor selection, dynamic group decision making

I. Introduction

Dynamic group multi-criteria decision making is a decision-making process that involves multiple criteria, stakeholders, and evolving scenarios over time. This approach integrates the complexities of group dynamics, changing environments, and various criteria that must be considered to reach an optimal decision. Multi-criteria decision making is indeed a critical branch in management science [1]. It involves evaluating and making decisions based on multiple conflicting criteria. This complexity is inherent in various real-world problems where decision-makers must consider several factors to arrive at the best possible solution [2]. A feature of several practical problems of multicriteria choice on a finite set of alternatives is not only a significant number of criteria and restrictions of various types, but also the presence of dependencies between the criteria, which appear when formalizing the preferences of the decision maker [3]. Additional difficulties arise when comparing qualitative criteria specified, for example, in linguistic scales [4]. Successful solution to the methodological and algorithmic problems that arise when forming a preference function based on criteria of various types was implemented during the development of dynamic multi-criteria decision-making approach [5]. The information used in the decision-making process of many multi-criteria decision-making problems generally belongs to the same time period. When evaluating alternatives and criteria, the same time period, that is, the time period in which the evaluation is made, is taken into account. However, in some cases, it is necessary to evaluate the current performance of the alternatives as well as their performance in previous time periods. For example, the information required for decision making in investment decisions, medical diagnosis, personnel evaluation, and evaluation of the effectiveness of military systems must be collected in different time periods. For this purpose, Xu and Yager developed dynamic multi-criteria decision-making procedures based on fuzzy sets and dynamic fuzzy weighted environment operator for combining information collected in different time periods [6]. This authors also proposed a dynamic intuitionistic fuzzy multi-attribute decision making process in which all criteria are explained by intuitionistic fuzzy numbers collected at different periods. The authors defined intuitionistic fuzzy and imprecise intuitionistic fuzzy variables and used the intuitionistic fuzzy weighted environment operator when all decision information about attributes in different periods is represented by intuitionistic fuzzy numbers. Su et al. investigated the dynamic intuitionistic fuzzy multi-attribute group decision making problem. In this study, evaluations are made using fuzzy sets by a group of decision makers in different periods [7]. The authors present an interactive method for solving intuitionistic fuzzy multi-attribute group decision making problems. In this method, firstly, decision makers use intuitionistic fuzzy set-in different periods for evaluation alternatives. Dynamic multi-criteria decision making is particularly relevant in environments where information and circumstances continuously change, necessitating ongoing reassessment and adjustment of decisions [8]. Exploring dynamic multi-criteria decision making involves understanding how decision-making processes can adapt to changes over time, incorporating evolving criteria, preferences, and conditions [9]. Dynamic multi-criteria decision making is an advanced approach that integrates the temporal evolution of criteria and preferences, allowing for more flexible, adaptive, and robust decision-making processes. By leveraging methodologies such as dynamic AHP [10], dynamic TOPSIS [11], and fuzzy logic [12], and by continuously incorporating real-time data and feedback, dynamic MCDM can significantly enhance decision quality in complex, changing environments across various fields. The decision-making environment is usually uncertain due to some uncontrollable factors. Lots of applications in the real world are researched based on uncertainty, such as fault diagnosis [13] and reliability analysis [14]. To cope with such uncertainty, many tools have been put forward, including fuzzy set [15], evidence theory [16], linguistic information [17] and Z-numbers [18]. Dynamic decision-making with Z-numbers is an advanced approach that incorporates the uncertainty and reliability of information in evolving scenarios. Z-numbers, introduced by Lotfi A. Zadeh, are an extension of fuzzy numbers that consider both the fuzzy value of a piece of information and its reliability [19]. Dynamic decision-making with Z-numbers is an advanced approach to decision-making that incorporates both uncertainty and reliability in information. It provides a robust framework for handling uncertainty and reliability in evolving scenarios. This framework is particularly useful in dynamic decision-making scenarios, where conditions evolve over time and decision-makers must handle both vague information and varying degrees of trust in that information. By incorporating both the fuzzy value and reliability of information, this approach allows for more adaptive and informed decision-making. It is particularly valuable in complex and dynamic fields such as vendor selection, where continuous updates and adjustments are necessary to respond to changing conditions and new data.

The structure of this article is formed as follows. Section 2 introduces the basic definitions of the dynamic fuzzy multi-attribute decision making approach with Z numbers that is employed in this problem. Section 3 proposed statement and solution of the supplier selection problem. Section 4 represents the main results achieved in this article.

II. Preliminaries

Definition 1. Z-numbers extend the concept of fuzzy numbers to include both the uncertainty of a value and the reliability of that value. A Z-number $Z=\left(\tilde{A},\tilde{B}\right)$ consists of two components - $\tilde{A}$ is a fuzzy number representing an uncertain value, $\tilde{B}$ is fuzzy number representing the reliability of $\tilde{A}$ [19].

Definition 2. Basic operations on Z numbers represented as below [20]. Suppose are given two Z numbers, $z_1=\left(\left(a_1^l,a_1^m,a_1^u\right),\left(b_1^l,b_1^m,b_1^u\right)\right)$ and $z_2=\left(\left(a_2^l,a_2^m,a_2^u\right),\left(b_2^l,b_2^m,b_2^u\right)\right)$.

$$z_1+z_2=\left(\left(\alpha_1a_1^l+\alpha_2a_2^l,\alpha_1a_1^m+\alpha_2a_2^m,\alpha_1a_1^u+\alpha_2a_2^u\right),\right.$$
$$\left.\left(\max\left\{b_1^l/b_1,b_2^l/b_2\right\},\max\left\{b_1^m/b_1,b_2^m/b_2\right\},\max\left\{b_1^u/b_1,b_2^u/b_2\right\}\right)\right) \tag{1}$$

$$z_1*z_2=\left(\left(\alpha_1\alpha_2a_1^la_2^l,\alpha_1\alpha_2a_1^ma_2^m,\alpha_1\alpha_2a_1^ua_2^u\right),\right.$$
$$\left.\left(\min\left\{r_1^l/r_1,r_2^l/r_2\right\},\min\left\{r_1^m/r_1,r_2^m/r_2\right\},\min\left\{r_1^u/r_1,r_2^u/r_2\right\}\right)\right) \tag{2}$$

$$\lambda z_1=\left(\left(\lambda\alpha_1a_1^\lambda,\lambda\alpha_1a_1^m,\lambda\alpha_1a_1^u\right),\left(r_1^l/r_1,r_1^m/r_1,r_1^u/r_1\right)\right),\lambda>0 \tag{3}$$

$$z_1^\lambda=\left(\left(\left(\alpha_1a_1^l\right)^\lambda,\left(\alpha_1a_1^m\right)^\lambda,\left(\alpha_1a_1^u\right)^\lambda\right),\left(b_1^l/b_1,b_1^m/b_1,b_1^u/b_1\right)\right),\lambda>0$$

Definition 3. Z-number-valued pairwise comparison matrix (Z_{ij}) is a matrix of Z-numbers [21]:

$$(Z_{ij}=(A_{ij},B_{ij}))=\begin{pmatrix} Z_{11}=(A_{11},B_{11}) & \dots & Z_{1n}=(A_{1n},B_{1n}) \\ . & \dots & . \\ Z_{n1}=(A_{n1},B_{n1}) & \dots & Z_{nn}=(A_{nn},B_{nn}) \end{pmatrix} \tag{4}$$

A Z-number $Z_{ij}=(A_{ij},B_{ij}),\ i,j=1,\dots,n$ describes partially reliable information on degree of preference for i-th criterion against j-th one. $\tilde{A}$ and $\tilde{B}$ can be linguistic terms selected from linguistic set $\tilde{A}_Z$ and $\tilde{B}_Z$, respectively. For example,

$\tilde{A}_Z=\left\{about\ a_1, nearly\ a_2, exactly\ a_3, over\ a_4\right\}and\ \tilde{B}_Z=\left\{very\ low\ sure, low\ sure, shure, very\ sure\right\}$

where $a_i\left(i=1,\cdots,4\right)$ can be certain values of evaluation, such as percentages.

Let $D=\lfloor D_{ij}\rfloor$ be the decision-making matrix, where D_{ij} is the evaluation of any alternative with respect to each attribute. $D_{ij}=Z_{ij}\left(\tilde{A},\tilde{B}\right)$ and the Z-number $Z_{ij}\left(\tilde{A},\tilde{B}\right), i=1,2,\cdots,m,\ j=1,2,\cdots,n$ is the evaluation of the j - th criteria for i - th alternative, which contains the opinion of evaluators, $\tilde{A}$, and reliability of the opinion, $\tilde{B}$. Thus, the decision-making problem can be modelled as below.

$$D=\begin{bmatrix} Z_{11}\left(\tilde{A},\tilde{B}\right) & Z_{11}\left(\tilde{A},\tilde{B}\right) & \cdots & Z_{11}\left(\tilde{A},\tilde{B}\right) \\ Z_{11}\left(\tilde{A},\tilde{B}\right) & Z_{11}\left(\tilde{A},\tilde{B}\right) & \cdots & Z_{11}\left(\tilde{A},\tilde{B}\right) \\ \vdots & \vdots & \cdots & \vdots \\ Z_{11}\left(\tilde{A},\tilde{B}\right) & Z_{11}\left(\tilde{A},\tilde{B}\right) & \cdots & Z_{11}\left(\tilde{A},\tilde{B}\right) \end{bmatrix} \tag{5}$$

Definition 4. Assume $z_1 = \left(\left(a_1^l, a_1^m, a_1^u\right), \left(b_1^l, b_1^m, b_1^u\right)\right)$ and $z_2 = \left(\left(a_2^l, a_2^m, a_2^u\right), \left(b_2^l, b_2^m, b_2^u\right)\right)$ are Z -numbers, then normalize their second components as below [21].

$$z_1 = \left(\left(a_1^l, a_1^m, a_1^u\right), \left(b_1^l / b_1, b_1^m / b_1, b_1^u / b_1\right)\right) \tag{6}$$

$$z_2 = \left(\left(a_2^l, a_2^m, a_2^u\right), \left(b_2^l / b_2, b_2^m / b_2, b_2^u / b_2\right)\right) \tag{7}$$

where $b_1 = b_1^l + b_1^m + b_1^u$, $b_2 = b_2^l + b_2^m + b_2^u$

Definition 5. Let $\alpha(t_1), \alpha(t_2), \cdots, \alpha(t_p)$ be the arguments collected from P different periods. Weight vector $\omega(t)$ of the time series $t_k \ (k = 1, 2, \cdots, p)$, introduced a basic unit-interval monotonic function based approach to determining $\omega(t)$ [22].

$$\omega(t_k) = \frac{e^{\frac{\alpha k}{P}}\left(1 - e^{\frac{\alpha}{P}}\right)}{e^{\alpha} - 1} \tag{8}$$

P is size of periods, k is observation of period, and $0 < \alpha < 1$. $\omega(t_k)$ show the importance degrees of the arguments of the different periods and can reflect the change of the importance degrees of the periods.

Definition 6. A Z-number $Z = (A, B)$ is characterized by fuzzy number A, fuzzy number B and an underlying set of probability distributions G [23]. Distance between Z-numbers $D(Z_1, Z_2)$ determined as follows. Distance between A_1 and A_2 is computed as below.

$$D(A_1, A_2) = \sup_{\alpha \in (0,1]} D(A_1^{\alpha}, A_2^{\alpha}) \tag{9}$$

$$D(A_1^{\alpha}, A_2^{\alpha}) = \left| \frac{A_{11}^{\alpha} + A_{12}^{\alpha}}{2} - \frac{A_{21}^{\alpha} + A_{22}^{\alpha}}{2} \right| \tag{10}$$

A_1^{α} and A_2^{α} denote α -cuts of A_1 and A_2 respectively, $A_{11}^{\alpha}, A_{12}^{\alpha}$ denote lower and upper bounds of A_1^{α} ($A_{21}^{\alpha}, A_{22}^{\alpha}$ are those of A_2^{α}). Distance between B_1 and B_2 is computed analogously. A distance between sets G_1 and G_2 of probability distributions p_1 and p_2 can be expressed as

$$D(G_1, G_2) = \inf_{p_1 \in G_1, p_2 \in G_2} \left\{ (1 - \int_R (p_1 p_2)^{\frac{1}{2}} dx)^{\frac{1}{2}} \right\} \tag{11}$$

where the expression in figure brackets is the Hellinger distance between two probability distributions p_1 and p_2. The use of inf operator implies that among all the possible pairs of distributions, the pair of the closest $p_1 \in G_1$ and $p_2 \in G_2$ is found to define distance $D(G_1, G_2)$. Given $D(A_1, A_2)$, $D(B_1, B_2)$ and $D(G_1, G_2)$, the distance for Z-numbers is defined as below.

$$D(Z_1, Z_2) = \beta D(A_1, A_2) + (1 - \beta) D_{total}(B_1, B_2) \tag{12}$$

$D_{total}(B_1, B_2)$ is a distance for reliability restrictions which is computed as:

$$D_{total}(B_1, B_2) = wD(B_1, B_2) + (1-w)D(G_1, G_2) \tag{13}$$

$\beta, w \in [0,1]$ are DM's assigned importance degrees.

Definition 7. The basic concept of the simple additive weighted method is to find the weighted sum of performance ratings on each alternative on all attributes. This method needs the procedure of normalization of the decision matrix to a scale suitable to all alternative ratings [23]. The formula for normalization represented as follows:

$$R_{ij} = \begin{cases} \dfrac{X_{ij}}{Max\ X_{ij}} \\ \dfrac{Min\ X_{ij}}{X_{ij}} \end{cases} \tag{14}$$

R_{ij} is performance rating, $Max\ X_{ij}$ is maximum value of each row and column, $Min\ X_{ij}$ is minimum value of each row. The preference value for each alternative (V_i) is represented as below.

$$V_i = \sum\nolimits_{j=1}^{n} W_j r_{ij} \tag{15}$$

V_i is final value of the alternative, W_j is specified weight, R_{ij} is normalization of the matrix. A larger value of V_i indicates that A_i alternatives are preferred.

Definition 8. An inconsistency index K for Z-number-valued pairwise comparisons method (Z_{ij}) is determined as follows [24]:

$$K(Z_{ij}) = \max_{i<j<k} \min \left\{ D\left(Z(1), \left(\frac{Z_{ik}}{Z_{ij}Z_{jk}} \right) \right) D\left(Z(1), \left(\frac{Z_{ij}Z_{ik}}{Z_{jk}} \right) \right) \right\} \tag{16}$$

where the components of Z-number $Z(1) = (A, B)$ are fuzzy singletons $A = 1$, $B = 1$.

Definition 9. For Z-numbers Z, Z' holds.

$$ZZ' \ if\ D(Z,\ (1,1)) \geq D(Z',\ (1,1)) \tag{17}$$

where D is distance defined above, (1,1) is a fuzzy singletons-based Z-number [24]. One can easily prove that ≤ is a partial order as it poses the following properties:
$Z \leq Z$ (reflexivity), If $Z \leq Z'$ and $Z' \leq Z$ then $Z = Z'$ (antisymmetric)
If $Z \leq Z'$ and $Z' \leq Z''$ then $Z \leq Z''$ (transitivity)

III. Case Study Example: Strategic vendor selection problem

The strategic vendor selection problem is a critical aspect of supply chain management and procurement, involving the evaluation and selection of suppliers that best meet an organization's strategic objectives. This problem is inherently multi-criteria and dynamic, as it must consider various factors such as cost, quality, delivery performance, technological capability, and more, which may change over time. Main steps in strategic vendor selection are problem definition, criteria identification, dynamic and multi-criteria approach utilization, data collection, applying appropriate MCDM methodologies to handle the complexity and dynamics of the vendor selection process and incorporating Z-numbers to manage uncertainty and reliability in vendor evaluation. Problem definition includes defining the strategic goals and objectives of the vendor selection process, identifying the criteria that will be used to evaluate potential vendors. Suppose a company which wants to select strategic vendor. Decision maker uses dynamic fuzzy multi-attribute decision making approach by using Z-numbers for estimation alternatives and criteria [25]. Vendor selection is a multi-faceted decision-making process that involves evaluating potential suppliers based on various criteria to ensure they align with an organization's strategic goals and operational requirements [26].

Statement of the problem: Suppose, linguistic scale is employed to estimate the performance of the vendors A, B, C in the years 2021, 2022, and 2023 according to the criteria C_1, C_2, C_3, and C_4 with Z numbers. C_1 is quality, C_2 is flexibility, C_3 is sustainability, and C_4 is reliability. Quality criteria include such characteristics as compliance with specifications-the degree to which the vendor's products meet the required specifications and standards, defect rates- the frequency of defects or non-conformance in the supplied goods, certifications- quality certifications held by the vendor, which indicate adherence to recognized quality management systems. Flexibility is the vendor's ability to respond to changes in demand, market conditions, or buyer requirements, to offer customized solutions tailored to the buyer's specific needs, the capacity to scale operations up or down based on the buyer's needs. Sustainability is the vendor's practices related to environmental sustainability, such as waste management, carbon footprint, and energy use, also, vendor's commitment to social responsibilities, including fair labour practices, community engagement, ethical sourcing, and vendor's adherence to relevant laws and regulations concerning environmental and social standards. Reliability is historical performance data including reliability, consistency, and ability to meet commitments, the vendor's reputation in the market, including feedback from other customers and industry recognition, and the vendor's ability to identify, manage, and mitigate risks. Dynamic fuzzy multi-attribute decision making is used for selecting the best supplier. Dynamic fuzzy multi-attribute decision making refers to a decision-making process that deals with multiple criteria or attributes under conditions of uncertainty and vagueness, with the added complexity of decision elements changing over time. It is a powerful method for making decisions in environments where uncertainty and change are common. By combining fuzzy logic with dynamic weighting, it offers a flexible and robust approach to evaluating multiple alternatives over time. It allows decision-makers to handle uncertainty and vagueness in complex, real-world problems, adapts to changes in information or preferences over time, making it more flexible than static models, provides a more nuanced and comprehensive framework for evaluating alternatives compared to traditional methods. Linguistic scales are often used in fuzzy logic to handle qualitative and imprecise information. These scales allow for the representation of subjective judgments using linguistic terms, which are then converted into fuzzy numbers for analysis and decision-making. Graphical representation of linguistic terms for restriction (first) component of Z-number defined by triangular fuzzy numbers that represented in figure 1.

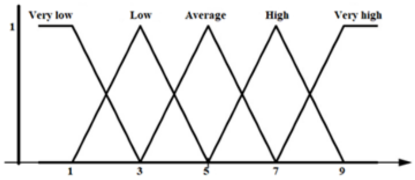

Figure 1: *Linguistic terms for restriction (first) component of Z-number*

Graphical representation of linguistic terms for reliability (second) component of Z-number defined by triangular fuzzy numbers is shown in figure 2.

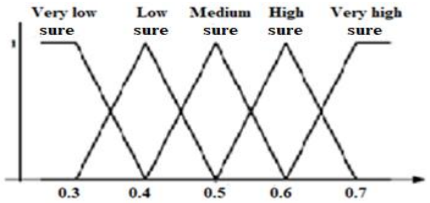

Figure 2: *Linguistic terms for reliability (second) component of Z-number*

Solution of the problem: In dynamic environments, supplier performance and market conditions can change over time. Traditional static decision models may not be effective in such scenarios because they don't account for the fluctuations and uncertainties that affect the performance of suppliers and the broader market. Dynamic approaches allow companies to be more responsive to changing supplier conditions and market dynamics, leading to more informed, by accounting for variability and uncertainty, dynamic models help reduce risks associated with poor supplier performance or unfavourable market conditions. The model incorporates time-dependent changes by updating Z-numbers periodically based on new information or performance reviews. Below is represented steps of how Z-numbers can be applied in this context.

Step 1: Construction decision matrices $D(t_1)$ for 2021, $D(t_2)$ for 2022, and $D(t_3)$ for 2023.

Construction of decision matrix $D(t_1)$ for 2021 represented in table 1.

Table 1: *Decision matrix* $D(t_1)$

Alternatives	Criteria C_i			
	C_1	C_2	C_3	C_4
A	5, 7, 9; 0.5,0.6,0.7	5, 7, 9; 0.5,0.6,0.7	3, 5, 7; 0.3,0.4,0.5	5, 7, 9; 0.5,0.6,0.7
B	3, 5, 7; 0.4,0.5,0.6	1, 3, 5; 0.5,0.6,0.7	3, 5, 7; 0.4,0.5,0.6	3, 5, 7; 0.4,0.5,0.6
C	5, 7, 9; 0.5,0.6,0.7	5, 7, 9; 0.5,0.6,0.7	3, 5, 7; 0.5, 0.6,0.7	5, 7, 9; 0.5,0.6, 0.7

Construction of decision matrix $D(t_2)$ for 2022 year represented in table 2.

Table 2: *Decision matrix* $D(t_2)$

Alternatives	Criteria C_i			
	C_1	C_2	C_3	C_4
A	3, 5, 7; 0.5,0.6,0.7	3, 5, 7; 0.5,0.6,0.7	1, 3, 5; 0.5,0.6,0.7	5, 7, 9; 0.3,0.4,0.5
B	5, 7, 9; 0.3,0.4,0.5	3, 5, 7; 0.5,0.6,0.7	3, 5, 7; 0.5,0.6,0.7	3, 5, 7; 0.5,0.6,0.7
C	3, 5, 7; 0.5,0.6,0.7	5, 7, 9; 0.3,0.4,0.5	3, 5, 7; 0.4,0.5,0.6	5, 7, 9; 0.3,0.4,0.5

Construction of decision matrix $D(t_3)$ for 2023 year represented in table 3.

Table 3: *Decision matrix* $D(t_3)$

Alternatives	Criteria C_i			
	C_1	C_2	C_3	C_4
A	1, 3, 5; 0.5,0.6,0.7	3, 5, 7; 0.5,0.6,0.7	1, 3, 5; 0.3,0.4,0.5	1, 3, 5; 0.5,0.6,0.7
B	3, 5, 7; 0.5,0.6,0.7	1, 3, 5; 0.5,0.6,0.7	3, 5, 7; 0.4,0.5,0.6	3, 5, 7; 0.4,0.5,0.6
C	5, 7, 9; 0.3,0.4,0.5	3, 5, 7; 0.5,0.6,0.7	5, 7, 9; 0.4,0.5,0.6	3, 5, 7; 0.4,0.5,0.6

Step 2. Construction comparative decision matrix of criteria for determining weight of each criterion as shown in table 4.

Table 4: *Comparative decision matrix of criteria*

Criteria C_i	Criteria C_i			
	C_1	C_2	C_3	C_4
C_1	1, 1, 1; 0.6, 0.7,0.8	0.2,0.25,0.33; 0.5, 0.6, 0.7	1, 2, 3; 0.4, 0.5, 0.6	3, 4, 5; 0.5, 0.6, 0.7
C_2	3, 4, 5; 0.5,0.6,0.7	1, 1, 1; 0.6, 0.7, 0.8	4.5, 5, 5.5; 0.6, 0.7, 0.8	3, 4, 5; 0.5, 0.6, 0.7
C_3	0.33, 0.5,1; 0.4, 0.5,0.6	0.18,0.2,0.23; 0.6, 0.7, 0.8	1, 1, 1; 0.6, 0.7, 0.8	0.2,0.25,0.3; 0.5, 0.6, 0.7
C_4	0.2,0.25,0; 0.5, 0.6,0.7	0.2,0.25,0.33; 0.5, 0.6, 0.7	3, 4, 5; 0.5, 0.6, 0.7	1, 1, 1; 0.6, 0.7, 0.8

When considering eigenvalues and eigenvectors in the context of Z-numbers, we deal with matrices whose entries are Z-numbers rather than real or complex numbers. The concept extends traditional linear algebra into the realm of uncertainty and fuzzy logic. Finding eigenvalues of Z-matrices involves solving the characteristic equation in the context of Z-numbers. This means extending the concept of determinants and characteristic polynomials to Z-numbers. The computational approach often involves the use of algorithms designed for fuzzy systems, where operations on Z-numbers are defined and used to compute the eigenvalues and eigenvectors. Once the eigenvalues are found, the corresponding eigenvectors are determined by solving the equation $Zv = \lambda v$ for each eigenvalue λ. In this equation, Z is $n \times n$ matrix, λ is a scalar (eigenvalue), and v is a vector (eigenvector).

Using Z-lab program we determine eigenvalues and weights of criteria [27]. The values of eigen vectors are represented below. Weights of criteria determined by defining eigenvalues.
For C_1 - [[0.0447, 0.3966, 0.3967] [0.3923, 0.483, 0.4998]],
For C_2 - [[0.1124, 0.9971, 0.9977] [0.3988, 0.411, 0.4241]],
For C_3 - [[0.0299, 0.2653, 0.2655] [0.442, 0.4709, 0.4877]],
For C_4 - [[0.0266, 0.2357, 0.2362] [0.4348, 0.498, 0.5011]]

Step 3. Calculation the time weight for each year. In dynamic fuzzy multi-attribute decision making, the decision-making process is influenced by the change of time. So, it is important to determine the weight of time. In dynamic fuzzy multi-attribute decision making, calculating the time weight for each year involves determining the importance or influence of each time period within the decision-making horizon. The time weight reflects how much each year's data impacts the final decision. Time weight is determined by using function (8) where P is size of periods, k is observation of period, and α is argument collected from different P periods ($0 < \alpha < 1$). $\omega(t_k)$ show the importance degrees of the arguments of the different periods and can reflect the change of the importance degrees of the periods. When size of periods (P) is 3 and $\alpha = 0.5$ time weights for different years will be calculated as below.

$$\omega(t_k) = \frac{e^{\frac{\alpha k}{P}}\left(1 - e^{\frac{\alpha}{P}}\right)}{e^{\alpha} - 1} = \frac{e^{\frac{0.5\times 1}{3}}\left(1 - e^{\frac{0.5}{3}}\right)}{e^{0.5} - 1} = 0.28,$$

$$\omega(t_2) = 0.33,\ \omega(t_3) = 0.39$$

Step 4. Determining weighted decision matrix. The weighted decision matrix is a powerful tool for making informed, transparent, and structured decisions. The weighted decision matrix is used to make more objective and transparent decisions, especially when multiple factors need to be considered, and it's difficult to rank options based on intuition alone. This matrix helps compare alternatives more objectively by considering both the performance of each alternative for each criterion and the relative importance of the criteria. By assigning weights and scoring alternatives, decision-makers can more easily evaluate multiple options based on the criteria that matter most. Each criterion is assigned a weight to reflect its relative importance, and each option is scored against these criteria. The option with the highest total score is typically the best choice. A weighted decision matrix is determined by multiplying matrix $D(t_1)$, $D(t_2)$, $D(t_3)$ into weights of criteria and time weights.

Weighted decision matrix of $D(t_1)$ represented in table 5.

Table 5: *Weighted decision matrix of* $D(t_1)$

Alternatives	Criteria C_i			
	C_1	C_2	C_3	C_4
A	0.06, 0.78, 0.9; 0.23, 0.32, 0.38	0.16,1.95, 2.5; 0.23, 0.28,0.32	0.03,0.37, 0.52; 0.17, 0.22, 0.28	0.04, 0.46, 0.6; 0.25, 0.33, 0.38
B	0.04,0.56,0.7; 0.19,0.27,0.33	0.03, 0.84, 1.4; 0.23, 0.28, 0.32	0.03,0.37, 0.52; 0.21, 0.27, 0.32	0.02,0.33, 0.46; 0.21, 0.28, 0.33
C	0.06,0.78,1; 0.23, 0.32, 0.38	0.16, 1.95, 2.51; 0.23, 0.28, 0.32	0.03,0.37, 0.52; 0.25, 0.31, 0.37	0.04, 0.46, 0.6; 0.25, 0.33, 0.38

Weighted decision matrix of $D(t_2)$ represented in table 6.

Table 6: *Weighted decision matrix of* $D(t_2)$

	Criteria C_i			
Alternatives	C_1	C_2	C_3	C_4
A	0.04,0.65,0.92; 0.23,0.32, 0.38	0.11,1.65, 2.30; 0.23, 0.28, 0.32	0.01, 0.26, 0.44; 0.25, 0.31, 0.37	0.04,0.54,0.7; 0.16,0.23,0.28
B	0.07,0.92,1.18; 0.15,0.22, 0.28	0.11, 1.65, 2.3; 0.23, 0.28, 0.32	0.03, 0.44, 0.61; 0.25, 0.31, 0.37	0.03,0.39,0.55; 0.25,0.33,0.38
C	0.04,0.65,0.92; 0.23,0.32, 0.38	0.19, 2.3, 2.4; 0.16, 0.2, 0.24	0.03, 0.44, 0.61; 0.21, 0.27, 0.32	0.04,0.54,0.7; 0.16,0.23, 0.28

Weighted decision matrix of $D(t_3)$ represented in table 7.

Table 7: *Weighted decision matrix of* $D(t_3)$

	Criteria C_i			
Alternatives	C_1	C_2	C_3	C_4
A	0.02,0.46,0.77; 0.23,0.32,0.38	0.13,1.94,2.72; 0.23,0.28, 0.32	0.01,0.31,0.52; 0.17, 0.22, 0.28	0.01,0.28, 0.46; 0.25, 0.33, 0.38
B	0.05,0.77,1.08; 0.23,0.32, 0.38	0.04, 1.17, 1.95; 0.23, 0.28, 0.32	0.03,0.52, 0.72; 0.21, 0.27, 0.32	0.03,0.46, 0.64; 0.21, 0.28, 0.33
C	0.09,1.08,1.39; 0.15,0.22, 0.28	0.13,1.94, 2.72; 0.23, 0.28, 0.32	0.06,0.72, 0.93; 0.21, 0.27, 0.32	0.03,0.46, 0.64; 0.21, 0.28, 0.33

Step 5. Aggregating decision matrices from different years is a way to combine decision-making data across multiple time periods to make more informed, long-term decisions. Aggregation of decision matrixes of different years represented in table 8.

Table 8: *Aggregated decision matrix*

	Criteria C_i			
Alternatives	C_1	C_2	C_3	C_4
A	0.12,1.89,2.69; 0.01, 0.05, 0.05	0.4, 5.54, 7.53; 0.01, 0.03,0.04	0.05,0.94,1.48; 0.01, 0.02, 0.09	0.09, 1.28, 1.76; 0.01, 0.03, 0.06
B	0.16,2.25, 3.04; 0.11, 0.15, 0.16	0.18,3.66,5.65; 0.01, 0.03,0.04	0.09,1.33,1.85; 0.02,0.05, 0.07	0.08, 1.18, 1.65; 0.06, 0.08, 0.1
C	0.19,2.51, 3.31; 0.01, 0.02, 0.05	0.48, 6.19, 7.63; 0.1, 0.2, 0.3	0.12, 1.53,2.06; 0.07, 0.09, 0.13	0.11, 1.46, 1.94; 0.01, 0.02, 0.05

Step 6. Normalizing aggregated decision matrix. Normalizing an aggregated decision matrix is an important step in multi-criteria decision-making processes. Normalization ensures that all criteria are comparable, especially when different criteria are measured on different scales (e.g., cost in dollars, quality in ratings, time in hours). This process converts the values of each criterion to a common scale, typically between 0 and 1, so that each criterion contributes proportionally to the final decision. It eliminates issues caused by different scales of measurement for different criteria, ensures each criterion contributes proportionally to the decision, preventing any single criterion from dominating because of its scale.

Normalized matrix represented as below.

Z_E_1_1- [[0.5696, 0.78, 0.9982], [0.01, 0.05, 0.07]]
Z_E_1_2- [[0.5422, 0.75, 0.9708], [0.01, 0.03, 0.04]]
Z_E_1_3- [[0.0, 0.0, 0.0, 0.0], [0.48, 0.48, 0.48, 0.49]]
Z_E_1_4- [[0.5643, 0.78, 0.9929], [0.01, 0.03, 0.06]]

Z_E_2_1- [[0.59, 0.92, 1.0], [0.45, 0.48, 0.49]]
Z_E_2_2- [[0.01, 0.21, 0.4274], [0.01, 0.03, 0.04]]
Z_E_2_3- [[0.01, 0.012, 0.013], [0.48, 0.48, 0.49]]
Z_E_2_4- [[0.59, 0.84, 1.0, 1.0], [0.45, 0.48, 0.48, 0.49]]

Z_E_3_1- [[0.5488, 0.92, 0.9774], [0.01, 0.02, 0.05]]
Z_E_3_2- [[0.532, 0.6392, 0.8534, 0.9606], [0.01, 0.02, 0.02, 0.03]]
Z_E_3_3- [[0.5372, 0.75, 0.9657], [0.07, 0.09, 0.13]]
Z_E_3_4- [[0.55, 0.76, 0.9827], [0.01, 0.02, 0.05]]

Table 9: *Normalizing aggregated decision matrix*

	Criteria C_i			
Alternatives	C_1	C_2	C_3	C_4
A	0.57,0.78,0.99; 0.01,0.05,0.07	0.54,0.5, 0.97; 0.01, 0.03, 0.04	0.01,0.012,0.013; 0.48, 0.481, 0.49	0.56, 0.78, 0.99; 0.01, 0.03, 0.06
B	0.59, 0.92, 1.0; 0.45, 0.48, 0.49	0.01,0.21,0.43; 0.01, 0.03, 0.04	0.01,0.012,0.013; 0.48, 0.481, 0.49	0.59, 0.92, 1.0; 0.45, 0.48, 0.49
C	0.55, 0.76, 0.98; 0.01, 0.02, 0.05	0.53,0.75,0.96; 0.01, 0.02, 0.03	0.54, 0.64, 0.96; 0.07, 0.09, 0.13	0.55, 0.76, 0.98; 0.01, 0.02, 0.05

Step 7. Using the SAW method, we get values for alternatives that are represented below.

A = [[1.67, 2.32, 2.96], [0.001, 0.01, 0.02]]

B = [[1.18, 2.05, 2.43], [0.001, 0.0011, 0.0012]]

C = [[2.17, 3.03, 3.89], [0.001, 0.0011, 0.0012]]

Step 8. Determining Hellinger distance for each alternative.

Hellinger distance between Alternative A and Z (1) is 2.1301350044600897

Hellinger distance between Alternative B and Z (1) is 2.015349684037576

Hellinger distance between Alternative C and Z (1) is 2.2936502538885915

$A = 2.13$, $B = 2.01$, $C = 2.29$

Comparison of alternatives represent that alternative B is best alternative.

IV. Conclusion

The strategic vendor selection problem refers to the process of choosing suppliers or vendors that align with a company's long-term strategic goals, rather than focusing solely on short-term needs like price or availability. Vendor selection is a critical decision that affects various aspects of an organization, including cost efficiency, product quality, innovation capacity, supply chain stability, and overall competitiveness. It requires a comprehensive and adaptive approach to handle the complexity and dynamics of modern supply chains. By integrating dynamic MCDM with Z-numbers, organizations can better manage uncertainty, incorporate real-time data, and continuously refine their vendor selection process. This approach ensures that the chosen vendors align with the organization's strategic goals and can adapt to changing conditions over time. In most of the methods developed to solve MCDM problems, the evaluation of alternatives is done over a certain period. However, in some MCDM problems, considering only the current performance of the alternatives may cause errors. For this reason, dynamic MCDM model with uncertainty factor was developed in the study, which allows the evaluation of alternatives in different time periods. Z-numbers theory was used to eliminate and express the uncertainty in the MCDM model and, it provides a closer evaluation to the human thinking structure. SAW methodology was used to select the most appropriate alternative. The proposed dynamic MCDM under model Z-information was applied as an example. The example showed that this model can be easily applied to MCDM problems and produce effective results. In today's competitive environment, selecting the right vendor is critical for the success of production systems. Dynamic fuzzy multi-attribute decision making utilization Z-numbers offers a sophisticated approach to handle the inherent uncertainties and complexities involved in this process. Dynamic fuzzy multi-attribute decision making approach is used to estimate the vendors A, B, C in the years 2021, 2022, and 2023 according to the criteria C_1, C_2, C_3 and C_4 with Z numbers. C_1 is technical capability, C_2 is quality, and C_3 is customer support and C_4 is reliability. Using Z-numbers in dynamic fuzzy multi-attribute decision making enhances the vendor selection process by effectively managing uncertainty and ensuring that decisions are based on reliable information. Comparison of alternatives represent that alternative B is best alternative for vendor selection.

References

[1] Aruldoss, M., Lakshmi, M.T., and Venkatesan, V.P. (2013). A survey on multi criteria decision making methods and its applications, *American Journal of Information Systems,* 1: 31-43.

[2] Velasquez, M., and Hester, P.T. (2013). An analysis of multi-criteria decision-making methods, *International Journal of Operations Research*, 10(2): 56-66.

[3] Ceballos, B., Lamata, M.T., and Pelta, D.A. (2016). A comparative analysis of multi-criteria decision-making methods, *Progress in Artificial Intelligence,* 5 (4): 315–322.

[4] Saaty, T.L. Decision Making with Dependence and Feedback, *The Analytic Network Process, RWS Publications,* 1996.

[5] Chen, Y., and Li, B. (2011). Dynamic Multi-Attribute Decision Making Model Based On Triangular İntuitionistic Fuzzy Numbers, *Scientia Iranica B,* 18 (2): 268–274.

[6] Xu, Z., and Yager, R.R. (2008). Dynamic Intuitionistic Fuzzy Multi-Attribute Decision Making, *International Journal of Approximate Reasoning*, 48 (1): 246–262.

[7] Su, Z., Chen, M., Xia, G., and Wang, L. (2011). An Interactive Method for Dynamic Intuitionistic Fuzzy Multi-Attribute Group Decision Making, *Expert Systems with Applications*, 38 (12): 15286–15295.

[8] Fei L., and Feng Y. (2021). A dynamic framework of multi-attribute decision making under Pythagorean fuzzy environment by using Dempster-Shafer theory, *Engineering Applications of Artificial Intelligence*, 101, 104213.

[9] Gao Y., and Li D. (2019). A consensus model for heterogeneous multi-attribute group decision making with several attribute sets, *Expert Systems with Applications*, 125, (1): 69-80.

[10] González-Prida, V., Barberá, L., Viveros, P., and Crespo A. (2012). Dynamic Analytic Hierarchy Process: AHP method adapted to a changing environment, *IFAC Proceedings Volumes*, 45 (31): 25-29.

[11] Yang, P., Liu, X., and Xu, G. (2018). A dynamic weighted TOPSIS method for identifying influential nodes in complex networks, *Modern Physics Letters B*, 32 (19).

[12] Wang, J., Nie, R., Zhang, H., Chen, X. (2013). Intuitionistic Fuzzy Multi-Criteria Decision-Making Method Based on Evidential Reasoning, *Applied Soft Computing*, 13 (4): 1823-1831.

[13] Yang, Y., and Han, D. (2016). A new distance-based total uncertainty measure in the theory of belief functions, *Knowledge Based Systems*, 94: 114-123.

[14] Pang, B., and Bai, S. (2013). An integrated fuzzy synthetic evaluation approach for supplier selection based on analytic network process, *Journal of Intelligent Manufacturing*, 24 (1): 163–174.

[15] Chen, C.-T., Lin, C.-T., and Huang, S.-F. (2006). A fuzzy approach for supplier evaluation and selection in supply chain management, *International Journal of Production Economics*, 102: 289–301.

[16] Hwang, C.L., and Yoon, K. Multiple attributes decision making methods and applications, *Springer-Verlag,* 1981.

[17] Kahraman, C., Cebeci, and U., Ulukan, Z. (2003). Multi-criteria supplier selection using Fuzzy AHP, *Logistics Information Management*, 16 (6):382–394.

[18] Aliyeva, K.R. (2019). Eigen solution of 2 by 2 Z-matrix, *International Conference on Theory and Application of Soft Computing, Computing with Words and Perceptions*: 758-762.

[19] Zadeh, L.A. (2011). A Note on a Z-Number, *Journal of Information Sciences*, 181: 2923-2932.

[20] Aliev, R.A., Alizadeh A.V., and Huseynov O.H. (2015). The arithmetic of discrete Z-numbers, *Information Sciences*, 290 (1): 134-155.

[21] Aliev R.A., Guirimov B. G., Huseynov O. H., and Aliyev, R. R. (2021). "A consistency-driven approach to construction of Z-number-valued pairwise comparison matrices", *Journal of Fuzzy Systems,* 18 (4), pp.37-49.

[22] Xu Z. S. (2009). A method based on the dynamic weighted geometric aggregation operator for dynamic hybrid multi-attribute group decision making, *International Journal of Uncertainty Fuzziness and Knowledge-Based Systems*, 17: 15-33.

[23] Aliev R., Huseynov O., and Aliyeva K. (2016). Z-valued t-norm and t-conorm operators-based aggregation of partially reliable information, *Procedia Computer Science*, 102: 12-17.

[24] Aliyeva, K., Aliyeva, A., Aliyev, R., and Özdeşer M. (2023). Application of Fuzzy Simple Additive Weighting Method in Group Decision-Making for Capital Investment, *Axioms,* 12 (8), 797.

[25] Aliev, R.A., Aliev, R.R., Guirimov B., and Uyar K. (2007). Dynamic data mining technique for rules extraction in a process of battery charging, *Applied Soft Computing*, 8 (3).

[26] Weber, C.A., Current, J.R., and Benton, W.C. (1991). Vendor selection criteria and methods, *European Journal of Operational Research*, 50: 2-18.

[27] Aliyeva, K.R. (2019). Toward eigenvalues and eigenvectors of matrices of Z-numbers, *International Conference on Theory and Application of Soft Computing, Computing with Words and Perceptions*: 309-317.