A WEIGHTED THREE-PARAMETER XGAMMA DISTRIBUTION WITH PROPRTIES AND ITS APPLICATION TO REAL LIFE DATA

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Abstract

In this article, a weighted three-parameter Xgamma distribution has been proposed. It is an extension of two-parameter Xgamma distribution. The weighted three-parameter Xgamma distribution designed for modelling real-life data. The density function and cumulative distribution function, moments, hazard and survival function, moment-generating function and characteristic function, Bonferroni and Lorenz curve, renyi entropy of this distribution have all been derived. The parameter of this distribution is estimated by maximum likelihood estimation method. Finally, an application of the model to a real-life data set is presented and compared with some other existing distributions.

Keywords: weighted three-parameter Xgamma distribution, Reliability analysis, moments, maximum likelihood estimation, order statistics, renyi entropy.

I. Introduction

Choosing a sampling unit with equal probability for observational research on the population like fish, insects, plants and wild- animals is impossible. Due to this, the recorded observations are skewed, and the sampling frame is not well defined. These observations deviate from the original distribution, and as a result, their modeling gives rise to the weighted distribution theory. These studies pertain to survival analysis, family data analysis, and reliability. The weighted distribution theory is often used to handle model specification and data interpretation. Fisher developed the idea of weighted distribution first in 1934, and Rao formalized and improved it in 1965. Zelen introduced the weighted distribution of a length-biased sample in 1974, while Patil and Ord examined a size-biased sampling and related topics in 1976. In a series of articles with other co-authors, Patil has extensively pursued weighted distribution for the purpose of the encountering data analysis.

In this article, we introduce a new weighted distribution (see more reference [9][11][15][16]) known as weighted three-parameter Xgamma distribution, is the extension of the two parameter Xgamma distribution (see reference [14]). The alternative form of weighted three parameter Xgamma distribution is introduced in Section 3. The survival properties are studied in Section 4.

Moments, moment generating-function and characteristic function are described in Section 5. In Section 6, order statistics of the distribution are derived. The likelihood ratio test is discussed in Section 7. Bonferroni and Lorenz curve and Renyi entropy are discussed in Section 8 and 9, respectively. Methods of estimating parameters are discussed in section 10. In section11, two real data sets are analyzed to show of the application of weighted three-parameter Xgamma distribution; finally, Section 12 concludes.

II.Two-parameter Xgamma distribution

Two-parameter Xgamma distribution is the generalization of Xgamma distribution [14] by adding additional parameter α.
The probability density function (PDF) of a two-parameter Xgamma distribution is given by

$$F(x;\alpha,\theta)=\left(\frac{\theta^2}{\alpha+\theta}\right)\left(1+\left(\frac{\alpha\theta}{2}\right)x^2\right)e^{-(\theta x)} \qquad x,\alpha,\theta>0 \tag{1}$$

The cumulative distribution function (CDF) is given by

$$F(x)=1-\frac{(\alpha+\theta+\alpha\theta x+\frac{1}{2}\alpha\theta^2x^2)}{(\alpha+\theta)}e^{-\theta x} \qquad x,\alpha,\theta > 0 \tag{2}$$

III. Weighted three-parameter Xgamma distribution

The concept of a weighted distribution, we have a definition, see Patil et al. (1988) as

$$f_w(X)=\frac{w(x)f(x)}{E\big(w(x)\big)} \qquad ;\ x>0 \tag{3}$$

Where W(x) is the non-negative weight function and f_w(X) is pdf. We have different choices of weight function based on the different weight model (see the references [1], [2], [5], [6], [7], [13]). In the paper, use w(x) = x^c as a weight function. Then the probability function of weighted distribution is

$$fw(\mathrm{X})=\frac{(\mathrm{x^c})\mathrm{f(x)}}{\mathrm{E(x^c)}} \qquad ;\ c>0 \tag{4}$$

Where

$$E(x^c)=\int_0^\infty (x^c)f(x;\alpha,\theta)dx$$

After simplification, we get

$$E(x^c)=\frac{c!}{2\theta^C(\alpha+\theta)})[2\theta+\alpha(1+c)(2+c)] \tag{5}$$

Substituting (1) and (5) in (4), we get

$$f\,w(x;\alpha,\theta,c)=\frac{x^{c}\left(\frac{\theta^{2}}{\alpha+\theta}\right)\left(1+\left(\frac{\alpha\theta}{2}\right)x^{2}\right)e^{-(\theta x)}}{\left(\frac{c!}{2\theta^{C}(\alpha+\theta)}\right)[2\theta+\alpha(1+c)(2+c)]}$$

Definition 1: A non-negative continuous random variable X, is said to follow weighted three -parameter Xgamma distribution with parameters α,θ *and* c if its pdf is given by

$$fw(x,\alpha,\theta,c)=\frac{x^{c}\theta^{c+2}\left(1+\left(\frac{\alpha\theta}{2}\right)x^{2}\right)e^{-(\theta x)}}{\frac{c!}{2}\;(2\theta+\alpha(1+c)(2+c))} \tag{6}$$

It is denoted by X~WTPXG (α,θ , c).

Note:

(1). When we put c = 0 in (6), we obtain the two-parameter Xgamma distribution with parameter α *and* θ [14].

(2). Weighted three-parameter Xgamma distribution is positive skewed distribution.

Figure 1 show the density function for weighted three-parameter Xgamma distribution for different value of α,θ *and* c.

Then the cumulative distribution function of weighted three-parameter Xgamma distribution is given by

$$Fw(x,\alpha,\theta,c)=\frac{\theta\gamma(c+1,\theta x)+\frac{\alpha}{2}\gamma(c+3,\theta x)}{\frac{c!}{2}[2\theta+\alpha(c+1)(c+2)]} \tag{7}$$

Where, $\gamma(c+1,\theta x)$ and $\gamma(c+4,\theta x)$ is a lower incomplete gamma function [4].

Figure 2 show the distribution function for weighted three-parameter Xgamma distribution for different value of α,θ *and* c.

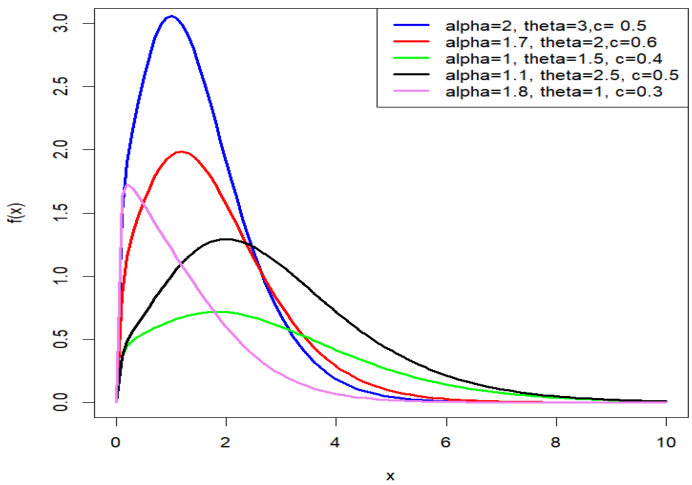

Figure 1: *probability density function of Weighted three-parameter Xgamma distribution*

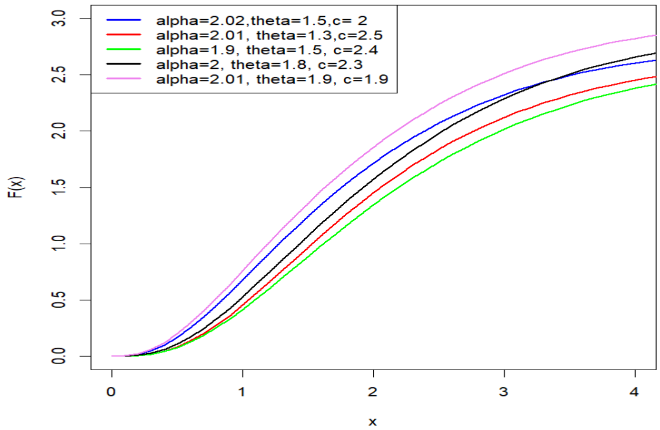

Figure 2: *cumulative distribution function of Weighted three-parameter Xgamma distribution*

IV. Reliability analysis

In this section, we study survival function, hazard function, reverse hazard function, mills ratio, odds rate function and cumulative hazard function for the weighted three-parameter Xgamma distribution with parameter α, θ and c given in (6).

I.Survival function:

The survival function is also known as the reliability function. The survival function of the weighted three-parameter Xgamma distribution is given by

$$s(x, \alpha, \theta) = 1 - F(x, \alpha, \theta, c) \tag{8}$$

Substituting equation (7) in (8)

$$s(x, \alpha, \theta, c) = 1 - \frac{\theta\gamma(c+1, \theta x) + \frac{\alpha}{2}\ \gamma(c+3, \theta x)\,]}{\frac{c!}{2}\ \big(2\theta + \alpha(1+c)(2+c)\big)} \tag{9}$$

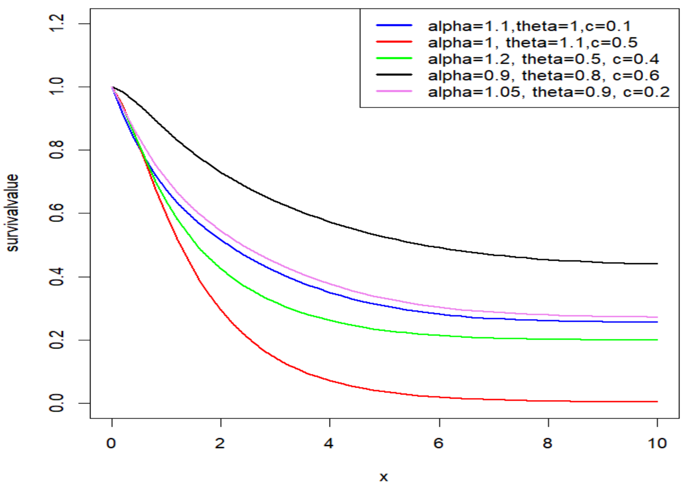

Figure 3: *survival function of Weighted three-parameter Xgamma distribution*

II. Hazard function

The hazard function of a weighted three-parameter Xgamma distribution is given

$$h\,(x,\alpha,\theta,c) \;=\frac{fw(x,\alpha,\theta,c)}{s(x,\alpha,\theta,c)}$$

$$h(x,\alpha,\theta,c)=\frac{\dfrac{x^c\theta^{c+2}\left(1+\left(\frac{\alpha\theta}{2}\right)x^2\right)e^{-(\theta x)}}{\frac{c!}{2}\;\big(2\theta+\alpha(1+c)(2+c)\big)}}{1-\dfrac{\theta\,\gamma(c+1,\theta x\,)+\frac{\alpha}{2}\;\gamma(c+3,\theta x\,)\,]}{\frac{c!}{2}\;\big(2\theta+\alpha(1+c)(2+c)\big)}}$$

After simplification, we get

$$h(x,\alpha,\theta,c)=\frac{x^c\theta^{c+2}\left(1+\left(\frac{\alpha\theta}{2}\right)x^2\right)e^{-(\theta x)}}{\frac{c!}{2}\;\big(2\theta+\alpha(1+c)(2+c)\big)-\left[\theta\,\gamma\,(c+1,\theta x\,)+\frac{\alpha}{2}\;\gamma(c+3,\theta x\,)\right]}$$

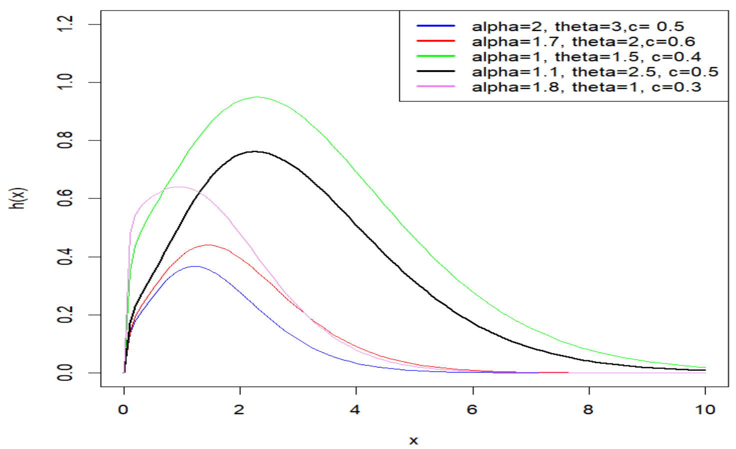

Figure 4*: hazard plot of Weighted three-parameter Xgamma distribution*

III. Reverse Hazard function

The reverse hazard function of a weighted three-parameter Xgamma distribution is given

$$hr(x)=\frac{fw(x;\alpha,\theta,c)}{Fw(x;\alpha,\theta,c)}$$

$$hr(x)=\frac{\dfrac{x^c\theta^{c+2}\left(1+\left(\frac{\alpha\theta}{2}\right)x^2\right)e^{-(\theta x)}}{\frac{c!}{2}\;\big(2\theta+\alpha(1+c)(2+c)\big)}}{\dfrac{[\theta\,\gamma(c+1,\theta x)+\frac{\alpha}{2}\,\gamma(c+3,\theta x)}{\frac{c!}{2}[2\theta+\alpha(c+1)(c+2)]}}$$

After simplification ,we get

$$hr(x) = \frac{x^c \theta^{c+2}\left(1+\left(\frac{\alpha\theta}{2}\right)x^2\right)e^{-(\theta x)}}{\theta\ \gamma(c+1,\theta x)+\frac{\alpha}{2}\gamma(c+3,\theta x)}$$

IV. Mills Ratio

The mills ratio of a weighted three-parameter Xgamma distribution is given by

$$\mathrm{m(x)} = \frac{1}{h(x,\alpha,\theta,c)}$$

$$m(x) = \frac{1}{\dfrac{x^c \theta^{c+2}\left(1+\left(\frac{\alpha\theta}{2}\right)x^2\right)e^{-(\theta x)}}{\frac{c!}{2}\ (2\theta+\alpha(1+c)(2+c)) - \ [\theta\ \gamma\,(c+1,\theta x\,)+\frac{\alpha}{2}\ \gamma(c+3,\theta x\,)\,]}}$$

V. Odds rate function

The odds rate function of a weighted three-parameter Xgamma distribution is given

$$O(x) = \frac{Fw(x;\alpha,\theta,c)}{1-Fw(x;\alpha,\theta,c)}$$

$$O(x) = \frac{\dfrac{[\theta\ \gamma(c+1,\theta x\,)+\frac{\alpha}{2}\ \gamma(c+3,\theta x\,)\,]}{\frac{c!}{2}\ \left(2\theta+\alpha(1+c)(2+c)\right)}}{1-\dfrac{[\theta\ \gamma(c+1,\theta x\,)+\frac{\alpha}{2}\ \gamma(c+3,\theta x\,)\,]}{\frac{c!}{2}\ \left(2\theta+\alpha(1+c)(2+c)\right)}}$$

After simplification, we get

$$O(x) = \frac{\theta\ \gamma(c+1,\theta x\,)+\frac{\alpha}{2}\ \gamma(c+3,\theta x)]}{\frac{c!}{2}\ \left(2\theta+\alpha(1+c)(2+c)\right)-[\theta\ \gamma(c+1,\theta x\,)+\frac{\alpha}{2}\ \gamma(c+3,\theta x\,)]}$$

VI. Cumulative hazard function

The cumulative hazard function of a weighted three-parameter Xgamma distribution is given

$$H(x) = -\ln\left(1-\frac{[\theta\ \gamma(c+1,\theta x\,)+\frac{\alpha}{2}\ \gamma(c+3,\theta x\,)\,]}{\frac{c!}{2}\ \left(2\theta+\alpha(1+c)(2+c)\right)}\right)$$

V. Statistical Properties

In this section, discuss the statistical properties of the weighted three-parameter Xgamma distribution, that is moment, moment-generating function, and characteristic function.

I. Moments

If X is a random variable that follows a weighted three-parameter Xgamma distribution with pdf (6), then the r[th] order moment, that is μ_r' can be obtained as

$$\mu_r' = E(X^r) = \int_0^\infty x^r\, f_w(x) d(x)$$

$$\mu_r' = E(X^r) = \int_0^\infty x^r \frac{x^c \theta^{c+2}\left(1+\left(\frac{\alpha\theta}{2}\right)x^2\right)e^{-(\theta x)}}{\frac{C!}{2}\ \left(2\theta+\alpha(1+c)(2+c)\right)} dx$$

After simplification we get r[th] order raw moment is

$$\mu_r' = \frac{\theta^{c+2}[2\Gamma(c+r+1)+\alpha\Gamma(c+1)]}{c!\left(2\theta+\alpha(1+c)(2+c)\right)\theta^{\,r+c+2}} \tag{10}$$

Putting r=1, 2 ,3, 4 in (10) we get 1[st], 2[nd], 3[rd] , 4[th] row moments of weighted three-parameter Xgamma distribution

$$\mu_1' = \frac{2(c+1)+\alpha(c+2)(c+1)}{2\theta+\alpha(c+1)(c+2)} \tag{11}$$

$$\mu_2' = \frac{2(c+1)\ (c+2)+\alpha(c+3)(c+2)(c+1)}{[2\theta+\alpha(c+1)(c+2)]\theta} \tag{12}$$

$$\mu_3' = \frac{2(c+1)\ (c+2)(c+3)+\alpha(c+3)(c+2)(c+1)(c+4)}{[2\theta+\alpha(c+1)(c+2)]\theta^{\,2}} \tag{13}$$

$$\mu_4' = \frac{2(c+1)\ (c+2)(c+3)((c+4)+\alpha(c+3)(c+2)(c+1)(c+4)(c+5)}{[2\theta+\alpha(c+1)(c+2)]\theta^{\,3}} \tag{14}$$

Note:

1.mean of the weighted three-parameter Xgamma distribution is given by

$$\text{Mean} = \frac{2(c+1)+\alpha(c+2)(c+1)}{2\theta+\alpha(c+1)(c+2)}$$

2. variance of the weighted three-parameter Xgamma distribution is given by

$$Variance = \frac{2(c+1)\ (c+2)+\alpha(c+3)(c+2)(c+1)}{[2\theta+\alpha(c+1)(c+2)]\theta} - \left(\frac{2(c+1)+\alpha(c+2)(c+1)}{2\theta+\alpha(c+1)(c+2)}\right)^2$$

$$= \frac{2(c+1)+\alpha(c+1)(c+2)}{\theta\left(2\theta+\alpha(c+1)(c+2)\right)^2}(\alpha(1-\theta)(c+1)(c+2)-2\theta c)$$

3. Standard deviation

$$\sigma = \sqrt{\left(\frac{2(c+1)+\alpha(c+1)(c+2)}{\theta\left(2\theta+\alpha(c+1)(c+2)\right)^2}(\alpha(1-\theta)(c+1)(c+2)-2\theta c)\right.}$$

4. Coefficient of variation (CV)

$$C\,V = \frac{\sqrt{2(c+1)+\alpha(c+2)(c+1)}}{\sqrt{\theta}\,(2(c+1)+\alpha(c+2)(c+1)}100$$

II. Moment generating function and Characteristic function

If a random variable X follows a weighted three-parameter Xgamma distribution, then the moment-generating function (MGF) is given as follows,

$$M_x(t) = E(e^{tx}) \tag{15}$$

$$M_x(t) = \int_0^\infty e^{tx} f(x)dx$$

$$M_x(t) = \int_0^\infty e^{tx} \frac{x^c\theta^{c+2}\left(1+\left(\frac{\alpha\theta}{2}\right)x^2\right)e^{-(\theta x)}}{\frac{c!}{2}\ \left(2\theta+\alpha(1+c)(2+c)\right)}dx$$

$$M_x(t)\ = \frac{\theta^{c+2}}{\frac{c!}{2}\left(2\theta+\alpha(c+1)(c+2)\right)}\int_0^\infty e^{-x(\theta-t)}\,x^c\left(1+\left(\frac{\alpha\theta}{2}\right)x^2\right)dx$$

After simplification, we get the moment-generating function is

$$M_x(t) = \frac{\theta^{c+2}(2(\theta-t)^2+\alpha\theta(c+1)(c+2)}{(\theta-t)^{c+3}(2\theta+\alpha(c+1)(c+2)} \tag{16}$$

Similarly, the characteristic function is obtained as

$$\phi_x(t) = \frac{\theta^{c+2}(2(\theta-it)^2+\alpha\theta(c+1)(c+2)}{(\theta-it)^{c+3}(2\theta+\alpha(c+1)(c+2)} \tag{17}$$

VI. Order Statistics

In this section, we derive the order statistics from weighted three-parameter Xgamma distribution. Let $x_{(1)}$, $x_{(2)}$, $x_{(3)}$,…. $x_{(n)}$ be the order statistics of the random variable x_1, x_2,…… x_n taken from weighted three parameter Xgamma distribution. Then the probability density function of the m th order statistics $X_{(m)}$ is defined as

$$fw(m)(x) = \frac{n!}{(m-1)!\,(n-m)!}f(x)[F\,(x)]m-1\ [1-F(x)]n-m \tag{18}$$

Inserting equation (6) & (7) in (18)

$$\mathrm{fw(m)(x)}\quad = \frac{\mathrm{n!}}{(\mathrm{m}-1)!\,(\mathrm{n}-\mathrm{m})!}\frac{\mathrm{x^c}\theta^{\mathrm{c}+2}\left(1+\left(\frac{\alpha\theta}{2}\right)\mathrm{x}^2\right)\mathrm{e}^{-(\theta\mathrm{x})}}{\frac{\mathrm{c!}}{2}\ \left(2\theta+\alpha(1+\mathrm{c})(2+\mathrm{c})\right)}\left(\frac{\theta\gamma(\mathrm{c}+1,\theta\mathrm{x})+\frac{\alpha}{2}\,\gamma(\mathrm{c}+3,\theta\mathrm{x})}{\frac{\mathrm{c!}}{2}[2\theta+\alpha(\mathrm{c}+1)(\mathrm{c}+2)]}\right)^{\mathrm{m}-1}$$

$$\left(\frac{\theta\gamma(c+1,\theta x)+\frac{\alpha}{2}\gamma(c+3,\theta x)}{\frac{c!}{2}[2\theta+\alpha(c+1)(c+2)]}\right)^{n-m} \tag{19}$$

Put m=1 in equation (19), we get first order statistics $X_{(1)}$=mini ($x_1, x_2, \ldots\ldots x_n$) and m=n, we get n^{th} order statistics $X_{(n)}$=max ($x_1, x_2, \ldots\ldots x_n$)

VII. Likelihood Ratio Test

The likelihood-ratio test is a hypothesis test that compares two competing statistical models' goodness of fit. Assume that $x_1, x_2, \ldots\ldots\ldots x_n$ be n random sample taken from a weighted three-parameter Xgamma distribution to test the hypothesis.

$$H_0: f(x) = f(x,\alpha,\theta,c)$$
against
$$H_1: f(x) = f_w(x,\alpha,\theta,c)$$

$$\Delta = \frac{L_1}{L_0}\prod_{i=1}^{n}\frac{f_w\ (xi)}{f(xi)}$$

$$\Delta = \prod_{i=1}^{n}\frac{\frac{x^c{}_i\theta^{c+2}\left(1+\left(\frac{\alpha\theta}{2}\right)x^c{}_i\right)e^{-(\theta x_i)}}{\frac{C!}{2}\ (2\theta+\alpha(1+c)(2+c))}}{\left(\frac{\theta^2}{\alpha+\theta}\right)\left(1+\left(\frac{\alpha\theta}{2}\right)x^c{}_i\right)e^{-(\theta x_i)}}$$

$$\Delta = \prod_{i=1}^{n}\frac{x^c{}_i\theta^c(\alpha+\theta)}{\frac{C!}{2}\ \big(2\theta+\alpha(1+c)(2+c)\big)}$$

The null hypothesis is rejected if

$$\Delta = \frac{\theta^c(\alpha+\theta)}{\frac{C!}{2}\ \big(2\theta+\alpha(1+c)(2+c)\big)}\prod_{i=1}^{n}x^c{}_i \quad > \quad \mathrm{K}$$

$$\Delta = \prod_{i=1}^{n}x^c{}_i > \frac{k}{\left[\frac{\theta^c(\alpha+\theta)}{\frac{C!}{2}\ \big(2\theta+\alpha(1+c)(2+c)\big)}\right]^n}$$

$$\Delta^* = \prod_{i=1}^{n}x^c{}_i > K^*$$

Where

$$K^* = \frac{k}{\left[\frac{\theta^c(\alpha+\theta)}{\frac{C!}{2}\ \big(2\theta+\alpha(1+c)(2+c)\big)}\right]^n}$$

A Chi-square variate with one degree of freedom is distributed as -2log Δ for large sample size (n). The null hypothesis is rejected when the probability value is provided by

$$p(\Delta^*) > \alpha^*$$

Where $\alpha^* = \prod_{i=1}^{n} x_i{}^c$ is less than level of significance. And $\prod_{i=1}^{n} x_i{}^c$ is the observed value of the statistic Δ^*

VIII. Bonferroni and Lorenz Curves

In this section, we derive the Bonferroni and Lorenz curves from the weighted three-parameter Xgamma distribution [2]. The Bonferroni and Lorenz curve is a powerful tool in the analysis of distributions and has applications in many fields, such as economies, insurance, income, reliability, and medicine. The Bonferroni and Lorenz cures is given as

$$B(p) = \frac{1}{p\mu_1{}'}\int_0^q x\, f_a(x)dx$$

And

$$L(p) = \frac{1}{\mu_1{}'}\int_0^q x\, f_a(x)dx$$

Where

$$\mu_1{}' = E(X) = \frac{2(c+1) + \alpha(c+2)(c+1)}{2\theta + \alpha(c+1)(c+2)} \qquad \text{and } q = F^{-1}(p)$$

$$B(p) = \frac{2\theta + \alpha(c+1)(c+2}{p\big(2(c+1) + \alpha(c+2)(c+1)\big)}\int_0^q x\,\frac{x^c\theta^{c+2}\left(1 + \left(\frac{\alpha\theta}{2}\right)x^2\right)e^{-(\theta x)}}{\frac{C!}{2}\ \big(2\theta + \alpha(1+c)(2+c)\big)}\,dx$$

$$B(p) = \frac{2\theta^{c+2}}{C!\,p\big(2(c+1) + \alpha(c+2)(c+1)\big)}\int_0^q x^c x\, e^{-\theta x}\left(1 + \left(\frac{\alpha\theta}{2}\right)x^2\right)dx$$

$$B(p) = \frac{2\theta^{c+2}}{C!\,p\big(2(c+1) + \alpha(c+2)(c+1)\big)}\{\int_0^q x^{c+1}e^{-\theta x}\,dx + \int_0^q \frac{\alpha\theta}{2}\,x^{c+3}e^{-\theta x}dx\,\}$$

After simplification we get

$$B(p) = \frac{2\theta\,\gamma(c+2,\theta q) + \alpha\gamma(c+4,\theta q)}{C!\,p\theta\big(2(c+1) + \alpha(c+2)(c+1)\big)} \tag{20}$$

Where $L(p) = pB(p)$

$$L(p) = \frac{2\theta\,\gamma(c+2,\theta q) + \alpha\gamma(c+4,\theta q)}{C!\,\theta\big(2(c+1) + \alpha(c+2)(c+1)\big)} \tag{21}$$

IX. Renyi Entropy

The Renyi entropy is important in ecology and statistics as an index of diversity. For a given probability distribution (6), Renyi entropy is given by

$$R(\lambda) = \frac{1}{1-\lambda}\log\int_0^\infty (f_w{}^\lambda(x))dx$$

Where $\lambda > 0$ *and* $\lambda \neq 1$

$$R(\beta) = \frac{1}{1-\lambda}\log\int_0^\infty \left(\frac{x^c\theta^{c+2}\left(1 + \left(\frac{\alpha\theta}{2}\right)x^2\right)e^{-(\theta x)}}{\frac{C!}{2}\ \big(2\theta + \alpha(1+c)(2+c)\big)}\right)^\lambda dx$$

$$R(\beta) = \frac{1}{1-\lambda}\log\left(\frac{\theta^{c+2}}{\frac{C!}{2}\ \left(2\theta+\alpha(1+c)(2+c)\right)}\right)^{\lambda}\int_0^{\infty} x^{c\lambda}\left(1+\left(\frac{\alpha\theta}{2}\right)x^2\right)^{\lambda} e^{-\theta\lambda x}dx$$

Using binomial expansion in above equation and can be obtain

$$R(\lambda) = \frac{1}{1-\lambda}log\left(\frac{\theta^{c+2}}{\frac{C!}{2}\ \left(2\theta+\alpha(1+c)(2+c)\right)}\right)^{\lambda}\sum_{i=0}^{\lambda}\binom{\lambda}{i}\left(\frac{\alpha\theta}{2}\right)^{i}\int_0^{\infty} x^{c\lambda+2i}e^{-\theta\lambda x}dx$$

$$R(\lambda) = \frac{1}{1-\lambda}log\left(\frac{\theta^{c+2}}{\frac{C!}{2}\ \left(2\theta+\alpha(1+c)(2+c)\right)}\right)^{\lambda}\sum_{i=0}^{\lambda}\binom{\lambda}{i}\left(\frac{\alpha\theta}{2}\right)^{i}\frac{(c\lambda+2i)!}{(\lambda\theta)^{c\lambda+2i+1}} \tag{22}$$

X. Estimations of Parameter

Let $x_1, x_2 \ldots\ldots x_n$ be the random sample of size n taken from weighted three-parameter Xgamma distribution, then the likelihood function is defined as below,

$$L(x,\alpha,\theta,c) = \prod_{i=1}^{n} f(x,\alpha,\theta,c)$$

$$L(x,\alpha,\theta,c) = \prod_{i=1}^{n}\frac{x^c\theta^{c+2}\left(1+\left(\frac{\alpha\theta}{2}\right)x^2\right)e^{-(\theta x)}}{\frac{C!}{2}\ \left(2\theta+\alpha(1+c)(2+c)\right)}$$

$$L(x,\alpha,\theta,c) = \left(\frac{\theta^{c+2}}{\frac{c!}{2}\left(2\theta+\alpha(c+1)(c+2)\right)}\right)^{n} e^{-\theta\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}x^c\left(1+\frac{\alpha\theta}{2}x_i^2\right)$$

Take logarithm on both sides we get

$$l\,n\,L(x,\alpha,\theta,c) = \ nln\left(\frac{\theta^{c+2}}{\frac{c!}{2}\left(2\theta+\alpha(c+1)(c+2)\right)}\right)-\theta\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}ln(\,(1+\frac{\alpha\theta}{2}x_i^2)x^C)$$

$$\begin{aligned} ln\ \ L(x,\alpha,\theta,c) &= n(c+2)ln\theta - n\ lnc! - nln\theta - n\ ln\frac{c!}{2} - n\ ln\alpha - n\ln(c^2+3c+2) \\ &\quad -\theta\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}c\ lnx_i+\sum_{i=1}^{n}\ln\left(1+\frac{\alpha\theta}{2}x_i^2\right) \end{aligned} \tag{23}$$

After differentiating equation (23) with respect to α, θ and c and equating to zero we get

$$\frac{\partial\ln L(x,\alpha,\theta,c)}{\partial\alpha} = \frac{\theta}{2}\sum_{i=1}^{n}\frac{x_i^2}{\left(1+\frac{\alpha\theta}{2}x_i^2\right)}-\frac{n}{\alpha} = 0 \tag{24}$$

$$\frac{\partial\ln L(x,\alpha,\theta,c)}{\partial\theta} = \frac{\alpha}{2}\sum_{i=1}^{n}\frac{x_i^2}{\left(1+\frac{\alpha\theta}{2}x_i^2\right)}+\frac{n(c+1)}{\theta}-\sum_{i=1}^{n}x_i = 0 \tag{25}$$

$$\frac{\partial \ln L(x,\alpha,\theta,c)}{\partial c} = n\ln\theta - \frac{2n}{c!}\psi(c+1)\Gamma(c+1 - \frac{4n}{c} + \sum_{i=1}^{n} ln\, x_i = 0 \tag{26}$$

Where ψ is the Digamma function [15]. On solving equations (24), (25) and (26) we obtain the maximum likelihood estimator of parameters in weighted three-parameter Xgamma distribution. But we have the above equations in complex form; it can't be solved directly, so we estimate the parameter of the weighted three-parameter Xgamma distribution using R and Wolfram Mathematica**.**

XI. Applications

In this section, we have considered two real data sets for the purpose of showing that the distribution of weighted three-parameter Xgamma distribution shows a better fit over two parameter Xgamma distribution and Xgamma distribution.

The Akaike information criterion (AIC), Bayesian information criterion (BIC), Akaike information criterion corrected (AICC), Hannan-Quinn information criterion (HQIC), Consistent Akaike information criterion (CAIC), and -2logl are using for model selection. It can be evaluated by using the formula as follows;

$AIC = 2k - 2logL$ $\quad$ $AICC = AIC + \frac{2k(k+1)}{(n-k-1)}$ $\quad$ $BIC = klog(n) - 2logL$

$CAIC = \frac{2kn}{n-k-1} - 2logL$ $\quad$ $HQIC = 2klog(\log(n)) - 2logL$

Where n is the sample size, k is the number of parameters, and -2logL is the maximal value of the log likelihood function. After the calculation of AIC, AICC, HQIC, CAIC, and -2logL which the model with the minimum value is chosen as the best model to fit the data.

Data set 1: **blood cancer (leukemia)**

The following real lifetime data set consists of 37 patients suffering from blood cancer (leukemia) reported from one of the ministries of health hospitals in Saudi Arabia (see Abouammah et al. [3]). The ordered lifetimes (in years) are given below in Data Set 1 as:

1.145,1.208,1.263,1.414,2.025,2.036,2.162,2.211,2.37,2.532,2.693,2.805,2.91,2.912,3.192,3.263,3.348, 3.348,3.427,3.499,3.534,3.767,3.751,3.858,3.986,4.049,4.244,4.323,4.381,4.392,4.397,4.647,4.753,4.929, 4.973,5.074,5.381

Table 1: *The summary of blood cancer data set*

Min	1st Qu	Median	Mean	Variance	3rd Qu	Max
1.145	2.532	3.427	3.357	1.359	4.323	5.381

Data set 2: lung cancer

The following real lifetime data set consists of 39 lung cancer patients' survival periods (measured in months) as reported by Pena (2002). The ordered lifetimes (in months) are given below in Data Set 2 as:

2.99,3.06,3.15,3.45,3.71,3.75,3.81,4.11,4.27,4.34,4.40,4.63,4.73,4.93,5.03,5.16,5.17,5.49,5.68,5.72,5.98, 8.15,8.62,8.48,8.61,9.46,9.53,10.05,10.15,10.95,5.85,11.24,11.63,12.26,12.65,12.78,13.18,13.4

Table 2: *The summary of lung cancer data set*

Min	1st Qu	Median	Mean	Variance	3rd Qu	Max
2.99	4.355	5.700	7.120	11.377	9.920	13.400

Table 3*: MLEs AIC, BIC, AICC, CAIC, HQIC and -2log L of the fitted distribution for the given data set 1*

Distribution	WTPXG	TPXG	XG
MLE	$\hat{\theta}$=2.063455 (4.860449 e-05) $\hat{\alpha}$ = 2.171212 e+05 (1.677722 e+04) $\hat{c}$ =3.926777 (1.573179)	$\hat{\theta}$ = 8.935533e-01 (8.482223e-02) $\hat{\alpha}$ =2.478485 e+03 (1.677841e+04)	$\hat{\theta}$ =0.69602623 (0.07361946)
-2logL	119.3462	129.9725	151.4200
AIC	125.3203	133.9725	153.4202
BIC	130.1531	137.1944	155.0311
AICC	125.7648	134.4170	153.8646
HQIC	127.05	135.1084	153.9881
CAIC	126.0735	134.3255	153.5345

Table 4*: MLEs AIC, BIC, AICC, CAIC, HQIC and -2log L of the fitted distribution for the given data set 2*

Distribution	WTPXG	TPXG	XG
MLE	$\hat{\theta}$= 0.6597119 (0.1542389) $\hat{\alpha}$ = 614.8364779 (2836.4541338) $\hat{c}$= 1.6986652 (1.0404110)	$\hat{\theta}$ = 4.211952e-01 (3.945576e-02) $\hat{\alpha}$ =1.113155e+03 (9.687653e+03)	$\hat{\theta}$=0.36242629 (0.03606323)
-2logL	192.5258	196.1067	208.8136
AIC	203.4356	203.4358	212.4512
BIC	198.5228	200.1067	210.8136
AICC	198.9552	200.5391	211.246
HQIC	200.2707	201.2719	211.3962
AIC	199.2287	200.4495	210.924

From tables 3 and 4 we can observe AIC, BIC, AICC, HQIC, CAIC, and -2logL values are lowest in the weighted three-parameter Xgamma distribution compared to the two-parameter Xgamma distribution and Xgamma distribution. Hence, it can be concluded that the weighted three-parameter Xgamma distribution model is the best model compared to the two-parameter Xgamma distribution and Xgamma distribution.

XII. Conclusion

This research deals with the selection of an appropriate model for fitting survival data. In this paper, the Two-parameter Xgamma distribution is extended to provide a new distribution, called weighted three-parameter Xgamma distribution, which is the lifetime model for a real- life data set. In the section 3 to 10, discussing the statistical property of weighted three-parameter Xgamma distribution. The effectiveness of the suggested model is demonstrated by an examination of two real cancer data sets. The result indicates that the weighted three parameter Xgamma distribution is more flexible and practical than the two-parameter Xgamma distribution and Xgamma distribution.

References

[1] Bartoszewicz, J. (2009). On a representation of weighted distributions. *Statistics & probability letters*, *79*(15), 1690-1694.

[2] Dey, S., Ali, S., & Park, C. (2015). Weighted exponential distribution: properties and different methods of estimation. *Journal of Statistical Computation and Simulation*, *85*(18), 3641-3661.

[3] Abouammoh, A. M. Ahmed, R. and Khalique, A. (2000). On new renewal better than used classes of life distribution. *Statistics and Probability Letters*, 48:189-194.

[4] Eric, U., Olusola, O. M. O., & Eze, F. C. (2021). A study of properties and applications of Gamma distribution. *African Journal of Mathematics and Statistics Studies*, *4*(2), 52-65.

[5] Gupta, R. C., & Kirmani, S. N. U. A. (1990). The role of weighted distributions in stochastic modeling. *Communications in Statistics-Theory and methods*, *19*(9), 3147-3162.

[6] Jain, K., Singla, N., & Gupta, R. (2014). A weighted version of gamma distribution. *Discussiones Mathematica Probability and Statistics*, *34*(1-2), 89-111.

[7] Kochar, S. C., & Gupta, R. P. (1987). Some results on weighted distributions for positive-valued random variables. *Probability in the Engineering and Informational Sciences*, *1*(4), 417-423.

[8] Lawless, J. F. (2011). *Statistical models and methods for lifetime data*. John Wiley & Sons.

[9] Para, B. A. and Jan, T. R. (2018). On Three Parameter Weighted Pareto Type II Distribution: Properties and Applications in Medical Sciences. *Applied Mathematics & Information Sciences Letters*, 6:13-26.

[10] Patil, G. P. (1977). THE WEIGHTED DISTRIBUTION: A SURVEY OF THEIR APPLICATIONS

[11] Ray, M., & Shanker, R. (2024). A PROBABILITY MODEL FOR SURVIVAL ANALYSIS OF CANCER PATIENTS. *Reliability: Theory & Applications*, *19*(3 (79)), 78-94.

[12] Ray, M., & Shanker, R. (2023). A COMPOUND OF GAMMA AND SHANKER DISTRIBUTION. *Reliability: Theory & Applications*, *18*(2 (73)), 87-99.

[13] Saghir, A., Hamedani, G. G., Tazeem, S., & Khadim, A. (2017). Weighted distributions: A brief review, perspective and characterizations. *International journal of Statistics and Probability*

[14] Sen, S., Chandra, N., & Maiti, S. S. (2018). On properties and applications of a two-parameter Xgamma distribution. *Journal of Statistical Theory and Applications*, *17*(4), 674-685.

[15] Sen, S., Maiti, S. S., & Chandra, N. (2016). The Xgamma distribution: statistical properties and application. *Journal of Modern Applied Statistical Methods*, *15*(1), 38.

[16] Shanker, R. and Shukla, K. K. (2017). Weighted Shanker distribution and its applications to model lifetime data‖. *Journal of Applied Quantitative Methods*, 12:1 – 17.