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A NEW GENERALIZED EXPONENTIATED FAMILY OF CONTINUOUS DISTRIBUTIONS WITH APPLICATIONS TO ENVIRONMENTAL DATA SETS

Ibrahim, Sule[1]
Olalekan Akanji, Bello[2]
Ismail Adekunle, Kolawole[3]

Department of Mathematical Sciences, Faculty of Pure and Applied Sciences, Kaduna State University, Kaduna, Nigeria[1].

Department of Statistics, Faculty of Basic and Applied Sciences, Osun State University, Osogbo, Osun State, Nigeria[2].
Department of Mathematics and Statistics, Kaduna Polytechnic, Kaduna, Kaduna State, Nigeria[3].

ibrahimsule76@gmail.com[1]
olalekan4sure@gmail.com[2]
talktoismak@gmail.com[3]

ABSTRACT

Different researchers in the field of distribution theory have derived new models for generalizing the classical ones to make them more flexible and to aid their application in various fields. This generalization and extension of the classical models is mostly done using families of distributions. This article presents a new family of distributions called the Exponentiated Pareto-G family of distributions with two positive shape parameters. Some statistical properties of the new family of distributions, such as explicit expressions for the quantile function, probability-weighted moments, moments, generating function, Reliability function, hazard function, and order statistics are discussed. A maximum likelihood estimation technique is employed to estimate the model parameters. Two submodels such as Weibull and Frechet distributions are employed to check the fit of the family of distributions with the aid of their pdf and hazard function plots. Also, a simulation study is presented to assess the performance of the maximum likelihood estimator. Furthermore, two real-life applications are carried out to assess the fit and flexibility of the new family using the Weibull model as the baseline. The results showed that the new distribution fits better in the two real data sets considered among the range of distributions considered.

Keywords: Exponentiated Pareto-G, maximum flood level, precipitation, consistent, flexibility

I. INTRODUCTION

Research in the field of statistical distribution theory has increased tremendously in the past few years and still growing rapidly. Different researchers in the field of distribution theory have derived new models for generalizing the classical ones to make them more flexible and to aid their application in various fields. This generalization and extension of the classical models is mostly done using families of distributions. These families of distributions developed have aided the fit of many classical distributions with the addition of extra parameters to the baseline distributions.

Several considerations motivate the development of new generalized families of distributions. More adaptable, flexible, and robust models are required since current distributions frequently fall short of capturing the variability and patterns found in modern data. New distributions that can explain high-dimensional data and adjust to different contexts become crucial as data dimensionality and complexity rise. By offering a more accurate representation of the underlying processes, such distributions improve the resilience and accuracy of statistical analysis. Some of the well-known recently proposed modified families of distributions in the literature by different researchers to improve the standard theoretical distribution and also add flexibility to the classical distributions are: the new generalized family of distributions by [2], Topp-Leone Kumaraswamy-G family of distributions by [12], Topp-Leone Exponentiated-G family of distributions by[11], Rayleigh-exponentiated odd generalized-X family of distributions by [17], Type I half-logistic exponentiated-G family of distributions by [6], new generalized family of distributions by [15], Exponentiated type II generalized Topp-Leone-G family of distributions by [1].

II. THE EXPONENTIATED PARETO-G FAMILY (ETP-G) OF DISTRIBUTIONS

A new two-parameter distribution, called the exponentiated Pareto distribution introduced by [9] with cdf and pdf given as

$$F(x;\tau,\rho)=\left[1-(1+x)^{-\tau}\right]^{\rho} \tag{1}$$

$$f(x;\tau,\rho)=\tau\rho(1+x)^{-\tau-1}\left[1-(1+x)^{-\tau}\right]^{\rho-1} \tag{2}$$

According to [3], the cdf of the T-X family of distribution is given as

$$F(x)=\int_{a}^{W[G(x)]} r(t)dt=R\left[W\left[G(x)\right]\right] \tag{3}$$

Where $W\left[G(x)\right]$ satisfies the following conditions

(i) $W\left[G(x)\right]\in\left[a,b\right]$

(ii) $W\left[G(x)\right]$ is differentiable and monotonically non-decreasing, and

(4)

(iii) $W\left[G(x)\right]\to a$ as $x\to -\infty$ and $W\left[G(x)\right]\to b$ as $x\to\infty$

Let $r(t)$ be the pdf of a random variable $T\in\left[c,d\right]$ for $-\infty\leq c<d<\infty$ and $W\left[G(x)\right]$ be a function of the cdf of a random variable X .

Then the pdf corresponding to equation (3) is given by;

$$f(x)=\left\{\frac{d}{dx}W\left[G(x)\right]\right\}r\{W\left[G(x)\right]\} \tag{4}$$

Proposition 1:

Let $G(x;\xi)$ be the cdf of any arbitrary random variable X . Also, let $T\in(c,d)$ be a random variable with a pdf, $r(t)$. Furthermore, let our proposed link function be given as $G(x)$, using the expoenentiated Pareto distribution as the generator, then the cdf of Exponentiated Pareto-G family of distributions is given as:

$$F(x;\tau,\rho,\zeta)=\left[1-\left(1+G(x;\zeta\right)^{-\tau}\right]^{\rho} \tag{5}$$

Proof:

$$F(x;\tau,\rho,\zeta)=\tau\rho\int\limits_{0}^{G(x;\zeta)}\left(1+t\right)^{-\tau-1}\left[1-\left(1+t\right)^{-\tau}\right]^{\rho-1}\partial t$$

Let $y=1+t$, when $t=0, y=1$ and when $t=G(x;\zeta), y=1+G(x;\zeta)$

So, $\partial t=\partial y$

Now,

$$F(x;\tau,\rho,\zeta)=\tau\rho\int\limits_{1}^{1+G(x;\zeta)}y^{-\tau-1}\left[1-y^{-\tau}\right]^{\rho-1}\partial y \tag{6}$$

From equation (6),

Let $k=1-y^{-\tau}$, when $y=1, k=0$ and when $y=1+G(x;\zeta), k=1-\left(1+G(x;\zeta)\right)^{-\tau}$

$\partial y=\frac{\partial k}{-\rho y^{-\rho-1}}$

So,

$$F(x;\tau,\rho,\zeta)=\tau\rho\int\limits_{0}^{1-(1+G(x;\zeta))^{-\tau}}y^{-\tau-1}k^{\rho-1}\frac{\partial k}{\rho y^{-\rho-1}}$$

$$F(x;\tau,\rho,\zeta)=\rho\int\limits_{0}^{1-(1+G(x;\zeta))^{-\tau}}k^{\rho-1}\partial k$$

$$F(x;\tau,\rho,\zeta)=\rho\int\limits_{0}^{1-(1+G(x;\zeta))^{-\tau}}\frac{k^{\rho}}{\rho}\partial k$$

$$F(x;\tau,\rho,\zeta)=\rho\left[\frac{k^{\rho}}{\rho}\right]_{0}^{1-(1+G(x;\zeta))^{-\tau}}$$

$$F(x;\tau,\rho,\zeta)=\left[k^{\rho}\right]_{0}^{1-(1+G(x))^{-\tau}}$$

$$F(x;\tau,\rho,\zeta)=\left[1-(1+G(x;\zeta))^{-\tau}\right]^{\rho},\ 0\le x\le\infty \tag{7}$$

Where $\tau,\rho>0$ are the shape parameters and $\zeta>0$ is a vector of parameters depending on the baseline distribution used.

The pdf to equation (7) is given as

$$f(x;\tau,\rho,\zeta)=\tau\rho g(x;\zeta)\left(1+G(x;\zeta)\right)^{-\tau-1}\left[1-\left(1+G(x;\zeta)\right)^{-\tau}\right]^{\rho-1} \tag{8}$$

III. EXPANSION OF DENSITY

This section presents the densities expansion which will be used to estimate some of the distributions properties.

$$\left(1+Z\right)^{-\omega}=\sum_{i=0}^{\infty}(-1)^{i}\binom{\omega+i-1}{i}z^{i} \tag{9}$$

$$1-Z^{\ \omega}=\sum_{j=0}^{\infty}\ -1^{\ j}\binom{\omega}{j}Z^{j} \tag{10}$$

For $|z|<1$ and ω is a positive real non integer.

Applying equation (9) and equation (10) on the last term in equation (8), we have

$$\left[1-\left(1+G(x;\zeta)\right)^{-\tau}\right]^{\rho-1}=\sum_{i=0}^{\infty}\left(-1\right)^{i}\binom{\rho-1}{i}\left[1+G(x;\zeta)\right]^{-\tau i}$$

$$\left[1+G(x;\zeta)\right]^{-\tau(i+1)-1}=\sum_{j=0}^{\infty}\left(-1\right)^{j}\binom{\tau\left(i+1\right)+j-2}{j}\left[G(x;\zeta)\right]^{j}$$

$$f(x;\tau,\rho,\zeta)=\tau\rho g(x;\zeta)\sum_{i,j=0}^{\infty}\ \left(-1\right)^{i+j}\binom{\rho-1}{i}\binom{\tau\left(i+1\right)+j-2}{j}\left[G(x;\zeta)\right]^{j} \tag{11}$$

In this vain, using equation (9) and equation (10) on equation (6), we have

$$\left[F(x;\tau,\rho,\zeta)\right]^{h}=\left[1-(1+G(x;\zeta))^{-\tau}\right]^{\rho h}$$

$$\left[1-(1+G(x;\zeta))^{-\tau}\right]^{\rho h}=\sum_{k=0}^{h}\left(-1\right)^{k}\binom{\rho h}{k}\left[1+G(x;\zeta)\right]^{-\tau k}$$

$$\left[1+G(x;\zeta)\right]^{-\tau k}=\sum_{d=0}^{\infty}\left(-1\right)^{d}\binom{\tau k+d-1}{d}\left[G(x;\zeta)\right]^{d}$$

$$\left[F(x;\tau,\rho,\zeta)\right]^h = \sum_{d=0}^{\infty}\sum_{k=0}^{h} (-1)^{d+k}\binom{\rho h}{k}\binom{\tau k+d-1}{d}\left[G(x;\zeta)\right]^d \tag{12}$$

IV. PROPERTIES OF ETP-G

I. PROBABILITY WEIGHTED MOMENTS (PWMS)

$$\xi_{r,s} = E\left[X^r F(X)^s\right] = \int_0^\infty x^r f(x)(F(x))^s dx \tag{13}$$

The PWMs of EtP-G is derive by substituting equation (11) and equation (12) into equation (13) by replacing h with s, we have

$$\xi_{r,s} = \int_0^\infty \tau\rho \sum_{i,j,d=0}^{\infty}\sum_{k=0}^{s}(-1)^{i+j+d+k}\binom{\rho-1}{i}\binom{\tau(i+1)+j-2}{j}\binom{\rho s}{k}\binom{\tau k+d-1}{d} x^r g(x;\zeta)\left[G(x;\zeta)\right]^{d+j} dx \tag{14}$$

II. MOMENTS

$$E(X^r) = \int_0^\infty x^r f(x)dx \tag{15}$$

The r^{th} moments for EtP-G distribution is derive by substituting equation (11) into equation (15) to obtain

$$E(X^r) = \int_0^\infty x^r \tau\rho \sum_{i,j=0}^{\infty} (-1)^{i+j}\binom{\rho-1}{i}\binom{\tau(i+1)+j-2}{j} g(x;\zeta)\left[G(x;\zeta)\right]^j dx \tag{16}$$

III. MOMENT GENERATING FUNCTION (MGF)

The Moment Generating Function of x is given as

$$M_x(t) = \int_0^\infty e^{tx} f(x)dx \tag{17}$$

The MGF for EtP-G distribution is derive by substituting equation (11) into equation (17) we obtain

$$M_x(t) = \int_0^\infty \tau\rho \sum_{i,j=0}^{\infty} (-1)^{i+j}\binom{\rho-1}{i}\binom{\tau(i+1)+j-2}{j} e^{tx} g(x;\zeta)\left[G(x;\zeta)\right]^j dx \tag{18}$$

where the expansion of $e^{tx} = \sum_{z=0}^{\infty}\frac{t^z x^z}{z!}$ and following the process of moments above, we have the MGF for EtP-G distribution in equation (18) given as

$$M_x(t) = \int_0^\infty \tau\rho \sum_{i,j=0}^{\infty}\sum_{q=0}^{\infty} \frac{t^z}{z!}(-1)^{i+j}\binom{\rho-1}{i}\binom{\tau(i+1)+j-2}{j} x^z g(x;\zeta)\left[G(x;\zeta)\right]^j dx \tag{19}$$

IV. RELIABILITY FUNCTION

$$R(x;\tau,\rho,\zeta)=1-\left[1-\left(1+G(x;\zeta\right)^{-\tau}\right]^{\rho} \tag{20}$$

V. HAZARD FUNCTION

$$T(x;\tau,\rho,\zeta)=\frac{\tau\rho g(x;\zeta)\left(1+G(x;\zeta)\right)^{-\tau-1}\left[1-\left(1+G(x;\zeta)\right)^{-\tau}\right]^{\rho-1}}{1-\left[1-\left(1+G(x;\zeta\right)^{-\tau}\right]^{\rho}} \tag{21}$$

VI. QUANTILE FUNCTION

$$\left[1-(1+G(x;\zeta))^{-\tau}\right]^{\rho}=U$$

$$1-(1+G(x;\zeta))^{-\tau}=U^{\frac{1}{\rho}}$$

$$1-U^{\frac{1}{\rho}}=\left[1+G(x;\zeta)\right]^{-\tau}$$

$$\left[1-U^{\frac{1}{\rho}}\right]^{\frac{1}{-\tau}}=1+G(x;\zeta)$$

$$G(x;\zeta)=\left[1-U^{\frac{1}{\rho}}\right]^{\frac{1}{-\tau}}-1$$

$$x=Q\left(u\right)=G^{-1}\left[\left[1-U^{\frac{1}{\rho}}\right]^{\frac{1}{-\tau}}-1\right] \tag{22}$$

VII. ORDER STATISTICS

$$f_{r:n}(x)=\frac{f(x)}{B(r,n-r+1)}\sum_{v=0}^{n-r}(-1)^{v}\binom{n-r}{v}F(x)^{v+r-1} \tag{23}$$

The pdf of r^{th} order statistic for distribution is obtained also replacing h with v+r-1 in cdf expansion, we have

$$f_{r:n}(x)=\frac{1}{B(r,n-r+1)}\sum_{i,j,d=0}^{\infty}\sum_{k=0}^{v+r-1}\sum_{v=0}^{n-r}(-1)^{v+i+j+d+k}\binom{\rho(v+r-1)}{k}$$
$$\binom{\tau k+d-1}{d}\binom{\rho-1}{i}\binom{\tau\left(i+1\right)+j-2}{j}\binom{n-r}{v}\left[G(x;\zeta)\right]^{j+d+v+r-1} \tag{24}$$

The pdf of the minimum order statistic of the EtP-G distribution is obtained by setting r=1 in equation (24)

$$f_{1:n}(x)=n\sum_{i,j,d=0}^{\infty}\sum_{k=0}^{v}\sum_{v=0}^{n-1}(-1)^{v+i+j+d+k}\binom{\rho v}{k}\binom{\tau k+d-1}{d}\binom{\rho-1}{i}\binom{\tau(i+1)+j-2}{j}\binom{n-1}{v}\left[G(x;\zeta)\right]^{j+d+v} \tag{25}$$

Also, the pdf of the maximum order statistic of the distribution is obtained by setting r = n in equation (24)

$$f_{n:n}(x)=n\sum_{i,j,d=0}^{\infty}\sum_{k=0}^{v+n-1}(-1)^{v+i+j+d+k}\binom{\rho(v+n-1)}{k}\binom{\tau k+d-1}{d}\binom{\rho-1}{i}\binom{\tau(i+1)+j-2}{j}\left[G(x;\zeta)\right]^{j+d+v+n-1} \tag{26}$$

VIII. MAXIMUM LIKELIHOOD ESTIMATION

This section explores the maximum likelihood estimation (mle) technique to estimate the unknown parameters of the EtP-G distribution. Let $x_1, x_2, ..., x_n$ be a random sample of size n from the EtP-G distribution. Then, the likelihood function based on observed sample for the vector of parameter $(\tau,\rho,\zeta)^T$ is given by

$$\log L = n\log\tau + n\log\rho + \sum_{i=1}^{n}\log g(x_i;\zeta) - (\tau-1)\sum_{i=1}^{n}\log(1+G(x;\zeta)) + (\rho-1)\sum_{i=1}^{n}\log\left[1-(1+G(x;\zeta))^{-\tau}\right] \tag{27}$$

The components of score vector $U = (U_\tau, U_\rho, U_\zeta)$ are given as

$$U_\tau = \frac{n}{\tau} - \sum_{i=1}^{n}\log(1+G(x;\zeta)) + (\rho-1)\sum_{i=1}^{n}\frac{(1+G(x;\zeta))^{-\tau}\log(1+G(x;\zeta))}{\left[1-(1+G(x;\zeta))^{-\tau}\right]} = 0 \tag{28}$$

$$U_\rho = \frac{n}{\rho} + \sum_{i=1}^{n}\log\left[1-(1+G(x;\zeta))^{-\tau}\right] = 0 \tag{29}$$

$$U_\zeta = \sum_{i=1}^{n}\left[\frac{g(x_i;\zeta)^{\zeta}}{g(x_i;\zeta)}\right] - (\tau-1)\sum_{i=1}^{n}\left[\frac{G(x;\zeta)^{\zeta}}{1+G(x;\zeta)}\right] + (\rho-1)\sum_{i=1}^{n}\left[\frac{\tau(1+G(x;\zeta))^{-\tau-1}G(x;\zeta)^{\zeta}}{\left[1-(1+G(x;\zeta))^{-\tau}\right]}\right] \tag{30}$$

Equations (28), (29) and (30) cannot be solved analytically, so we have to resort to numerical method to estimate the unknown parameters.

V. SUB MODELS

I. EXPONENTIATED PARETO-WEIBULL (ETPW) DISTRIBUTION

The cdf and pdf of the Weibull distribution are given as

$$G(x;\theta,\beta)=1-e^{-(\theta x)^{\beta}} \tag{31}$$

$$g(x;\theta,\beta)=\theta\beta^{\theta}x^{\beta-1}e^{-(\theta x)^{\beta}} \tag{32}$$

Where $x\geq 0,\theta,\beta>0$.
The cdf for ETPW distribution is obtained by inserting equation (31) into equation (7) as

$$F(x;\tau,\rho,\theta,\beta)=\left[1-\left[2-e^{-(\theta x)^{\beta}}\right]^{-\tau}\right]^{\rho} \tag{33}$$

And the pdf for ETPW distribution is obtained by differentiating equation (33) with respect to *x* as

$$f(x;\tau,\rho,\theta,\beta)=\tau\rho\theta\beta^{\theta}x^{\beta-1}e^{-(\theta x)^{\beta}}\left[2-e^{-(\theta x)^{\beta}}\right]^{-\tau-1}\left[1-\left[2-e^{-(\theta x)^{\beta}}\right]^{-\tau}\right]^{\rho-1} \tag{34}$$

Where $x\geq 0,\tau,\rho,\theta,\beta>0$

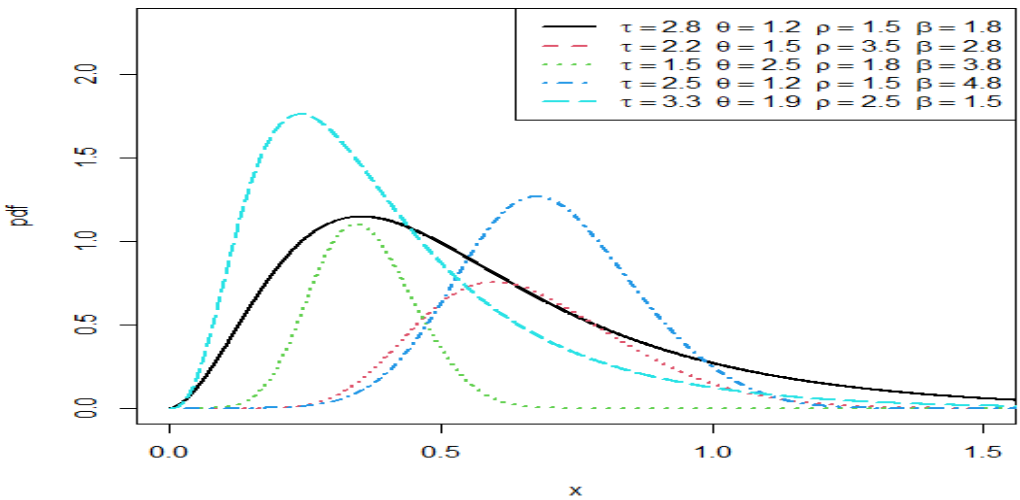

Figure 1: *Plots of pdf of ETPW distribution with different parameter values*

Reliability function for the ETPW distribution is given as

$$R(x;\tau,\rho,\theta,\beta)=1-\left[1-\left[2-e^{-(\theta x)^{\beta}}\right]^{-\tau}\right]^{\rho} \tag{35}$$

Hazard function for the ETPW distribution is given as

$$T(x;\tau,\rho,\theta,\beta)=\frac{\tau\rho\theta\beta^{\theta}x^{\beta-1}e^{-(\theta x)^{\beta}}\left[2-e^{-(\theta x)^{\beta}}\right]^{-\tau-1}\left[1-\left[2-e^{-(\theta x)^{\beta}}\right]^{-\tau}\right]^{\rho-1}}{1-\left[1-\left[2-e^{-(\theta x)^{\beta}}\right]^{-\tau}\right]^{\rho}} \tag{36}$$

Quantile function for the ETPW distribution is given as

$$x=Q(u)=\left[\frac{-1}{\theta}\log\left[1-\left[\left[1-U^{\frac{1}{\rho}}\right]^{\frac{1}{-\tau}}-1\right]\right]\right]^{\frac{1}{\beta}} \tag{37}$$

II. EXPONENTIATED PARETO-FRECHET (ETPFr) DISTRIBUTION

The Frechet distribution's cdf and pdf are provided as

$$G(x;\theta,\delta)=e^{-(\frac{\theta}{x})^{\delta}}\quad,\quad x>0,\theta,\delta>0 \tag{38}$$

$$g(x;\theta,\delta)=\delta\theta^{\delta}x^{-\delta-1}e^{-(\frac{\theta}{x})^{\delta}}\quad,\quad x>0,\theta,\delta>0 \tag{39}$$

The cdf for ETPFr distribution is given as

$$F(x;\tau,\rho,\theta,\delta)=\left[1-\left[1+e^{-(\frac{\theta}{x})^{\delta}}\right]^{-\tau}\right]^{\rho},x>0,\tau,\rho,\theta,\delta>0 \tag{40}$$

The pdf for ETPFr distribution is given as

$$f(x;\tau,\rho,\theta,\delta)=\tau\rho\delta\theta^{\delta}x^{-\delta-1}e^{-(\frac{\theta}{x})^{\delta}}\left[1+e^{-(\frac{\theta}{x})^{\delta}}\right]^{-\tau-1}\left[1-\left[1+e^{-(\frac{\theta}{x})^{\delta}}\right]^{-\tau}\right]^{\rho-1},x>0,\tau,\rho,\theta,\delta>0 \tag{41}$$

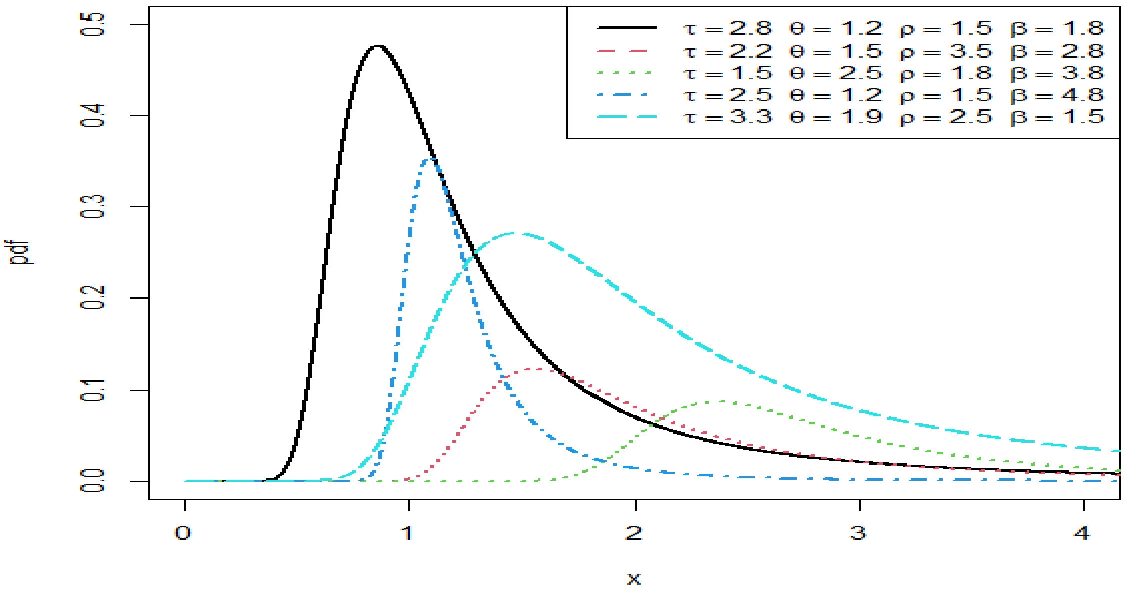

Figure 2: *Plots of pdf of ETPFr distribution with different parameter values*

Reliability function of the ETPFr distribution is given as

$$R(x;\tau,\rho,\theta,\beta)=1-\left[1-\left[1+e^{-(\frac{\theta}{x})^{\beta}}\right]^{-\tau}\right]^{\rho} \tag{42}$$

Hazard function of the ETPFr distribution is given as

$$T(x;\tau,\rho,\theta,\beta)=\frac{\tau\rho\beta\theta^{\beta}x^{-\beta-1}e^{-(\frac{\theta}{x})^{\beta}}\left[1+e^{-(\frac{\theta}{x})^{\beta}}\right]^{-\tau-1}\left[1-\left[1+e^{-(\frac{\theta}{x})^{\beta}}\right]^{-\tau}\right]^{\rho-1}}{1-\left[1-\left[1+e^{-(\frac{\theta}{x})^{\beta}}\right]^{-\tau}\right]^{\rho}} \tag{43}$$

Quantile function of the ETPFr distribution is given as

$$x=Q(u)=\frac{\theta}{\left[-\log\left[\left[1-U^{\frac{1}{\rho}}\right]^{\frac{1}{-\tau}}-1\right]\right]^{\frac{1}{\beta}}} \tag{44}$$

VI. SIMULATION STUDY

This section addresses a numerical analysis to evaluate the performance of MLE for ETPW Distribution.

Table 1: *MLEs, biases and RMSE for some values of the parameters of ETPW distribution*

		(25,2,6,5)			(27,4,8,6)		
N	Parameters	Estimated Values	Bias	RMSE	Estimated Values	Bias	RMSE
	τ	25.7129	0.7129	3.7808	27.7166	0.7166	4.1678
20	ρ	2.7524	0.7524	2.0073	5.5891	1.5891	4.0788
	θ	6.7789	0.7789	2.2489	9.1209	1.1209	2.8826
	β	5.0041	0.0041	0.4382	6.0493	0.0493	0.4292
	τ	25.5242	0.5242	3.1134	27.4919	0.4919	3.2601
50	ρ	2.2234	0.2234	0.9125	4.5111	0.5111	1.8794
	θ	6.4049	0.4049	1.5416	8.5580	0.5580	1.7238
	β	5.0226	0.0226	0.2663	6.0445	0.0445	0.2728
100	τ	25.3816	0.3816	2.3289	27.4562	0.4562	2.2990
	ρ	2.0852	0.0852	0.4054	4.2018	0.2018	0.9604
	θ	6.2106	0.2106	1.0451	8.2503	0.2503	1.1375
	β	5.0193	0.0193	0.1648	6.0314	0.0314	0.1609
250	τ	25.5657	0.5657	1.6176	27.3854	0.3854	1.4242
	ρ	2.0123	0.0123	0.2027	4.0360	0.0360	0.4619
	θ	5.9821	-0.0179	0.5937	8.0288	0.0288	0.6385
	β	5.0174	0.0174	0.0796	6.0192	0.0192	0.0843
500	τ	25.4954	0.4954	1.2777	27.2583	0.2583	0.8795
	ρ	1.9991	-0.0009 -	0.1293	4.0074	0.0074	0.2968
	θ	5.9229	0.0771	0.4329	7.9697	-0.0303	0.4076
	β	5.0078	0.0078	0.0553	6.0076	0.0076	0.0495
1000	τ	25.4001	0.4001	0.9152	27.1578	0.1578	0.5850
	ρ	1.9956	-0.0044 -	0.0896	4.0003	0.0003	0.1969
	θ	5.9244	0.0756	0.2979	7.9761	-0.0239	0.2774
	β	5.0045	0.0045	0.0323	6.0031	0.0031	0.0342

Table 1 displays the values of biases, estimated values and RMSEs It is noticed from the table that the RMESs approach zero and the estimates tend to the true parameter values as the sample increases. This is an indication that that the maximum likelihood estimates are efficient and consistent.

VII. APPLICATION

The fit of ETPW distribution is tested with applications to environmental data sets to assess its flexibility and robustness. The fit of the new model is compared with some existing distributions having Weibul distribution as the baseline. The comparators are: the Type I Half-Logistic Exponentiated Weibull (TIHLEtW) Distribution by [7], Type II Exponentiated Half Logistic Weibull (TIIEHLW) distribution by [4], Half-Logistic Generalized Weibull (HLGW) Distribution by [13], Exponentiated Weibull (EW) by [14]vand Weibull Distribution by [16].

The data set 1 consists of 20 observations with respect to maximum flood level data to see how the new model works in practice. The data has been obtained from [8] and is given as: 0.654, 0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379, 0.3235, 0.269, 0.740, 0.418, 0.412, 0.494, 0.416, 0.338, 0.392, 0.484, 0.265.

The data set 2 is obtained from [10] and also reported in [5]. It consists of thirty successive values of March precipitation (in inches) in Minneapolis/St Paul. The data are:

0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

Table 2: *The models' MLEs and performance requirements based on data set 1*

Models	$\hat{\tau}$	$\hat{\theta}$	$\hat{\rho}$	$\hat{\beta}$	ll	AIC	BIC
EtPW	38.9937	0.0354	206.5286	0.5620	16.3103	-24.6205	-20.6376
TIHLEtW	13.8158	2.3621	37.6306	0.5298	13.9359	-19.8717	-15.8888
TIIEtHLW	4.2059	0.6008	2.7424	6.0126	13.3035	-18.6071	-14.6242
HLGW	-	0.2951	6.5128	6888.7174	14.9716	-23.9432	-20.0560
EtW	-	1.4919	3.0333	2.3652	13.9497	-21.8993	-18.9121
W	-	3.5083	-	14.2303	13.2633	-22.5261	-20.5352

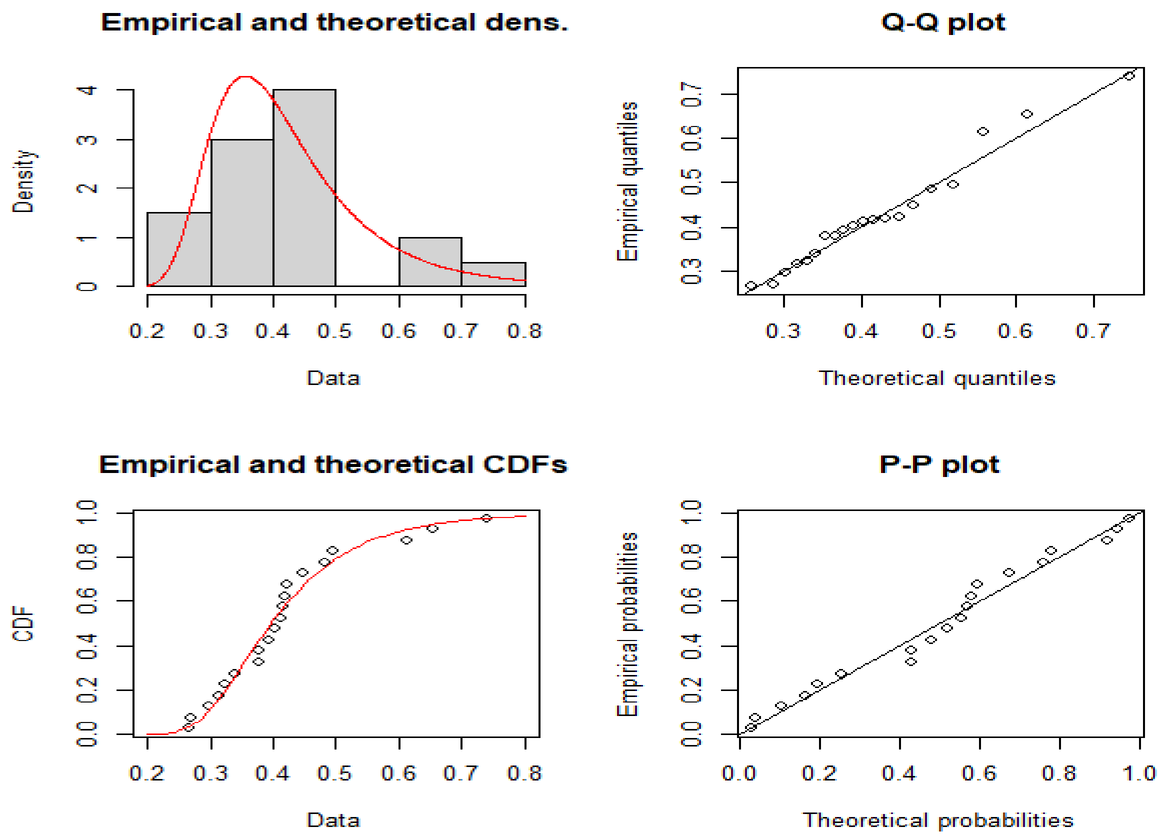

Figure 3: *Fitted cdf, pdf, Q-Q, and P-P plots for data set 1*

Table 3: *The models' MLEs and performance requirements based on data set 1*

Models	$\hat{\tau}$	$\hat{\theta}$	$\hat{\rho}$	$\hat{\beta}$	ll	AIC	BIC
EtPW	48.1901	0.0022	2.9823	1.0738	-38.0910	84.1820	89.7868
TIHLEtW	4.1437	0.6170	13.5135	0.5563	-38.4067	84.8135	90.4183

TIIEtHLW	0.7691	1.6782	0.4909	1.5904	-38.1060	84.2121	89.8168
HLGW	-	0.3416	2.7617	2.7351	-40.1181	86.2362	90.4398
EtW	-	2.4241	1.1680	0.8941	-39.8193	85.6386	89.8422
W	-	1.8088	-	0.3154	-41.6433	87.2866	90.0891

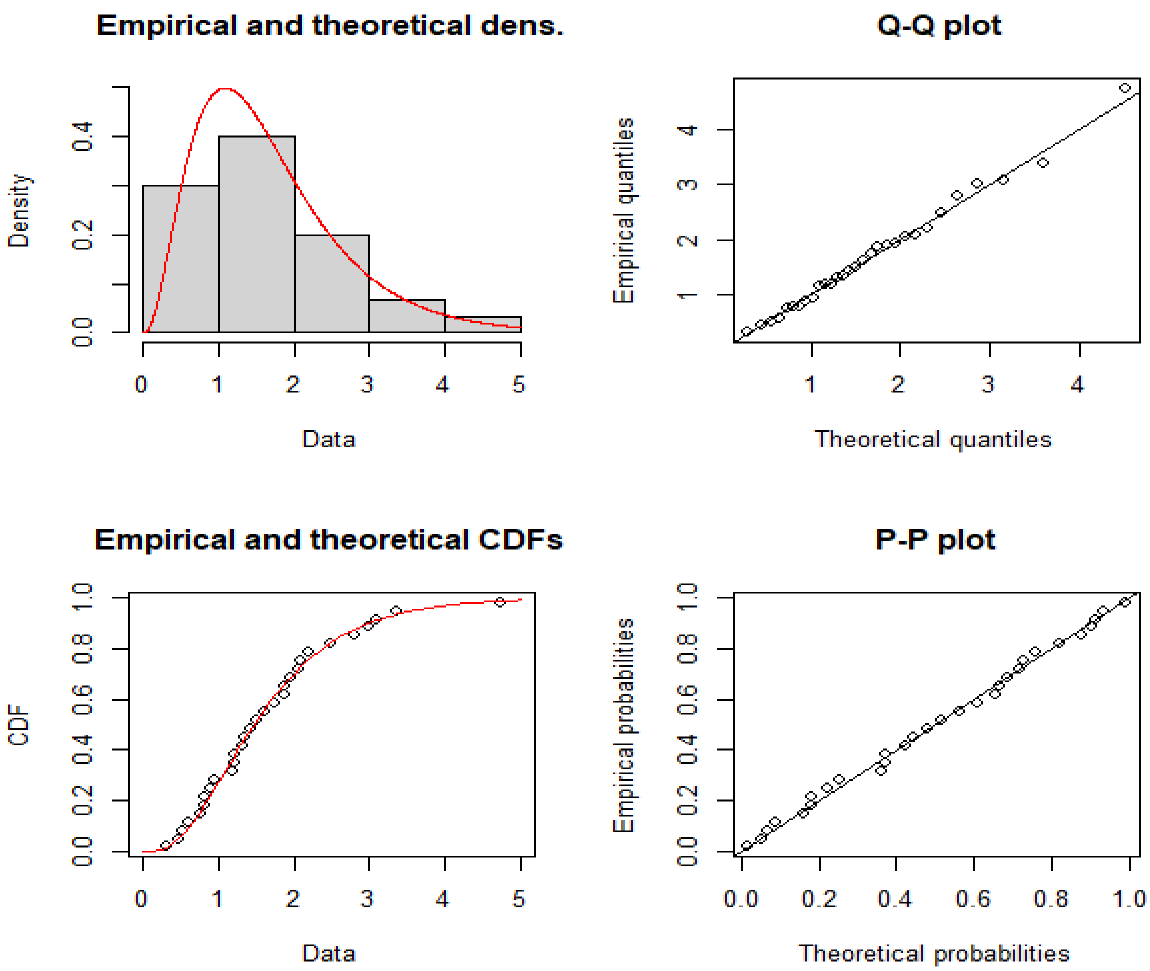

Figure 4: *Fitted cdf, pdf, Q-Q, and P-P plots for data set 2*

Tables 2 and 3 outline the results of the mle of the parameters of the EtPW distribution together with the comparator distributions. Based on the goodness of fit statistic AIC and BIC, the new probability model recorded the lowest AIC as well the lowest BIC value suggesting that the EtPW is best fits the two data sets. Figures 3 and 4 also buttress and reaffirm the fit of the EtPW distribution as it follows the pattern and shape of the data.

VIII. CONCLUSION

This research article proposed and studied a new family of distributions called the Exponentiated Pareto-G family of distributions. The family was derived from the exponentiated Pareto distribution using the T-X methodology proposed by [3]. The properties of the new family such as quantile function, probability-weighted moments, moments, generating function, reliability function, hazard function, and order statistics were examined as statistical components of the newly proposed family of distributions. The parameters of the family are estimated using the method of maximum likelihood technique. Two submodels such as Weibull and Frechet are used to show the shape of the family as baseline distributions. A simulation results to evaluate the new distribution's performance is carried out using Weibull as the baseline distribution. This is to assess the efficiency of the estimation method used. Two real data sets are applied to ascertain the importance and flexibility of the new family of distributions. The results reveal that the new exponentiated Pareto Weibull distribution appears to be superior to the existing models considered. This implies that the new family has added flexibility to the baseline distribution and it can be used to model data in a variety of fields.

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