TOWARD RISK FORMALISM. THE SCORE MATCHING METHOD IN A GROUP CHOICE PROBLEM

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Abstract

The task of coordinating expert assessments inevitably arises in the expert construction of integral indicators of objective judging in sports, analysis of the state and risks of functioning of social, economic, environmental, transportation systems and for many other subject areas of human activity. A decision-maker, as a rule, sets a criterion by which an object is evaluated, and an expert defines a set of comparable objects according to this criterion and evaluates each object, ranking them in descending (ascending) order of one or another quality. It is assumed that the expert has his own opinion that does not diverge from the generally accepted one and is not built only on the basis of measurable data. This opinion is based on his/her personal experience and knowledge gained in the process of work. As a result, there is often a situation when the transitivity of expert evaluations is violated. The proposed procedures for finding an integral indicator can be used in the tasks of decision-making, harmonization of expert assessments of the state of objects, construction of environmental and social indicators, as well as quality indicators, such as the integral indicator of the quality of life, the index of the quality of the transport system, the index of human development, etc. The methods of matrix theory are applied. The methods of matrix theory, graph theory and combinatorics are applied. The presented algorithm differs from existing methods in that it measures the contribution of the total error of experts to the collective measure of their consistency. The method under discussion offers a solution for decision makers in the so-called group selection problem (which means the task of analyzing and aggregating heterogeneous information about the preferences of compared objects into a single "group" preference) of critical objects that require increased attention of the security service and expenditure of resources of the state and the owner of the system to ensure their safety, security and sustainability of operation.

Keywords: expert evaluations, consistency, integral evaluation, group choice problem, permutations

I. Introduction

The method of recovering a consistent solution in expert ranking of objects involves the use of various techniques and approaches. One approach is to extend the use of Kendall's concordance coefficient (W) to incomplete rankings, allowing for the inclusion of objects that are not easily compared with others [1]. Another approach is to reconstruct expert preference functions using available data and expert judgment, which can be achieved using machine learning and matching techniques [2]. Decision analysis techniques can also be used to systematically analyze complex problems and improve the objectivity of remediation decisions by quantifying expert opinions and prioritizing remediation activities based on several criteria [3]. In addition, the rank analysis method can be applied to calculate the ranking of managed objects by replacing ordinal rankings with effective rankings that take into account the degree of difference in the estimated values [4]. Finally, decision making using fuzzy and fuzzy preference relations can be improved by introducing the concepts of additive consistency and group consensus analysis [5].

The following method is designed to find the most consistent solution in the task of ranking objects by means of expert evaluations. Such tasks are not uncommon in various fields of activity, where it is necessary to analyze several available alternatives and decide on their relative importance. As a rule, each of the experts involved in the evaluation has its own evaluation scales (most often irregular, since the experts have different experience and knowledge). Linking disparate opinions into one is an extremely important task for the decision maker (LPR).

The main obstacle to combining different opinions into one is the need for consistent assessments. The requirement for consistency in expert judgment is that the experts involved in the evaluation process make consistent and coherent assessments of the issue or subject under study. This means that experts must adhere to the same evaluation criteria and methods to avoid contradictory or inconsistent results.

Consistency of expert assessments is important to ensure the objectivity and reliability of the assessment results. Inconsistent assessments can lead to distorted conclusions and incorrect decisions based on these assessments. This fact clearly reflects the so-called Condorcet paradox.

The Condorcet paradox is a situation in which uncertainty or contradiction in preference determination arises in group preference selection using a voting method. This paradox was discovered by the French mathematician and philosopher Marquis de Condorcet in the 18th century.

The essence of the paradox is that it is possible that, given three or more alternatives to choose from, there is no voting method that satisfies all the basic principles of democracy at the same time:

- transitivity (if A is preferable to B and B is preferable to C, then A is preferable to C),
- independence from alternatives (a change in preferences for alternatives should not affect the voting result),
- absence of a dictator (the decision is made collectively, not by one person).

Condorcet's paradox emphasizes the complexity of group choice and shows that even the most democratic voting methods can lead to imperfect or inconsistent results in certain situations.

Condorcet's paradox of preference conflict in social choice has been studied by scholars and scientists for more than 200 years [6]. Various mechanisms have been proposed to resolve this paradox, including the use of rules in coordination games [7]. To understand the solution to Condorcet's paradox, strategic models of majority bargaining have also been analyzed [8]. It was found that in these models, there exists a stationary perfect equilibrium in the subgame that ensures agreement within a finite expected time [9]. In addition, mixed and consistent perfect equilibria in subgames have been found in the simplest Condorcet cycle, leading to immediate agreement in some scenarios [10]. These studies provide insights to resolve the Condorcet paradox and shed light on the factors affecting collective decision making.

Various methods such as consistency analysis method, Delphi method, expert judgment aggregation method and others can be used to ensure consistency of expert judgments. These methods help to identify discrepancies between expert assessments and make adjustments to achieve consistency.

There are many methods for assessing the consistency of expert judgment. The most common ones are:

1. Kendall's W coefficient of concordance: This method is used to measure the degree of consistency between multiple experts when evaluating the same item. It takes into account not only the consistency between pairs of experts, but also the average consistency between all experts.
2. Pearson's correlation coefficient (Pearson's r): This method is used to measure the degree of linear relationship between the scores of different experts. The closer the correlation coefficient is to 1, the higher the level of consistency.
3. Fleiss' Kappa consistency index (Fleiss' Kappa): This method is used to assess the degree of consistency between multiple experts when classifying objects into categories. It takes into account the random agreement between experts.
4. Consistency Ratio (C.R. - Consistency Ratio) in Thomas Saaty's method of hierarchy

analysis. This indicator allows assessing the degree of consistency between pairs of compared alternatives and identifying possible errors or contradictions in the experts' assessments.

The choice of method depends on the problem to be solved. The methods allow identifying inconsistencies in the estimates, but they do not offer to correct these inconsistencies by asking experts to refine their estimates repeatedly. Sometimes such a process is looped and the LPR is forced to make a decision at his/her own risk.

In 1951, C. Arrow formulated [11] the theorem "On the Impossibility of Collective Choice within the Ordinalist Method", mathematically generalizing Condorcet's paradox [12]. The theorem states that within the framework of this approach there is no method of combining individual preferences for three or more alternatives that would satisfy some quite fair conditions (axioms of choice) and would always give a logically consistent result.

II. Problem statement

The proposed method not only states the existing inconsistency in the experts' assessments, but also allows obtaining an optimal solution for the available assessments and the alternatives under consideration, "restoring" their correct ordering. The original author's algorithm for processing expert preferences in a collective choice problem based on the concept of the total "error" of experts and measuring their contribution to the collective measure of their consistency is presented in [13]. The authors called it the Pit Finding Method (PF-Method).

Let us recall the problem formulation from [13].

Considered N comparison objects $O_1, \dots, O_n, \dots, O_N$ whose indices are the first N members of the natural series $\langle 1, \dots, n, \dots, N \rangle$ - correspond to the order of submission of objects for examination (initial order). The examination of objects involves M equal experts $E_1, \dots, E_m, \dots, E_M$. Each of the experts E_m has its own idea about the order of objects placement $g_m = \langle g_{m,1}, \dots, g_{m,n}, \dots, g_{m,N} \rangle$. The indices of which increase as some quality of objects decreases from the point of view of this expert. That is, the value $g_{m,1}$ corresponds to the index of the object O_{k_1}participating in the examination with the maximum evaluated quality in the expert's opinion E_m, a $g_{m,N}$ - the worst object with the assessed quality with the index of O_{k_N}:

$$g_{m,n} = \begin{pmatrix} g_{1,1} & \cdots & g_{1,N} \\ \cdots & \ddots & \cdots \\ g_{M,1} & \cdots & g_{M,N} \end{pmatrix}.$$

Thus g_m - is a permutation of object ratings whose argument is the initial order: $g_m = \begin{pmatrix} 1 & \dots & n & \dots & N \\ g_{m,1} & \cdots & g_{m,n} & \cdots & g_{m,N} \end{pmatrix}$. Places $p_m = \langle p_{m,1}, \dots, p_{m,k}, \dots, p_{m,N} \rangle$ by values inverse to permutations of object ratings $g_m (p_m = g_m^{-1})$ are permutations of object indices with argument $E_{\text{ПИО}}$:

$$p_{m,n} = \begin{pmatrix} p_{1,1} = g_{1,1}^{-1} & \cdots & p_{1,N} = g_{1,N}^{-1} \\ \cdots & \ddots & \cdots \\ p_{M,1} = g_{M,N}^{-1} & \cdots & p_{M,N} = g_{M,N}^{-1} \end{pmatrix}.$$

It is necessary to find the compression of all private ratings of permutations of object ratings $g_m (m = 1, \dots, M)$ in the form of permutation of object ratings $g_m^* = \langle g_1^*, \dots, g_N^* \rangle$which would minimize the total inconsistency of expert evaluations $g_{m,n} \to g_m^*$ (on the basis of equality of all participants of the expertise), measured in inversions of transitions from $g_{m,n}$ к g_m^*i.e.

$$K^* = \min K(g) = \min_{g_m} \left(\sum_{m=1}^{M} K_m(\langle g_1, \dots, g_N \rangle) \right),$$

where $K_m(\langle g_1, \dots, g_N \rangle)$ - is the sum of inversions in the estimations $m$of -th expert, K^*- is the marginal measure of disagreement between the experts' opinions.

III. Algorithm description

Finding an optimum in permutations of object ratings is equivalent to finding a permutation of an object index p^*: $p^* = \langle p_1^*, \dots, p_N^* \rangle$, since $K(g_m^*) = K(p_m^*)$where p^*=$(g^*)^{-1}$ (the lengths of the back paths ($E \to g$) coincide with the forward paths ($p = g^{-1} \to E$) at any g) (Table 1).

Table 1: *Solution search table $P(g)$ with inversion table $B(g, P)$*

	$Arg\ E$	1	...	n	...	N	$Arg\ E$	1	...	k	...	N	Criterion inconsistencies
	Func g	g_1	...	g_n	...	g_N	Func p	g_1^{-1}	...	g_k^{-1}	...	g_N^{-1}	
	$Arg\ E$	1	...	k	...	N	$Arg\ E$	1	...	k	...	N	
1	$p_1(g)$	p_{1,g_1}	...	p_{1,g_k}	...	p_{1,g_N}	$B_1(p_1(g))$	$B_{1,1}$	...	$B_{1,k}$	...	$B_{1,N}$	$K_1(g) = \sum_{k=1}^{N} B_{1,k}$
...	...	...	...	...	...	...		...	...	...	...	...	...
m	$p_m(g)$	p_{m,g_1}	...	p_{m,g_k}	...	p_{m,g_N}	$B_m(p_m(g))$	$B_{1,1}$	...	$B_{m,k}$	...	$B_{m,N}$	$K_m(g) = \sum_{k=1}^{N} B_{m,k}$
...	...	...	...	...	...	...		...	...	...	...	...	...
M	$p_M(g)$	p_{M,g_1}	...	p_{M,g_k}	...	p_{M,g_N}	$B_M(p_M(g))$	$B_{M,1}$	...	$B_{M,k}$	...	$B_{M,N}$	$K_M(g) = \sum_{k=1}^{N} B_{M,k}$
													$K(g) = \sum_{m=1}^{M} K_m(g)$

After the experts have expressed their opinions, we assume that the order of submitting objects for examination corresponds to their correct ranking. Then, when pairwise comparisons of all experts' evaluations are made, the evaluations for the i-th object should be "better" from the point of view of its quality than the assessments of the -th object. $i + 1$of the -th object. If this assumption is violated, a "penalty" is fixed for this compared pair and the total "experts' error" is increased by one. If the total "experts' error" turns out to be higher than some specified value (e.g., $\frac{N}{2}$), a contradiction arises, reflecting the inconsistency of experts' opinions when comparing this pair. Obviously, the lower is the set value of the total "error", the more stringent are the requirements to the consistency of experts' opinions.

For N for the objects, it is necessary to perform $N \cdot (N - 1)$ pairwise comparisons. As a result of all comparisons, we obtain a square matrix of dimensionality $N \times N$in which the main diagonal consists of zeros (object's evaluations are compared in pairs with its own evaluations), and the matrix elements represent the corresponding total errors of experts, obtained as a result of the above described pairwise comparisons.

Then we rearrange the columns and rows of the obtained matrix in pairs, trying to move the maximum number of matrix elements, whose values are greater than the specified value of the total error, below the main diagonal. At such permutations the order of columns of the initial error matrix will change. The final order of columns will reflect the maximally consistent decision on the ranking of the compared objects. If such a solution is not the only one, it is necessary to calculate the value of the inconsistency criterion for all obtained solutions $K(g)$taking the obtained solution as the initial order of objects. The solution that has received the minimum value $K(g)$will be the final one.

Let us illustrate the work of the method on numerical examples.

IV. Case of study

Let seven experts $E_m (m = 1,2, \ldots, 7)$ compare 4 alternatives on some basis $O_n (n = 1, \ldots, 4)$. Let also let the initial order of the alternatives being compared be given by the mapping $\langle 1\ 2\ 3\ 4 \rangle \leftrightarrow \langle O_1\ O_2\ O_3\ O_4 \rangle$.

For a given set of evaluations of the alternatives being compared $\langle 1\ 2\ 3\ 4 \rangle$ it is necessary to find such a group $\langle k_1 \ldots k_4 \rangle$for which there is no possibility to improve the optimality criterion $K(g^*)$. The total number of consecutive inversions of each expert's evaluations is taken as the optimality criterion $K_m(g^*)$, restoring the current working order to the original one. In other words, it is necessary to find such a group for which $K(g^*) \to K_{\min}$.

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	1	2	3	4
	O_1	O_2	O_3	O_4
E_1	1	4	2	3
E_2	2	3	1	4
E_3	3	2	1	4
E_4	4	2	3	1
E_5	1	4	3	2
E_6	2	4	1	3
E_7	2	1	4	3

	1	2	3	4	Inversions				$K_m(g^*)$
	O_1	O_2	O_3	O_4	1→1	2→2	3→3	4→4	
E_1	1	4	2	3	0	1	1	0	2
E_2	2	3	1	4	2	0	0	0	2
E_3	3	2	1	4	2	1	0	0	3
E_4	4	2	3	1	3	1	1	0	5
E_5	1	4	3	2	0	2	1	0	3
E_6	2	4	1	3	2	0	1	0	3
E_7	2	1	4	3	1	0	1	0	2
								Optimality criterion $K(g^*)$:	20

Fig. 1 shows all pairwise comparisons of object evaluations and the scheme of forming the matrix of "errors". Bringing the matrix of pairwise comparisons of evaluations assigned to alternatives by experts to the form in which the upper triangular matrix contains no contradictions, we obtain the desired solution ⟨3 1 4 2⟩ (Fig. 2).

1	→	1	1	→	2	1	→	3	1	→	4
1		1	1		4	1		2	1		3
2		2	2		3	2	1	1	2		4
3		3	3	1	2	3	1	1	3		4
4		4	4	1	2	4	1	3	4	1	1
1		1	1		4	1		3	1		2
2		2	2		4	2	1	1	2		3
2		2	2		1	2		4	2		3
	0			**2**			**4**			**1**	

2	→	1	2	→	2	2	→	3	2	→	4
4	1	1	4		4	4	1	2	4	1	3
3	1	2	3		3	3	1	1	3		4
2		3	2		2	2	1	1	2		4
2		4	2		2	2		3	2	1	1
4	1	1	4		4	4	1	3	4	1	2
4	1	2	4		4	4	1	1	4	1	3
1		2	1		1	1		4	1		3
	4			**0**			**5**			**4**	

3	→	1	3	→	2	3	→	3	3	→	4
2	1	1	2		4	2		2	2		3
1		2	1		3	1		1	1		4
1		3	1		2	1		1	1		4
3		4	3	1	2	3		3	3	1	1
3	1	1	3		4	3		3	3	1	2
1		2	1		4	1		1	1		3
4	1	2	4	1	1	4		4	4	1	3
	3			**2**			**0**			**3**	

4	→	1	4	→	2	4	→	3	4	→	4
3	1	1	3		4	3	1	2	3		3
4	1	2	4	1	3	4	1	1	4		4
4	1	3	4	1	2	4	1	1	4		4
1		4	1		2	1		3	1		1
2	1	1	2		4	2		3	2		2
3	1	2	3		4	3	1	1	3		3
3	1	2	3	1	1	3		4	3		3
	6			**3**			**4**			0	

	1	2	3	4
1	0	2	4	1
2	4	0	5	4
3	3	2	0	3
4	6	3	4	0

Figure 1: *Scheme of formation of the "errors" matrix*

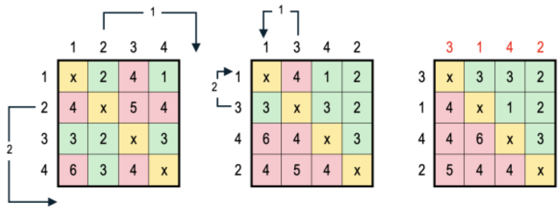

Figure 2: *Elementary transformations of the "error" matrix*

Since there are no contradictions in the upper triangular matrix, this solution is the only and non-improvable one. Let us check its compliance with the optimality criterion:

	3	1	4	2	Inversions				$K_m(g^*)$
	O_1	O_2	O_3	O_4	1→1	2→2	3→3	4→4	
E_1	2	1	3	4	1	0	0	0	1
E_2	1	2	4	3	0	0	1	0	2
E_3	1	3	4	2	0	2	0	0	2
E_4	3	4	1	2	2	2	0	0	4
E_5	3	1	2	4	1	1	0	0	2
E_6	1	2	3	4	0	0	0	0	0
E_7	4	2	3	1	3	1	1	0	5
								Optimality criterion $K'(g^*)$:	15

As can be seen from the table, the expert E_6 "guessed" the optimal solution, since. $K_6(g^*) = 0$. Experts E_1 и E_2 made only 1 error each, E_3 и E_5 - 2 errors each, and E_4 и E_7 - too many errors. The optimality criterion is *met* $K'(g^*) < K(g^*)$.

By reconstructing the $g^* = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$ the optimal sequence as $p^* = (g^*)^{-1}$ we obtain the desired locations $p^*(E) = \langle 2, 4, 1, 3 \rangle$.

V. Concluding remarks

The situation arising after permutations in the "errors" matrix will not always be as shown in the demo example. The problem of finding the optimal order of the evaluated objects may have several possible solutions. In this case, the best solution is determined by calculating the optimality criterion for all obtained solutions and selecting the solution that has the smallest value of this criterion. In most of the cases tested by the authors, the solution resulted in a global optimum, which favorably distinguishes the method from Schulze's method, for example. In addition, unlike the Schulze method, the PF-method is computationally much simpler and clearer.

Further development of this method implies its application in ranking definitions that allow equality of scores of the compared objects when determining the weight coefficients of the compared objects (similar to pairwise comparisons in the method of hierarchy analysis [14] and solving problems of fusion of heterogeneous scales [15, 16]).

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