GENERALIZATION OF RAYLEIGH DISTRIBUTION THROUGH A NEW TRANSMUTATION TECHNIQUE

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Abstract

In our research paper, we introduce an innovative statistical distribution known as the New Transmuted Rayleigh Distribution. This distribution serves as a versatile expansion of the traditional Rayleigh distribution and has been developed using a novel transmutation technique. We provide an in-depth analysis of several statistical properties of this new distribution. The resulting model has the ability to represent complex shapes, making it suitable for a wide range of applications. Our manuscript thoroughly examines the fundamental characteristics of the new model, outlining the methodology for estimating its unknown parameters through maximum likelihood estimation. Additionally, we demonstrate the practical significance of the model by applying it to an empirical dataset and conclusively establishing its superiority over some existing prominent models.

Keywords: New Transmuted Rayleigh distribution, Rayleigh distribution, reliability, stress strength reliability, parameter estimation.

1. Introduction

Innumerable methods to extend the probability models are available in the literature. A prominent technique called Quadratic Rank Transmutation map was put forth by Shaw and Buckley [11] that has been in use for a very long time. It enables us to obtain extended models with improved flexibility. Lately, some new ways to obtain the transmuted version of distribution are being introduced by various researchers. The introduction of these methods is triggered by the inability of the conventional method to model data that arise in different fields that are of random nature.

A novel transmutation technique introduced by Mansour et.al . [5] to derive an extended version of a probability model. Let the cdf of baseline distribution be the $G(x)$, then the new cdf, $M(x)$ is defined as

$$M(x) = (1+\lambda)G^{\delta}(x) - \lambda G^{\alpha}(x) \quad ; x > 0, \tag{1}$$

where $\alpha, \delta > 0$ if $0 > \lambda > -1$ and $\alpha > 0, \frac{\alpha}{2} \le \delta \le \alpha + \frac{\alpha}{2}$ if $0 < \lambda < 1$.
The pdf associated with equation (1) is presented in (2).

$$m(x) = [\delta(1+\lambda)G^{\delta-1}(x) - \alpha\lambda G^{\alpha-1}(x)]g(x). \tag{2}$$

Rayleigh Distribution (RD) was propounded by Rayleigh [7]. Balakrishnan [1] approximated the likelihood function to derive an exact estimator to estimate the scale parameter of RD. Sarti et.al. [10] studied maximum likelihood segmentation of ultrasound images with RD. The distribution followed by the product of independent Rayleigh variates was studied by EL-Sallabi and Vainikainen [9]. The cdf and pdf are respectively given by equations (3) and (4).

$$G(x) = 1 - e^{-\frac{x^2}{2\theta^2}} \quad ; x, \theta > 0, \tag{3}$$

$$g(x) = \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}}. \tag{4}$$

The failure rate of RD accelerates with time. Hence can be used in situations where the components have no manufacturing defect but intense aging. However, this distribution fails to provide an acceptable fit when the failure rate takes complex shapes which prove to be a motivation for extending this distribution. Some researchers who have obtained flexible extensions of RD are : Merovci [6], Reshi et.al. [8], Malik and Ahmad [4], Bhat and Ahmad [2] etc. In this paper, we exploit the generalization suggested by Mansour et.al. [5] to define a new generalization of RD. The new model so obtained is named as New Transmuted Rayleigh Distribution. The main motivation for considering this generalization is given as the hazard rate of NTRD exhibits several complex shapes such as constant, increasing-decreasing, decreasing-increasing, etc. thus, overcoming the shortcomings of RD. Also, the new distribution outperforms the base distribution and some well-known models in view of a real-life data set. The remaining manuscript is arranged as: in section 2, the pdf and cdf of the NTRD are defined and its special cases are discussed. The alternative form of the pdf and cdf of NTRD are given in section 3 and the prominent properties of the NTRD are expounded in section 4. In section 5, the parameter estimation is executed utilizing a very powerful method. Finally, the applicability of NTRD and its conclusion are given in sections 6 and 7 respectively.

2. New Transmuted Rayleigh Distribution (NTRD)

The cdf of NTRD can be obtained upon substituting equation (3) in equation (1) and is given as

$$M(x) = (1+\lambda)\left[1 - e^{-\frac{x^2}{2\theta^2}}\right]^{\delta} - \lambda\left[1 - e^{-\frac{x^2}{2\theta^2}}\right]^{\alpha} \quad ; x > 0. \tag{5}$$

Also, the pdf corresponding to equation (5) is presented below:

$$m(x) = \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} \left(\delta(1+\lambda)\left[1 - e^{-\frac{x^2}{2\theta^2}}\right]^{\delta-1} - \alpha\lambda\left[1 - e^{-\frac{x^2}{2\theta^2}}\right]^{\alpha-1}\right), \tag{6}$$

where $\alpha, \delta, \theta > 0$ if $0 > \lambda > -1$ and $\alpha, \theta > 0, \frac{\alpha}{2} \leq \delta \leq \alpha + \frac{\alpha}{2}$ if $0 < \lambda < 1$.

2.1. Special cases of NTRD

The special cases of NTRD are as follows:

- RD: For $\lambda = 0$ and $\delta = 1$, equation (5) reduces to equation (3), which is the cdf of RD.
- Transmuted Rayleigh Distribution (TRD): For $\alpha = 2$ and $\delta = 1$, equation (5) becomes equation (7)

$$M(x) = (1+\lambda)\left[1 - e^{-\frac{x^2}{2\theta^2}}\right] - \lambda\left[1 - e^{-\frac{x^2}{2\theta^2}}\right]^{2} \quad ; \theta, x > 0, -1 \leq \lambda \leq 1, \tag{7}$$

which is the cdf of TRD.

- Transmuted Exponentiated Rayleigh Distribution (TERD): For $\delta = \frac{\alpha}{2}$, equation (5) becomes equation (8)

$$M(x) = (1+\lambda)\left[1-e^{-\frac{x^2}{2\theta^2}}\right]^{\frac{\alpha}{2}} - \lambda\left[1-e^{-\frac{x^2}{2\theta^2}}\right]^{\alpha} \quad ;\theta,\alpha,x>0,-1\leq\lambda\leq 1, \tag{8}$$

which is the cdf of TERD

- Exponentiated Rayleigh Distribution (ERD): For $\lambda = 0$, equation (5) becomes equation (9)

$$M(x) = \left[1-e^{-\frac{x^2}{2\theta^2}}\right]^{\delta} \quad ;\theta,\delta,x>0, \tag{9}$$

which is the cdf of ERD.

The graphical overview of possible shapes of the pdf of NTRD is shown in Figure 1. It is clear from the Figure 1 that PDF plots of NTRD are unimodal, Decreasing, Increasing, Symmetric and postively skewed

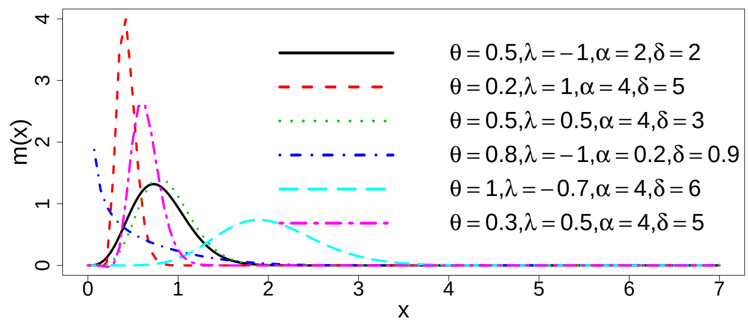


Figure 1: *Pdf plots of NTRD*

The survival function ($S(x)$), hazard rate function ($h(x)$)(hrf) and cummulative hrf ($H(x)$) are respectively given as:

$$S(x) = 1 - \left((1+\lambda)\left[1-e^{-\frac{x^2}{2\theta^2}}\right]^{\delta} - \lambda\left[1-e^{-\frac{x^2}{2\theta^2}}\right]^{\alpha}\right),$$

$$h(x) = \frac{\frac{x}{\theta^2}e^{-\frac{x^2}{2\theta^2}}\left(\delta(1+\lambda)\left[1-e^{-\frac{x^2}{2\theta^2}}\right]^{\delta-1} - \alpha\lambda\left[1-e^{-\frac{x^2}{2\theta^2}}\right]^{\alpha-1}\right)}{1-\left((1+\lambda)\left[1-e^{-\frac{x^2}{2\theta^2}}\right]^{\delta} - \lambda\left[1-e^{-\frac{x^2}{2\theta^2}}\right]^{\alpha}\right)},$$

$$H(x) = -\,ln\left[1-\left((1+\lambda)\left[1-e^{-\frac{x^2}{2\theta^2}}\right]^{\delta} - \lambda\left[1-e^{-\frac{x^2}{2\theta^2}}\right]^{\alpha}\right)\right].$$

The shapes of hrf for different parameter values are displayed in Figure 2. Figure 2 clearly shows that the proposed model is flexible and can exhibit different types of shapes such as U-Shaped, J shaped, Increasing and Bathtub over the parameter space.

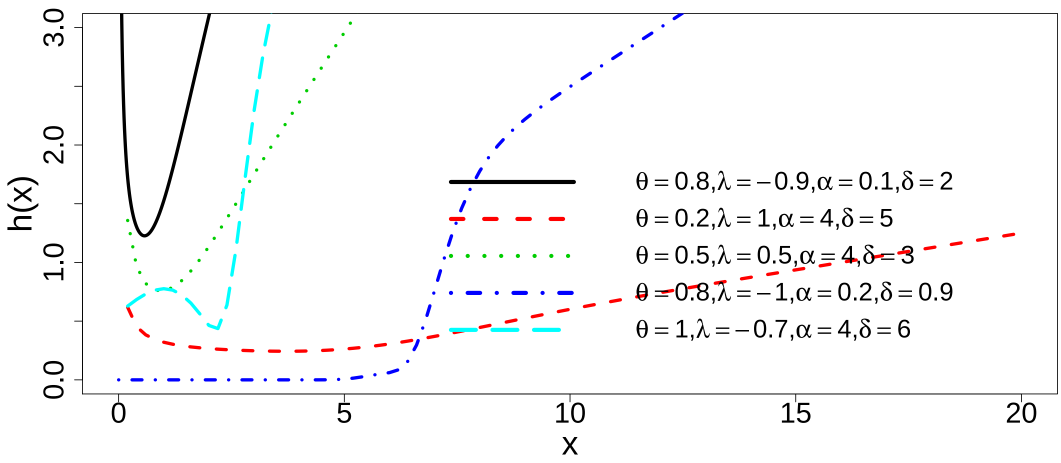


Figure 2: *Hrf plots of NTRD*

3. Mixture representation of NTRD

Using the expansions

$$(a-b)^n = \sum_{j=0}^{n} \binom{n}{j} (-1)^j b^j a^{n-j},$$

in equation (6), we get

$$m(x) = \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} \sum_{j=0}^{1} (-1)^j \{\delta(1+\lambda)\}^{1-j} \{\lambda\alpha\}^j \left(1 - e^{-\frac{x^2}{2\theta^2}}\right)^{(\delta-1)(1-j)+(\alpha-1)j}. \tag{10}$$

Now, using

$$(1-y)^k = \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(k+1)}{\Gamma(k-m+1)m!} y^m \quad ; \quad |y| < 1, k > 0,$$

in equation (10), we obtain the alternative form of pdf given as

$$\begin{aligned} m(x) &= \sum_{j=0}^{1} \sum_{m=0}^{\infty} \binom{(\delta-1)(1-j)+(\alpha-1)j}{m} (-1)^{j+m} \{\delta(1+\lambda)\}^{1-j} \\ &\quad \{\lambda\alpha\}^j \frac{x}{\theta^2} e^{-(m+1)\frac{x^2}{2\theta^2}}, \\ &= \sum_{m=0}^{\infty} \eta(j) \frac{x}{\theta^2} e^{-(m+1)\frac{x^2}{2\theta^2}}, \end{aligned} \tag{11}$$

where

$$\eta(j) = \sum_{j=0}^{1} \binom{(\delta-1)(1-j)+(\alpha-1)j}{m} (-1)^{j+m} \{\delta(1+\lambda)\}^{1-j} \{\lambda\alpha\}^j.$$

Similarly, the cdf of NTRD takes the alternative expression given in equation (12).

$$M(x) = \sum_{s=0}^{\infty} \eta'(p) e^{-s\frac{x^2}{2\theta^2}}, \tag{12}$$

where $\eta'(p) = \sum_{p=0}^{1} (-1)^{p+s} \lambda^p (1+\lambda)^{1-p} \binom{\lambda p+\delta(1-p)}{s}$.

4. Statistical Properties of NTRD

This section brings forth several prominent belonging such as moments, Generating functions, mean deviation about mean and median, mean residual life and mean waiting time and Stress Strength Reliability to NTRD .

4.1. Moments of NTRD

The r^{th} moment about origin can be expounded as

$$\mu_r^{'} = \int_0^\infty x^r m(x) dx.$$

Using equation (11), we get

$$= \sum_{m=0}^{\infty} \eta(j) \int_0^\infty \frac{x^{r+1}}{\theta^2} e^{-(m+1)\frac{x^2}{2\theta^2}} dx,$$

which upon simplification yields equation (13).

$$\mu_r^{'} = \sum_{m=0}^{\infty} \eta(j) \frac{(2\theta^2)^{\frac{r}{2}} \Gamma\left(\frac{r}{2}+1\right)}{(m+1)^{\left(\frac{r}{2}+1\right)}}. \tag{13}$$

The mean and variance are respectively given by equations (14) and (15) respectively.

$$Mean = \sum_{m=0}^{\infty} \eta(j) \frac{(2\theta^2)^{\frac{1}{2}} \Gamma\left(\frac{3}{2}\right)}{(m+1)^{\left(\frac{3}{2}\right)}}, \tag{14}$$

$$Variance = \left(\sum_{m=0}^{\infty} \eta(j) \frac{2\theta^2}{(m+1)^2} \right) - \left(\sum_{m=0}^{\infty} \eta(j) \frac{(2\theta^2)^{\frac{1}{2}} \Gamma\left(\frac{3}{2}\right)}{(m+1)^{\left(\frac{3}{2}\right)}} \right)^2. \tag{15}$$

Also, for NTRD, the n^{th} incomplete moment about origin is

$$\begin{aligned} \psi_n(s) &= \int_0^s x^n m(x) dx, \\ &= \sum_{m=0}^{\infty} \eta(j) \int_0^s \frac{x^{n+1}}{\theta^2} e^{-(m+1)\frac{x^2}{2\theta^2}} dx, \\ &= \sum_{m=0}^{\infty} \eta(j) \frac{(2\theta^2)^{\frac{n}{2}} \gamma\left(\frac{n}{2}+1, \frac{(m+1)s^2}{2\theta^2}\right)}{(m+1)^{\left(\frac{n}{2}+1\right)}}. \end{aligned} \tag{16}$$

4.2. Generating functions of NTRD

The moment, characteristic and cummulant generating function are respectively given by equations (17), (18) and (19) respectively.

$$M_X(t) = \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{t^r \eta(j)}{r!} \frac{(2\theta^2)^{\frac{r}{2}} \Gamma\left(\frac{r}{2}+1\right)}{(m+1)^{\left(\frac{r}{2}+1\right)}}, \tag{17}$$

$$\phi_X(t) = \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(it)^r \eta(j)}{(i)^r r!} \frac{(2\theta^2)^{\frac{r}{2}} \Gamma\left(\frac{r}{2}+1\right)}{(m+1)^{\left(\frac{r}{2}+1\right)}}, \tag{18}$$

$$K_X(t) = ln \left[\sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{t^r \eta(j)}{r!} \frac{(2\theta^2)^{\frac{r}{2}} \Gamma\left(\frac{r}{2}+1\right)}{(m+1)^{\left(\frac{r}{2}+1\right)}} \right]. \tag{19}$$

4.3. Mean Deviation about Mean and Median of NTRD

The mean Deviation about mean of NTRD is discussed below

$$\begin{aligned}D(\mu) =&E[|X-\mu|],\\ =&2\mu M(\mu) - 2\psi_1(\mu),\\ =&2\mu\left((1+\lambda)\left[1-e^{-\frac{\mu^2}{2\theta^2}}\right]^{\delta} - \lambda\left[1-e^{-\frac{\mu^2}{2\theta^2}}\right]^{\alpha}\right) -\\ &2\left(\sum_{m=0}^{\infty}\eta(j)\frac{(2\theta^2)^{\frac{1}{2}}\gamma\left(\frac{3}{2},\frac{(m+1)\mu^2}{2\theta^2}\right)}{(m+1)^{\left(\frac{3}{2}\right)}}\right).\end{aligned}$$

Also, for NTRD the mean deviation about Median is

$$\begin{aligned}D(M) =&E[|X-M|],\\ =&\mu - 2\psi_1(M),\\ =&\mu - 2\left(\sum_{m=0}^{\infty}\eta(j)\frac{(2\theta^2)^{\frac{1}{2}}\gamma\left(\frac{3}{2},\frac{(m+1)M^2}{2\theta^2}\right)}{(m+1)^{\left(\frac{3}{2}\right)}}\right).\end{aligned}$$

4.4. Mean Residual Life (MRL) and Mean Waiting Time (MWT) of NTRD

The formula for computing MRL is

$$MRL = \frac{E(t)-\psi_1(t)}{1-M(t)} - t. \tag{20}$$

Using equation (16) for $n=1$, equation (14) and equation (5) in equation (20), we get

$$\begin{aligned}MRL =&\frac{\left(\sum_{m=0}^{\infty}\eta(j)\frac{(2\theta^2)^{\frac{1}{2}}\Gamma\left(\frac{3}{2}\right)}{(m+1)^{\left(\frac{3}{2}\right)}}\right) - \left(\sum_{m=0}^{\infty}\eta(j)\frac{(2\theta^2)^{\frac{1}{2}}\gamma\left(\frac{3}{2},\frac{(m+1)t^2}{2\theta^2}\right)}{(m+1)^{\left(\frac{3}{2}\right)}}\right)}{1-\left((1+\lambda)\left[1-e^{-\frac{t^2}{2\theta^2}}\right]^{\delta} - \lambda\left[1-e^{-\frac{t^2}{2\theta^2}}\right]^{\alpha}\right)} - t,\\ =&\frac{\left(\sum_{m=0}^{\infty}\eta(j)\frac{(2\theta^2)^{\frac{1}{2}}\Gamma\left(\frac{3}{2},\frac{(m+1)t^2}{2\theta^2}\right)}{(m+1)^{\left(\frac{3}{2}\right)}}\right)}{1-\left((1+\lambda)\left[1-e^{-\frac{t^2}{2\theta^2}}\right]^{\delta} - \lambda\left[1-e^{-\frac{t^2}{2\theta^2}}\right]^{\alpha}\right)} - t,\end{aligned}$$

where $\Gamma(a,b) = \int_0^b t^{a-1}e^{-t}dt$.

Similarly, we can obtain MWT for NTRD using the formula given as

$$\begin{aligned}MWT =&t - \frac{\psi_1(t)}{M(t)},\\ =&t - \frac{\left(\sum_{m=0}^{\infty}\eta(j)\frac{(2\theta^2)^{\frac{1}{2}}\gamma\left(\frac{3}{2},\frac{(m+1)t^2}{2\theta^2}\right)}{(m+1)^{\left(\frac{3}{2}\right)}}\right)}{\left((1+\lambda)\left[1-e^{-\frac{t^2}{2\theta^2}}\right]^{\delta} - \lambda\left[1-e^{-\frac{t^2}{2\theta^2}}\right]^{\alpha}\right)}.\end{aligned}$$

4.5. Stress Strength Reliability(SSR) of NTRD

Let $X_1 \sim NTRD(\alpha, \lambda, \delta, \theta_1)$ be the strength of a system and $X_2 \sim NTRD(\alpha, \lambda, \delta, \theta_2)$ denote the stress that the system is subjected to. Then, the SSR can be computed as

$$
\begin{aligned}
R &= \int_0^\infty M_1(x) m_2(x) dx, \\
&= \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \eta^{'}(p)\eta(j) \int_0^\infty \frac{x}{\theta_1^2} e^{-(m+1)\frac{x^2}{2\theta_1^2} - s\frac{x^2}{2\theta_2^2}} dx, \\
&= \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \frac{\eta^{'}(p)\eta(j)}{\theta_1^2 \Gamma\left(\frac{m+1}{\theta_1^2} + \frac{s}{\theta_2^2}\right)}.
\end{aligned}
$$

5. Estimation of Parameters

The technique of MLE is employed to estimate the parameters are not known. For a sample $X_1, X_2, ..., X_n$ taken from NTRD randomly, the log-likelihood function denoted by l can be obtained as

$$
\begin{aligned}
l = \sum_{i=1}^{n} log \left(\delta(1+\lambda) \left[1 - e^{-\frac{x_i^2}{2\theta^2}} \right]^{\delta-1} - \alpha\lambda \left[1 - e^{-\frac{x_i^2}{2\theta^2}} \right]^{\alpha-1} \right) + \\
\sum_{i=1}^{n} log x_i - 2n log\theta - \sum_{n=1}^{n} \frac{x_i}{2\theta^2}.
\end{aligned} \tag{21}
$$

Equation (21) is differentiated w.r.t α, λ, δ and θ respectively and equated to zero to obtain the normal equations. These normal equations will not be in closed form. Therefore, a number of iterative procedures are available in literature which can be followed to solve such equations and obtain the parameter estimates.

6. Application

The applicability of NTRD from practical standpoint is assessed through a real-life data set which consists of the waiting period (in minutes) of 100 bank clients prior service which has been extracted from Ghitany et.al. [3]. The conventional models that are being used to compare the fits are

- TRD with cdf given by equation (7).
- ERD with cdf given by equation (9).
- Exponential Distribution (ED) with pdf given as

$$
m(x) = \frac{e^{-\frac{x}{\theta}}}{\theta} \qquad ; \qquad x, \theta > 0.
$$

- RD with pdf given in equation (4).

To choose the best model among the compared models, performance comparing tools such as AIC, AICc, SIC, and HQIC are exploited. These Criterion choose the superior distribution as the one which gives the minimal value. Furthermore, the Kolmogorov-Smirnov (KS)-Statistic (which is used to test if the sample comes from a specific population) and associated p- value (for decision making regarding the hypothesis) is obtained to assess the goodness of fit. All the calculations are performed using R software. The formulas for computing these criterion are given as:

$$AIC = -2\hat{l} + 2w, \qquad AICc = AIC + \frac{2w(w+1)}{n-w-1},$$
$$SIC = -2\hat{l} + logn, \qquad HQIC = -2\hat{l} + 2wlog(logn),$$
$$KS - statistic = Maximum|M_0(x) - M_n(x)|.$$

where c represents number of parameters, $M_0(x)$ is the observed cdf and $M_n(x)$ is the theoretical cdf.

Table 1 displays the MLE's and corresponding Standard Error (SE) and Table 2 presents comparison of performance of NTRD and compared distributions for waiting time data.

Table 1: *MLE's of NTRD and compared distributions with corresponding standard error (given in parenthesis) for waiting time data*

Model	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\delta}$
NTRD	12.089	1.000	1.269	0.754
	(1.675)	(0.990)	(0.340)	(0.119)
TRD	10.127	0.647		
	(0.778)	(0.172)		
ERD	8.185			0.629
	(0.774)			(0.0777)
RD	8.643			
	(0.432)			
ED	9.877			
	(0.987)			

Table 2: *Comparison of NTRD and compared distributions for waiting time data*

Model	-$\hat{l}$	AIC	SIC	AICc	HQIC	KS-statistic	p-value
NTRD	318.71	645.43	655.85	645.86	649.65	0.0655	0.7831
TRD	323.83	651.66	656.87	651.79	653.77	0.1317	0.0621
ERD	321.51	647.03	658.24	647.45	658.25	0.0945	0.3337
RD	329.24	660.48	663.08	660.52	661.53	0.1736	0.0048
ED	329.02	660.04	662.64	660.08	661.09	0.1730	0.0050

The values reported in Table 2 reveal that NTRD has a minimum value of SIC, AIC, AICc, and HQIC, consequently outperforms the base model and some well-known models for the given data set. Figure 3 further justifies the claim. Also, the QQ-plots are plotted for the proposed and compared distributions for the waiting data in Figure 4.

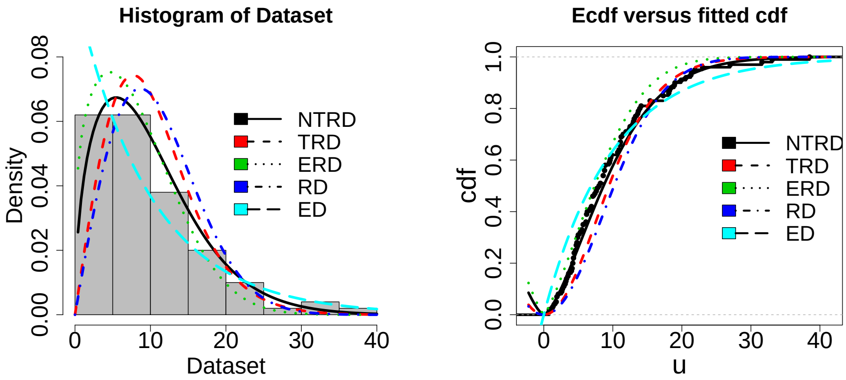
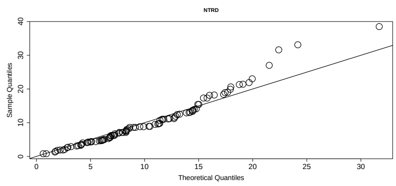
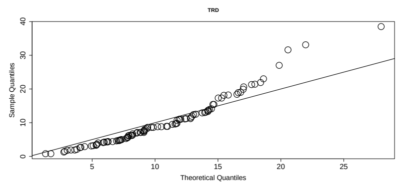


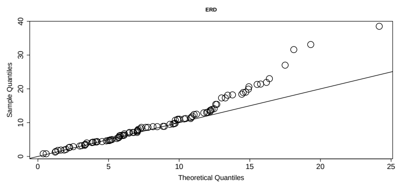
Figure 3: *Fitted densities*



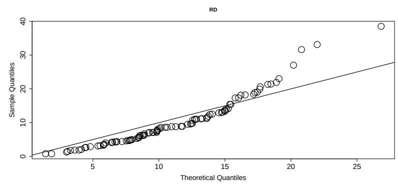
(a) *QQ plots of NTRD*



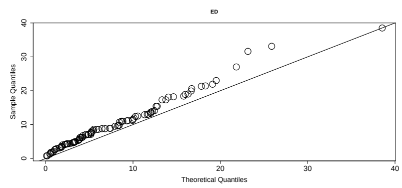
(b) *QQ plots of TRD*



(c) *QQ plots of ERD*



(d) *QQ plots of RD*



(e) *QQ plots of ED*

Figure 4: *QQ-plot of NTRD and competitive models for given dataset*

7. Conclusion

In this manuscript, a flexible generalization of RD is proposed which can acts as a potential substitute for the base model in various situations. Some of its important properties are expounded and parameters are estimated utilizing a very powerful estimation procedure. A data set is incorporated for the illustration of the applicability of NTRD from a practical standpoint and the results prove the superiority of NTRD over various standard Distributions namely TRD, ERD, RD and ED. Additionally, we present graphs to visually illustrate the results.

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