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A CLASS OF CONTROL CHARTS FOR PROCESS LOCATION PARAMETER OF EXPONENTIAL DISTRIBUTION

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Abstract

Control charts are essential in production processes to maintain quality of the products. Inspite of numerous control charts existing for process location under normal model, there is a need for developing control charts when situations demand production process under other distributions. In this paper, a class of control charts based on various midranges is proposed for monitoring location parameter of an ongoing process when process variables follow exponential distribution. The midranges are defined and their distributions are obtained. The performance of some members of the proposed class are evaluated in terms of their power, average run length (ARL), median run length (MRL) and standard deviation of run length (SDRL). Also, optimality and effectiveness of members of the class are discussed along with their illustration through an example.

Keywords: ARL, control limits, exponential distribution, location parameter, process control, r^{th} midrange

1. Introduction

Control charts are indispensable tools in statistical process control (SPC) for monitoring and optimizing ongoing manufacturing processes. They provide a visual representation of process variation over time, enabling practitioners to distinguish between chance cause of variation and assignable cause of variation. The monitoring of process location parameter helps in maintaining ongoing process and gives an indication about when corrective actions are required. Many control charts are developed with the assumption that process variables are taken from normal distribution, which usually do not represent some real-world scenarios. For example, chemical process, lifetime process and cutting tool wear process do not follow a normal distribution as highlighted in [3]. Therefore, studies focusing on non-normal distributions, particularly skewed distributions are of importance in decision making in process control.

In manufacturing operations, downtime often follows a two parameter exponential distribution, characterized with probability density function (pdf)

$$f(x) = \frac{1}{\lambda} e^{-\frac{x-\mu}{\lambda}}, \quad x \geq \mu, \quad -\infty < \mu < \infty, \quad \lambda > 0 \tag{1}$$

where μ represents the average downtime duration. Monitoring μ is crucial for understanding and managing production interruptions effectively. Focusing on μ, which signifies the central tendency of downtime durations, allows companies to implement proactive measures such as predictive maintenance scheduling and process optimization. Statistical process control techniques enable the monitoring of μ through tools like process average control charts. These charts aid in identifying deviations from expected downtime durations and facilitate corrective actions to maintain operational efficiency and minimize disruptions. They enhance production reliability, resource utilization and support continuous improvement efforts aimed at reducing overall downtime resulting in improvement of productivity.

Numerous control charts have been developed for process location and scale parameters based on the assumption of normal distribution. The control charts for process location based on various approaches under normal model are discussed in [11], [13], [15], [18], [21] and [22]. The nonparametric control charts offer an alternative method to monitor ongoing processes suitable for scenarios where the distributional form of process variables is not known or it is non-normal. The nonparametric control charts based on distribution-free statistics are deliberated in [2],[4], [7], [8], [16],[17] and [23].

The quantile based statistics offering robust tools are helpful in assessing the location parameter. Various approaches of defining quantiles are discussed in [5] , [12] and [20] elaborates on r^{th} midrange and discuss its asymptotic variance. The sampling distribution of quantiles is explored in [10] along with providing foundational insights into their statistical properties. The control charts whose control limits are depending on quantiles of non-normal distributions are discussed in [3].

In recent years, the prime focus has been towards developing control charts for exponential distribution. For instance, a control chart to jointly monitor both process location and scale is proposed by [9]. An in-depth examination of the theoretical foundations and practical techniques associated with exponential distribution is provided in [1]. A control chart to monitor process stability using exponential distribution modelling for event times is studied in [19]. The control charts for joint monitoring of origin and scale parameters for ongoing processes are proposed in [14]. A median control chart is suggested to monitor process median under non normality including exponential distribution in [6].

The motivation for proposing a class of control charts based on r^{th} midrange when process variables are taken from exponential distribution is to intensify either sensitivity or robustness of control charts. Also, identifying an optimal control chart among the proposed class of control charts that rationalizes the application of the control chart under exponential model.

In this paper, we propose a class of control charts based on r^{th} midrange and obtain the optimal control chart to detect shift in the process location parameter. Section 2 elaborates on the features of r^{th} midrange. Section 3 deals with the proposed class of control charts and their evaluation. Section 4 and 5, respectively, deal with the illustration of control charts and conclusions based on our observations.

2. Role of r^{th} Midrange and its Distribution

Suppose $X_1,\ X_2,\ldots,X_n$ is a random sample of size n taken from exponential distribution $E\left(\mu,\ \lambda^{-1}\right)$ whose pdf is given in (1) and μ, λ are location and scale parameters respectively. The r^{th} midrange due to [20] is given by

$$M_r = \frac{1}{2}\left(X_{(r)} + X_{(n-r+1)}\right), \quad r = 1, 2, \ldots, \left\lfloor \frac{n}{2} + \frac{1}{2} \right\rfloor \tag{2}$$

where $X_{(r)}$ is the r^{th} order statistic and [y] is the largest integer $\leq\ y$.
The r^{th} midrange is known for its wide range of sensitivity as well as robustness to outliers. It also provides a reliable measure of central tendency which is ideal for skewed or heavy-tailed distributions. Its flexibility allows it to adapt to various statistical measures, including median, midquartile and midpercentiles. Its minimal data requirement and computational simplicity compared to measures like arithmetic mean enhances its utility. These features make it important for real time applications and statistical analysis that need quick-reliable summaries of the data.

$$\text{For } r = 1, \text{ we have,} \quad M_1 = \frac{1}{2}\left(X_{(1)} + X_{(n)}\right) \tag{3}$$

which is midrange given by the average of extreme order observations of a sample. It is a sensitive measure and offers a simple, yet effective summary of the location.
For $r = \frac{n+1}{2}, \frac{n+1}{4}, \frac{n+1}{10}$ and $\frac{n+1}{100}$ we get respectively, the

$$\text{median,} \quad M_d = X_{\left(\frac{n+1}{2}\right)} \tag{4}$$

$$\text{midquartile or midhinge,} \quad M_h = \frac{X_{\left(\frac{n+1}{4}\right)} + X_{\left(3\frac{n+1}{4}\right)}}{2}, \tag{5}$$

$$\text{middecile,} \quad M_{D_i} = \frac{X_{\left(i\frac{n+1}{10}\right)} + X_{\left((10-i)\frac{n+1}{10}\right)}}{2}, \quad i = 1, 2, \ldots, 5 \tag{6}$$

$$\text{and midpercentile,} \quad M_{P_i} = \frac{X_{\left(i\frac{n+1}{100}\right)} + X_{\left((100-i)\frac{n+1}{100}\right)}}{2}, \quad i = 1, 2, \ldots, 50. \tag{7}$$

Also, M_r is called as quasi midrange for $r \geq 2$.

Further, the p^{th} sample quantile is given by $X_{(r)}$ with

$$r = \begin{cases} np, & \text{if } np \text{ is an integer} \\ [np] + 1, & \text{if } np \text{ is not an integer} \end{cases} \tag{8}$$

where 0 < p < 1. Here, we get the median at $p = \frac{1}{2}$.

When $r = [np] + 1$, the midquantile is given by

$$M_q = \frac{X_{([np]+1)} + X_{(n-[np])}}{2}$$

$$= \frac{X_{(p(n+1))} + X_{((1-p)(n+1))}}{2} \tag{9}$$

As given in [10], suppose z_p is a sample quantile, then

$$z_p \sim N\left(\xi_p, \frac{p(1-p)}{nf^2\left(\xi_p\right)}\right) \tag{10}$$

where ξ_p is a population quantile given by $\xi_p = F^{-1}(p)$ and $f\left(\xi_p\right)$ is the pdf evaluated at ξ_p.
Hence,

$$M_q = \frac{z_p + z_{1-p}}{2} \sim N\left(\frac{\xi_p + \xi_{1-p}}{2}, \sigma^2_{M_q}\right)$$

where

$$\sigma^2_{M_q} = \frac{1}{4}\left[\frac{p(1-p)}{nf^2\left(\xi_p\right)} + \frac{(1-p)p}{nf^2\left(\xi_{1-p}\right)} + \frac{2p^2}{nf\left(\xi_p\right)f\left(\xi_{1-p}\right)}\right]. \tag{11}$$

Under exponential distribution,

$$\xi_p = \mu - \lambda \log(1-p) \tag{12}$$

and

$$f\left(\xi_p\right) = \frac{1-p}{\lambda}. \tag{13}$$

Therefore,

$$E\left(M_q\right) = E\left(\frac{z_p + z_{1-p}}{2}\right)$$
$$= \frac{\mu - \lambda \log(1-p) + \mu - \lambda \log(p)}{2}$$

$$= \mu - \frac{\lambda}{2}\log(p(1-p)) \tag{14}$$

Defining $M_q^* = M_q + \frac{\lambda}{2}\log(p(1-p))$,

$$\text{we get} \quad E\left(M_q^*\right) = \mu \quad \text{and} \quad \sigma^2_{M_q^*} = \sigma^2_{M_q}. \tag{15}$$

$$\begin{aligned}
\text{Therefore,}\ \ \sigma^2_{M_q^*} &= \frac{1}{4}\left[\frac{p(1-p)\lambda^2}{n(1-p)^2} + \frac{(1-p)p\lambda^2}{np^2} + \frac{2p^2\lambda^2}{n(1-p)p}\right] \\
&= \frac{\lambda^2}{4n}\left[\frac{p}{(1-p)} + \frac{(1-p)}{p} + \frac{2p}{(1-p)}\right] \\
&= \frac{\lambda^2}{4n}\left[\frac{p^2 + (1-p)^2 + 2p^2}{p(1-p)}\right] \\
&= \frac{\lambda^2}{4n}\left[\frac{4p^2 - 2p + 1}{p(1-p)}\right], \quad 0 < p \le \frac{1}{2}.
\end{aligned} \tag{16}$$

However, to obtain the mean and variance of M_1, we consider the following results due to [1].

$$E\left(X_{(r)}\right) = \mu + \lambda \sum_{j=1}^{r} \frac{1}{n-j+1}, \tag{17}$$

$$\text{Var}\left(X_{(r)}\right) = \lambda^2 \sum_{j=1}^{r} \frac{1}{(n-j+1)^2}, \quad r = 1, 2, \ldots, n \tag{18}$$

$$\text{and}\ \ \text{Cov}\left(X_{(r)}, X_{(s)}\right) = \lambda^2 \sum_{j=1}^{r} \frac{1}{(n-j+1)^2}, \quad 1 \le r \le s \le n. \tag{19}$$

Now,

$$E\left(X_{(1)}\right) = \mu + \frac{\lambda}{n} \quad \text{and} \quad E\left(X_{(n)}\right) = \mu + \lambda\log(n).$$

Hence,

$$\begin{aligned}
E\left(M_1\right) &= \frac{E\left(X_{(1)}\right) + E\left(X_{(n)}\right)}{2} \\
&= \frac{\mu + \frac{\lambda}{n} + \mu + \lambda\log(n)}{2} \\
&= \mu + \frac{\lambda}{2}\left[\frac{1}{n} + \log(n)\right].
\end{aligned}$$

$$\therefore E\left(M_1^*\right) = E\left(M_1 - \frac{\lambda}{2}\left[\frac{1}{n} + \log(n)\right]\right) = \mu. \tag{20}$$

$$\text{Also,} \quad \text{Var}\left(X_{(1)}\right) = \frac{\lambda^2}{n^2}, \quad \text{Var}\left(X_{(n)}\right) = \lambda^2, \quad \text{and} \quad \text{Cov}\left(X_{(1)}, X_{(n)}\right) = \frac{\lambda^2}{n^2}.$$

$$\begin{aligned}
\text{Therefore,}\ \ \text{Var}\left(M_1^*\right) = \sigma^2_{M_1^*} &= \frac{1}{4}\left[\text{Var}\left(X_{(1)}\right) + \text{Var}\left(X_{(n)}\right) + 2\cdot\text{Cov}\left(X_{(1)}, X_{(n)}\right)\right] \\
&= \frac{1}{4}\left[\frac{\lambda^2}{n^2} + \lambda^2 + \frac{2\lambda^2}{n^2}\right] \\
&= \frac{\lambda^2}{4}\left[\frac{3}{n^2} + 1\right].
\end{aligned} \tag{21}$$

3. Control charts and their evaluation

In this section, we propose a class of shewhart type control charts based on r^{th} midrange, using the mean and standard deviation (sd) of the appropriate statistics. The control limits of M_r^* control charts are given by

$$UCL_{M_r^*} = E\left(M_r^*\right) + 3\sigma_{M_r^*}, \quad CL_{M_r^*} = E\left(M_r^*\right), \quad LCL_{M_r^*} = E\left(M_r^*\right) - 3\sigma_{M_r^*}. \tag{22}$$

where $UCL_{M_r^*}$, $CL_{M_r^*}$ and $LCL_{M_r^*}$ is the upper control limit, center line and lower control limit of M_r^*control chart. $E\left(M_r^*\right)$ and $\sigma_{M_r^*}$ represent respectively the mean and sd of M_r^* control chart. The variance of some members of the proposed class of control charts, M_r^* are furnished in Exhibit 1.

Exhibit 1: Variance of various members of r^{th} midrange

M_r	M_r^*	$\sigma^2_{M_r^*}$
M_1	$M_1 - \frac{\lambda}{2}\left[\frac{1}{n} + \log(n)\right]$	$\frac{\lambda^2}{4}\left[\frac{3}{n^2} + 1\right]$
M_{D_1}	$M_{D_1} - 1.2040\lambda$	$2.3333\frac{\lambda^2}{n}$
$M_{D_2} = M_{P_{20}}$	$M_{D_2} - 0.9163\lambda$	$1.1875\frac{\lambda^2}{n}$
M_h	$M_h - 0.8370\lambda$	$\frac{\lambda^2}{n}$
$M_{D_3} = M_{P_{30}}$	$M_{D_3} - 0.7803\lambda$	$0.9048\frac{\lambda^2}{n}$
$M_{D_4} = M_{P_{40}}$	$M_{D_4} - 0.7136\lambda$	$0.8750\frac{\lambda^2}{n}$
$M_{D_5} = M_{P_{50}} = M_d$	$M_{D_5} - 0.6931\lambda$	$\frac{\lambda^2}{n}$

Since $E\left(M_d^*\right) = E(M_h^*)$ and $\sigma^2_{M_d^*} = \sigma^2_{M_h^*}$, the M_d^* and M_h^* control charts and their performances will be the same.
The performance of proposed M_r^* control chart is evaluated using some performance measures viz. power, $P_{M_r^*}$; average run length, $ARL_{M_r^*}$; median run length, $MRL_{M_r^*}$ and sd of run length, $SDRL_{M_r^*}$. These measures are defined as

$$P_{M_r^*} = 1 - \beta_{M_r^*} \tag{23}$$

$$\text{where,} \quad \beta_{M_r^*} = P\left(LCL_{M_r^*} < M_r^* < UCL_{M_r^*} \mid \mu'\right) \tag{24}$$

is the operating characteristic (OC) function,

$$ARL_{M_r^*} = \frac{1}{P_{M_r^*}} \tag{25}$$

$$MRL_{M_r^*} = \frac{\log(0.5)}{\log\left(1 - P_{M_r^*}\right)} \tag{26}$$

$$\text{and} \quad SDRL_{M_r^*} = \sqrt{ARL_{M_r^*}\left(1 - ARL_{M_r^*}\right)} \tag{27}$$

The $\beta_{M_r^*}$ represents the probability of not detecting the shift a in the process location parameter in the first subsequent sample if the process location shifts from μ to $\mu' = \mu + a$. The $P_{M_r^*}$ indicates the effectiveness of control chart in detecting a shift in the process location parameter. The $ARL_{M_r^*}$ represents the average number of samples needed to detect the shift and measures how quickly the control chart responds to process shifts, with a lower ARL indicating faster detection. $MRL_{M_r^*}$ provides the median number of samples required to detect a shift and it complements the ARL by offering a central tendency measure that is less influenced by extreme values. The $SDRL_{M_r^*}$ measures the variability in the number of samples needed to detect a shift and asses the consistency of control chart's performance, with lower SDRL indicating higher consistency. The values of $P_{M_r^*}$, $ARL_{M_r^*}$, $MRL_{M_r^*}$ and $SDRL_{M_r^*}$ are computed by setting $\lambda^2 = 1$. The values of $P_{M_r^*}$, $ARL_{M_r^*}$ are presented in Table 1 and $MRL_{M_r^*}$, $SDRL_{M_r^*}$ in Table 2.

When a random sample is taken from exponential distribution, the distributional form of midrange is not known. Hence using R program, we evaluate various performance measures of M_1^* for different values of n and a. The computed values of these performance measures are presented in Table 3. All the tables are given in appendix.

For various values of a and $n = 10$, we plot Figure 1 using Table 1 and 2. Figure 2 is plotted using Table 3 for different values on n.

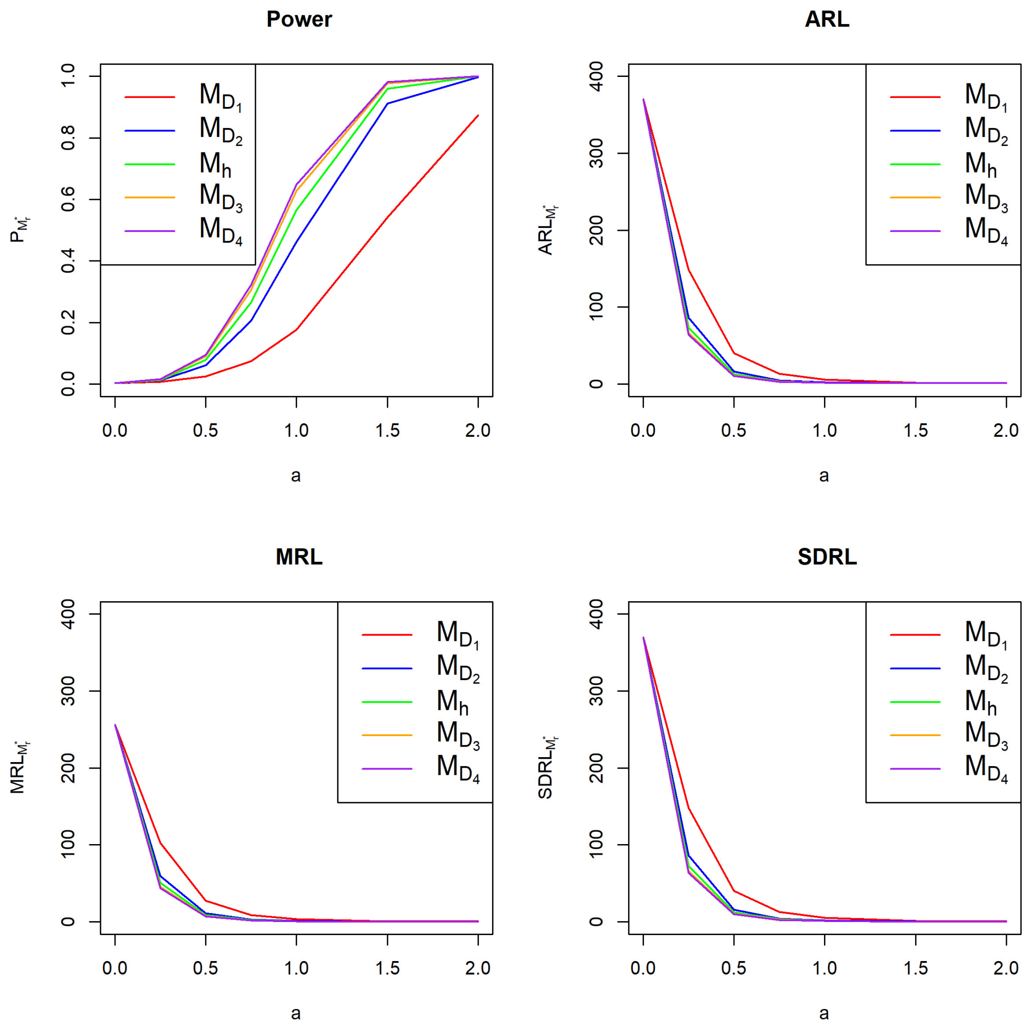

Figure 1: *Performance measures of M_r^* control charts*

From Table 1, Table 2 and Figure 1, we observe that, for fixed n and increasing a, $P_{M_r^*}$ increases, where as $ARL_{M_r^*}$, $MRL_{M_r^*}$ and $SDRL_{M_r^*}$ decreases. At $a = 0$ for various values of n, $P_{M_r^*}$ is 0.0027, $ARL_{M_r^*}$, $MRL_{M_r^*}$ and $SDRL_{M_r^*}$ are approximately 370, 256 and 369 respectively. Additionally, across different control charts, it is observed that, the $M_{D_2}^*$ control chart performs better than $M_{D_1}^*$, M_h^* is better than $M_{D_2}^*$ and $M_{D_3}^*$ outperforms M_h^* indicating a progressive improvement in performance of control charts as decile value increases.

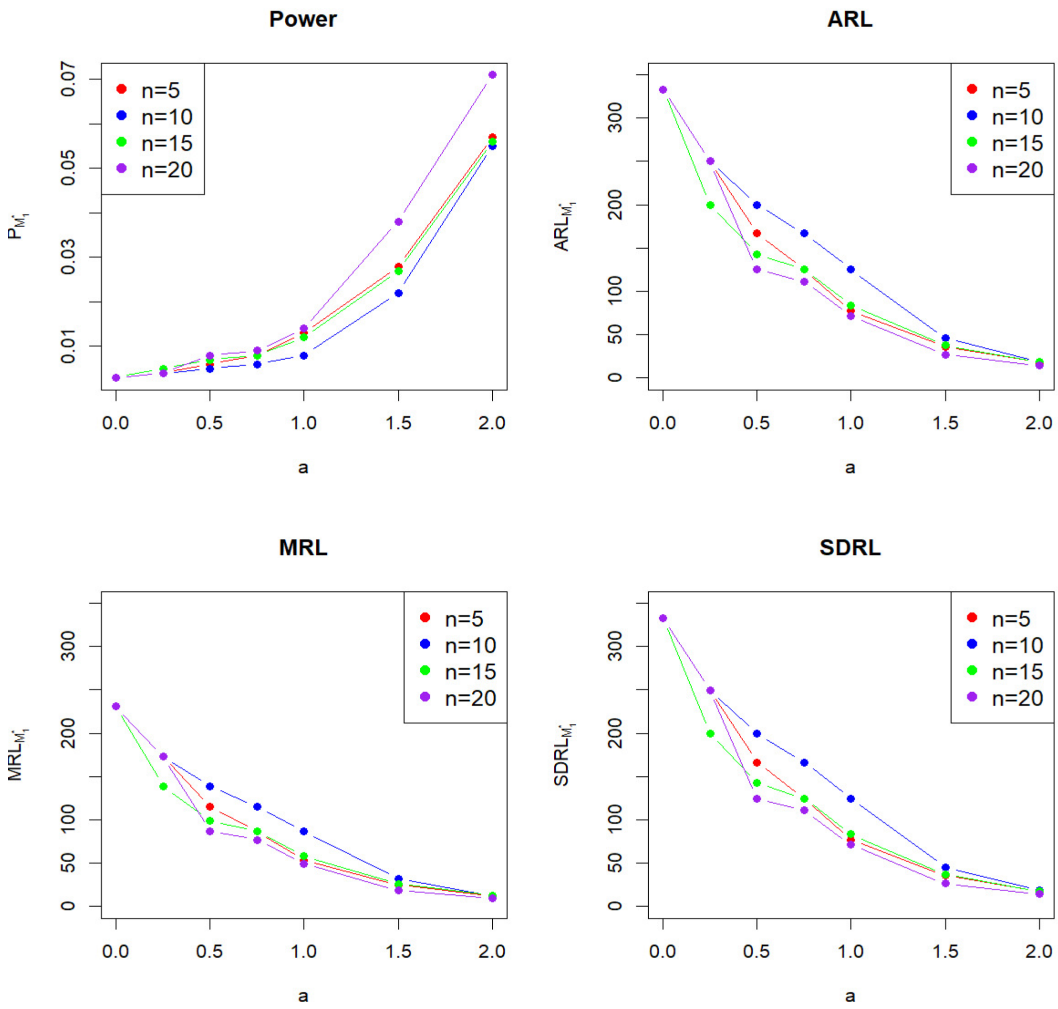

Figure 2: *Performance measures of M_1 control chart for various values of n and a*

From Table 3 and Figure 2, we observe that, as n increases, for a specified shift a, $P_{M_1^*}$ increases, $ARL_{M_1^*}$, $MRL_{M_1^*}$ and $SDRL_{M_1^*}$ decrease.

We also observe from exhibit 1 that, $\sigma^2_{M^*_{P_{30}}} > \sigma^2_{M^*_{P_{40}}}$, which reflects that there is a decreasing trend in the variance of 30^{th} percentile to 40^{th} percentile. Hence, we obtain $\sigma^2_{M^*_{P_{35}}} = 0.8681\frac{\lambda^2}{n} > \sigma^2_{M^*_{P_{40}}}$. Also, we evaluate the values of 36^{th} to 38^{th} percentile and are given by $\sigma^2_{M^*_{P_{36}}} = 0.8663\frac{\lambda^2}{n}$, $\sigma^2_{M^*_{P_{37}}} = 0.8662\frac{\lambda^2}{n}$, $\sigma^2_{M^*_{P_{38}}} = 0.8676\frac{\lambda^2}{n}$. We see that, $\sigma^2_{M^*_{P_{36}}} > \sigma^2_{M^*_{P_{37}}} < \sigma^2_{M^*_{P_{38}}}$ yielding minimum variance for 37^{th} percentile. Hence, we compute various performance measures of $M^*_{P_{37}}$ control chart for various values of n and present in Exhibit 2.

Exhibit 2: Performance measures for the $M^*_{P_{37}}$ control chart

a \ n	5	10	15	20	5	10	15	20
	$P_{M^*_{P_{37}}}$				$\mathrm{ARL}_{M^*_{P_{37}}}$			
0.00	0.0027	0.0027	0.0027	0.0027	370.3983	370.3983	370.3983	370.3983
0.25	0.0084	0.0158	0.0250	0.0360	119.4651	63.2330	39.9287	27.7430
0.50	0.0360	0.0966	0.1790	0.2751	27.7430	10.3510	5.5877	3.6349
0.75	0.1154	0.3257	0.5482	0.7270	8.6621	3.0699	1.8243	1.3755
1.00	0.2751	0.6546	0.8773	0.9645	3.6349	1.5277	1.1399	1.0368
1.50	0.7270	0.9820	0.9994	1.0000	1.3755	1.0183	1.0006	1.0000
2.00	0.9645	0.9999	1.0000	1.0000	1.0368	1.0001	1.0000	1.0000
	$\mathrm{MRL}_{M^*_{P_{37}}}$				$\mathrm{SDRL}_{M^*_{P_{37}}}$			
0.00	256.3938	256.3938	256.3938	256.3938	369.8980	369.8980	369.8980	369.8980
0.25	82.4598	43.4823	27.3284	18.8813	118.9640	62.7310	39.4255	27.2384
0.50	18.8813	6.8223	3.5151	2.1544	27.2384	9.8383	5.0630	3.0948
0.75	5.6505	1.7586	0.8725	0.5339	8.1468	2.5208	1.2262	0.7186
1.00	2.1544	0.6521	0.3304	0.2077	3.0948	0.8978	0.3994	0.1954
1.50	0.5339	0.1726	0.0933	0.0616	0.7186	0.1367	0.0244	0.0036
2.00	0.2077	0.0728	0.0413	0.0281	0.1954	0.0086	0.0002	0.0000

From Exhibit 2 and Tables 1, 2 and 3, we observe that, $M^*_{P_{37}}$ control chart displays the highest power and the lowest ARL, MRL and SDRL when compared to other control charts within the proposed class. Hence, we consider $M^*_{P_{37}}$ as the optimal control chart among the class of M^*_r control charts.

4. Illustration

In this section, we provide an example to illustrate the class of M_r control charts using partial data from [3]. The dataset comprises the failure times of light bulbs, recorded in units across 10 samples, each consisting of 10 observations. Various values of M_r for each sample is computed in Exhibit 3 (a). Here, $n = 10$, μ is estimated by $\overline{M}_r = \frac{1}{10}\sum_{i=1}^{10} M_{r_i}$ and σ_{M_r} by $\widehat{\sigma}_{M_r} = \delta\widehat{\lambda}$, where $\delta = \sqrt{\frac{n-1}{2}}\left(\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right)$, $\widehat{\lambda} = \frac{1}{10}\sum_{i=1}^{10} s_i$, $s_i^2 = \frac{1}{9}\sum_{i=1}^{10}(x_i - \overline{x})^2$.

Exhibit 3 (a): CL_{M_r} of various control charts

Sl. no	M_1	$M_{(D_1)}$	$M_{(D_2)}$	M_h	$M_{(D_3)}$	$M_{(P_{37})}$	$M_{(D_4)}$	$\bar{x}$	s_i
1	0.6600	1.0025	0.8960	0.8438	0.7965	0.7155	0.6750	0.8550	0.5355
2	0.6500	1.1275	1.0340	1.0112	1.0135	0.9474	0.8920	1.0140	0.4709
3	1.1700	1.0020	0.9840	0.9725	0.9590	0.9368	0.9260	0.9720	0.5548
4	1.6100	0.6885	0.6320	0.5925	0.5340	0.4851	0.4770	0.6380	0.4327
5	0.6500	0.9640	0.9200	0.8887	0.8505	0.7970	0.7740	0.8870	0.4436
6	0.9500	0.7370	0.7460	0.7688	0.7935	0.8578	0.8970	0.8420	0.4411
7	0.9800	0.7125	0.6460	0.6512	0.6895	0.7150	0.7150	0.7010	0.3990
8	1.1700	1.0760	0.9140	0.8700	0.8700	0.8485	0.8310	0.9170	0.4607
9	1.6100	0.8310	0.8360	0.7888	0.6965	0.6136	0.5960	0.7590	0.5363
10	1.2500	1.0230	0.9890	1.0100	1.0640	1.0719	1.0490	1.0250	0.5394
$CL_{M_r} = \bar{M}_r$	1.0700	0.9164	0.8597	0.8398	0.8267	0.7989	0.7832	$\bar{x} = 0.8610$	$\hat{\lambda} = 0.4814$

Exhibit 3 (b): σ_{M_r}, UCL_{M_r}, CL_{M_r}, LCL_{M_r} and w_{M_r} of various control charts

Statistic	σ_{M_r}	UCL_{M_r}	CL_{M_r}	LCL_{M_r}	w_{M_r}
M_1	0.2511	1.8234	1.0700	0.3166	1.5069
M_{D_1}	0.2391	1.6336	0.9164	0.1992	1.4344
M_{D_2}	0.1706	1.3714	0.8597	0.3480	1.0233
M_h	0.1565	1.3093	0.8398	0.3703	0.9391
M_{D_3}	0.1489	1.2733	0.8267	0.3801	0.8932
$M_{P_{37}}$	0.1457	1.2359	0.7989	0.3619	0.8740
M_{D_4}	0.1464	1.2224	0.7832	0.3440	0.8784

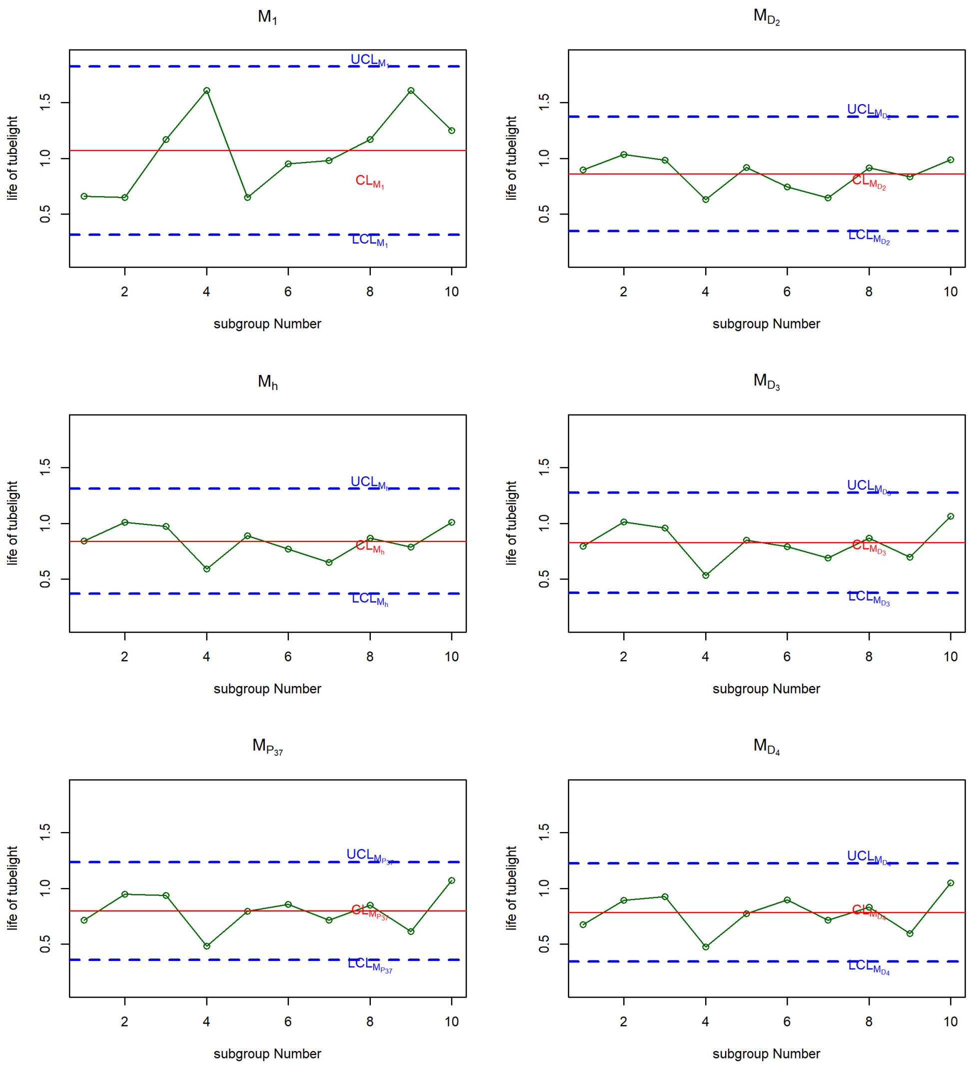

Figure 3: *M_1, M_{D_2}, M_h, M_{D_3}, $M_{P_{37}}$, M_{D_4} control charts*

From Exhibit 3 (b) and Figure 3 it is observe that, the M_1, M_{D_2}, M_h, M_{D_3}, $M_{P_{37}}$ and M_{D_4} control charts show that the process is in control. Further, the w_{M_r} is largest for M_{D_1} control chart and smallest for $M_{P_{37}}$ control chart.

5. Conclusions

In this section, we record our conclusions about the proposed class of M_r control charts based on our findings.

- A class of control charts based on r^{th} sample midrange is proposed for location parameter when process is under exponential model.
- The proposed class of control charts includes midrange, mid quantile, middecile, midpercentile, midhinge and median control charts as its members.
- M_r is biased estimator. Hence, after adjusting the bias, it is renamed as M_r^*.
- When bias of M_h and M_d estimators are adjusted, the M_h^* and M_d^* control charts are the same.
- The power of the proposed class increases for smaller shifts as the sample size increases, exhibiting greater ability to detect shifts in process location.
- ARL and MRL decrease as sample size increases indicating improved performance of the control charts.
- As sample size increases, the performance of the control charts stabilizes.
- Among the various members of the proposed class, the control chart based on 37^{th} percentile, $M_{P_{37}}^*$ outperforms other control charts establishing its optimality.

Appendix

Table 1: *$P_{M_r^*}$ and $ARL_{M_r^*}$ of various control charts*

Statistic	a \ n	5	10	15	20	5	10	15	20
		$\mathbf{P_{M_r^*}}$				$\mathbf{ARL_{M_r^*}}$			
M_{D_1}	0.00	0.0027	0.0027	0.0027	0.0027	370.3983	370.3983	370.3983	370.3983
	0.25	0.0046	0.0067	0.0091	0.0118	217.3808	148.3252	109.5630	85.0522
	0.50	0.0118	0.0247	0.0416	0.0623	85.0522	40.4195	24.0249	16.0627
	0.75	0.0286	0.0739	0.1360	0.2106	34.9655	13.5312	7.3520	4.7475
	1.00	0.0623	0.1762	0.3211	0.4712	16.0627	5.6741	3.1139	2.1223
	1.50	0.2106	0.5419	0.7891	0.9180	4.7475	1.8452	1.2673	1.0894
	2.00	0.4712	0.8729	0.9808	0.9979	2.1223	1.1455	1.0196	1.0022
M_{D_2}	0.00	0.0027	0.0027	0.0027	0.0027	370.3983	370.3983	370.3983	370.3983
	0.25	0.0067	0.0116	0.0174	0.0242	150.0911	86.4698	57.4181	41.2916
	0.50	0.0242	0.0607	0.1107	0.1716	41.2916	16.4773	9.0355	5.8291
	0.75	0.0720	0.2051	0.3690	0.5311	13.8877	4.8759	2.7098	1.8830
	1.00	0.1716	0.4609	0.7102	0.8652	5.8291	2.1695	1.4080	1.1558
	1.50	0.5311	0.9119	0.9901	0.9992	1.8830	1.0966	1.0100	1.0008
	2.00	0.8652	0.9975	1.0000	1.0000	1.1558	1.0025	1.0000	1.0000
M_h	0.00	0.0027	0.0027	0.0027	0.0027	370.3983	370.3983	370.3983	370.3983
	0.25	0.0075	0.0136	0.0211	0.0299	133.1594	73.2735	47.3362	33.4008
	0.50	0.0299	0.0780	0.1438	0.2225	33.4008	12.8251	6.9553	4.4953
	0.75	0.0929	0.2649	0.4621	0.6384	10.7611	3.7749	2.1643	1.5665
	1.00	0.2225	0.5645	0.8087	0.9295	4.4953	1.7716	1.2366	1.0758
	1.50	0.6384	0.9594	0.9975	0.9999	1.5665	1.0424	1.0025	1.0001
	2.00	0.9295	0.9996	1.0000	1.0000	1.0758	1.0004	1.0000	1.0000
M_{D_3}	0.00	0.0027	0.0027	0.0027	0.0027	370.3983	370.3983	370.3983	370.3983
	0.25	0.0081	0.0151	0.0238	0.0340	123.5647	66.1805	42.0789	29.3723
	0.50	0.0340	0.0905	0.1675	0.2581	29.3723	11.0505	5.9706	3.8744
	0.75	0.1081	0.3062	0.5214	0.7006	9.2538	3.2656	1.9177	1.4273
	1.00	0.2581	0.6272	0.8581	0.9556	3.8744	1.5943	1.1654	1.0465

	1.50	0.7006	0.9765	0.9991	1.0000	1.4273	1.0240	1.0009	1.0000
	2.00	0.9556	0.9999	1.0000	1.0000	1.0465	1.0001	1.0000	1.0000
$\mathbf{M_{D_4}}$	0.00	0.0027	0.0027	0.0027	0.0027	370.3983	370.3983	370.3983	370.3983
	0.25	0.0083	0.0156	0.0247	0.0356	120.4118	63.9092	40.4202	28.1145
	0.50	0.0356	0.0952	0.1762	0.2711	28.1145	10.5096	5.6742	3.6889
	0.75	0.1137	0.3211	0.5419	0.7210	8.7961	3.1140	1.8453	1.3870
	1.00	0.2711	0.6483	0.8729	0.9625	3.6889	1.5426	1.1456	1.0389
	1.50	0.7210	0.9808	0.9993	1.0000	1.3870	1.0196	1.0007	1.0000
	2.00	0.9625	0.9999	1.0000	1.0000	1.0389	1.0001	1.0000	1.0000

Table 2: *$MRL_{M_r^*}$ and $SDRL_{M_r^*}$ of various control charts*

Statistic	**n** / **a**	**5**	**10**	**15**	**20**	**5**	**10**	**15**	**20**
		$\mathbf{MRL_{M_{r^*}}}$				$\mathbf{SDRL_{M_{r^*}}}$			
M_{D_1}	0.00	256.3938	256.3938	256.3938	256.3938	369.8980	369.8980	369.8980	369.8980
	0.25	150.3300	102.4642	75.5962	58.6064	216.8802	147.8243	109.0618	84.5507
	0.50	58.6064	27.6686	16.3037	10.7835	84.5507	39.9164	23.5195	15.5546
	0.75	23.8880	9.0281	4.7410	2.9305	34.4619	13.0216	6.8337	4.2179
	1.00	10.7835	3.5752	1.7895	1.0879	15.5546	5.1498	2.5657	1.5433
	1.50	2.9305	0.8878	0.4454	0.2772	4.2179	1.2489	0.5820	0.3120
	2.00	1.0879	0.3360	0.1753	0.1128	1.5433	0.4083	0.1412	0.0465
M_{D_2}	0.00	256.3938	256.3938	256.3938	256.3938	369.8980	369.8980	369.8980	369.8980
	0.25	103.6882	59.5891	39.4516	28.2732	149.5902	85.9683	56.9159	40.7885
	0.50	28.2732	11.0710	5.9096	3.6830	40.7885	15.9695	8.5208	5.3056
	0.75	9.2753	3.0199	1.5052	0.9153	13.3784	4.3472	2.1525	1.2895
	1.00	3.6830	1.1218	0.5596	0.3459	5.3056	1.5929	0.7579	0.4244
	1.50	0.9153	0.2853	0.1501	0.0972	1.2895	0.3254	0.1004	0.0283
	2.00	0.3459	0.1159	0.0640	0.0429	0.4244	0.0504	0.0045	0.0003
M_h	0.00	256.3938	256.3938	256.3938	256.3938	369.8980	369.8980	369.8980	369.8980
	0.25	91.9521	50.4419	32.4632	22.8033	132.6585	72.7718	46.8335	32.8970
	0.50	22.8033	8.5384	4.4655	2.7548	32.8970	12.3150	6.4359	3.9639
	0.75	7.1068	2.2523	1.1180	0.6815	10.2489	3.2365	1.5874	0.9420
	1.00	2.7548	0.8340	0.4191	0.2613	3.9639	1.1692	0.5409	0.2856
	1.50	0.6815	0.2164	0.1155	0.0756	0.9420	0.2101	0.0499	0.0102
	2.00	0.2613	0.0898	0.0503	0.0340	0.2856	0.0211	0.0010	0.0000
M_{D_3}	0.00	256.3938	256.3938	256.3938	256.3938	369.8980	369.8980	369.8980	369.8980
	0.25	85.3015	45.5254	28.8189	20.0108	123.0637	65.6786	41.5759	28.8680
	0.50	20.0108	7.3076	3.7814	2.3217	28.8680	10.5387	5.4477	3.3371
	0.75	6.0611	1.8959	0.9405	0.5747	8.7396	2.7201	1.3267	0.7809
	1.00	2.3217	0.7024	0.3550	0.2226	3.3371	0.9734	0.4390	0.2205
	1.50	0.5747	0.1847	0.0995	0.0655	0.7809	0.1569	0.0307	0.0050
	2.00	0.2226	0.0776	0.0438	0.0297	0.2205	0.0115	0.0004	0.0000
$\mathbf{M_{D_4}}$	0.00	256.3938	256.3938	256.3938	256.3938	369.8980	369.8980	369.8980	369.8980
	0.25	83.1160	43.9510	27.6691	19.1388	119.9107	63.4073	39.9171	27.6100
	0.50	19.1388	6.9323	3.5753	2.1922	27.6100	9.9971	5.1500	3.1495
	0.75	5.7435	1.7896	0.8878	0.5431	8.2810	2.5657	1.2489	0.7327
	1.00	2.1922	0.6634	0.3360	0.2110	3.1495	0.9149	0.4083	0.2011
	1.50	0.5431	0.1753	0.0947	0.0625	0.7327	0.1412	0.0258	0.0039
	2.00	0.2110	0.0739	0.0419	0.0285	0.2011	0.0092	0.0003	0.0000

Table 3: *$MRL_{M_r^*}$ and $SDRL_{M_r^*}$ of various control charts*

a \ n	5	10	15	20	5	10	15	20
	$\mathbf{P_{M_{r^*}}}$				$\mathbf{ARL_{M_{r^*}}}$			
0.00	0.0030	0.0030	0.0030	0.0030	333.3333	333.3333	333.3333	333.3333
0.25	0.0040	0.0040	0.0050	0.0040	250.0000	250.0000	200.0000	250.0000
0.50	0.0060	0.0050	0.0070	0.0080	166.6667	200.0000	142.8571	125.0000
0.75	0.0080	0.0060	0.0080	0.0090	125.0000	166.6667	125.0000	111.1111
1.00	0.0130	0.0080	0.0120	0.0140	76.9231	125.0000	83.3333	71.4286
1.50	0.0280	0.0220	0.0270	0.0380	35.7143	45.4545	37.0370	26.3158
2.00	0.0570	0.0550	0.0560	0.0710	17.5439	18.1818	17.8571	14.0845
	$\mathbf{MRL_{M_{1^*}}}$				$\mathbf{SDRL_{M_{1^*}}}$			
0.00	230.7023	230.7023	230.7023	230.7023	332.8330	332.8330	332.8330	332.8330
0.25	172.9400	172.9400	138.2826	172.9400	249.4995	249.4995	199.4994	249.4995
0.50	115.1776	138.2826	98.6741	86.2964	166.1659	199.4994	142.3563	124.4990
0.75	86.2964	115.1776	86.2964	76.6693	124.4990	166.1659	124.4990	110.6100
1.00	52.9717	86.2964	57.4150	49.1631	76.4214	124.4990	82.8318	70.9268
1.50	24.4070	31.1588	25.3240	17.8919	35.2107	44.9518	36.5336	25.8109
2.00	11.8105	12.2528	12.0277	9.4118	17.0365	17.6747	17.3499	13.5753

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