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CHARACTERIZATION OF GENERALIZED DISTRIBUTIONS BASED ON CONDITIONAL EXPECTATION OF ORDER STATISTICS

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Abstract

Characterization of probability distributions plays a significant role in the field of probability and statistics and attracted many researchers these days. Characterization refers to the process of identifying distributions uniquely based on certain statistical properties or functions. The various characterization results have been established by using different methods. The paper aims to characterize two general forms of continuous distributions using the conditional expectation of order statistics. Further, the results obtained are applied to some well-known continuous distributions. Finally, some numerical calculations are performed.

Keywords: Order statistics, conditional expectation; truncated moments, characterization, continuous distributions.

I. Introduction

The characterization of probability distributions is indeed crucial in statistical studies as it allows us to understand and utilize various distributions effectively. The distributions can be characterized using different statistical properties like moments, truncated moments, order statistics, record values, reliability functions, characteristic function etc. Each of these approaches leverages different statistical properties to uniquely identify or characterize specific probability distributions. This allows statisticians and researchers to model data effectively, understand underlying processes, and make informed decisions based on statistical analyses.

The various characterization results using truncated moment and conditional expectation of order statistics are available in the literature. Grudzien and Szynal [15] characterized the uniform distribution in terms of moments of order statistics when the sample size is random whereas Balasubramanian and Beg [11] focused on distribution characterizations using moments and order statistics. Khan and Abu-Salih [18] characterized a general class of distribution through conditional expectations of order statistics. Further, Khan and Abouammoh [17] extended the results of Khan and Abu-Salih [18] by characterized the general form of distribution for higher order gap. Khan and Athar [23] characterized some continuous distributions by examining the linearity of regression when conditioned on a pair of order statistics. Nassar [27] characterized a mixture of two generalized power function distributions based on the conditional expectation of order statistics. Similarly, Lee *et al.* [25] provided a characterization of mixtures of Weibull and Pareto distributions through the conditional expectation of order statistics and upper record values. The recent interest on characterizing probability distributions via truncated moments has led to significant contributions from various authors. For example, Ahsanullah *et al*. [3] explore the characterization of Lindley distribution based on a relation between truncated moments and failure rate function. Kilany [24] established the characterization of the Lindley distribution using the truncated moments of order statistics. Athar and Abdel-Aty [6] characterized a class of continuous distributions based on left and right truncated moments whereas Athar *et al.* [9] studied the characterization of some generalized continuous distribution by doubly truncated moments. Bashir and Khan [12] employed a range of techniques to characterize the weighted power function distribution, using mean inactivity times, mean residual function, conditional moments, conditional variance, doubly truncated mean, incomplete moments, and the reverse hazard function. For more studies on characterization, one can refer to Khan and Khan [21], Khan and Masoom Ali [22], Franco and Ruiz [14], Ali and Khan [4], Khan and Athar [20], Khan and Alzaid [19], Athar *et al*. [10], Ahsanullah and Hamdani [1], Huang and Su [16], Ahsanullah and Shakil [2], Athar and Akhtar [7], Athar *et al.* [8], Ansari *et al.* [5] and references given therein.

Let $X_1, X_2, ..., X_n$ be a random sample of size $n(>2)$ from a continuous population having probability density function (*pdf*) $f(x)$ and cumulative distribution function (*cdf*) $F(x)$ and its corresponding order statistics be the $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$ respectively. It is well Known that the conditional *pdf* of r^{th} order statistic $(X_{r:n} = x)$ given that s^{th} order statistic $(X_{s:n} = y)$ for $s > r$, is the same as the distribution of the r^{th} order statistics obtained from a sample of size $(s-1)$ from a population whose distribution is truncated on the right side at y while the conditional *pdf* of s^{th} order statistic $(X_{s:n} = y)$ given that r^{th} order statistic $(X_{r:n} = x)$ for $r < s$, is the same as the distribution of the $(s-r)^{th}$ order statistics from a sample of size $(n-r)$ from a population whose distribution is simply truncated on the left side at x (David and Nagaraja [13]).

Therefore, the conditional *pdf* of r^{th} order statistic $(X_{r:n} = x)$ given that s^{th} order statistic $(X_{s:n} = y), 1 \le r < s \le n$ is,

$$f(x \mid y) = \frac{(s-1)!}{(r-1)!(s-r-1)!} \frac{\left(F(x)\right)^{r-1}\left(F(y)-F(x)\right)^{s-r-1}}{\left(F(y)\right)^{s-1}} f(x), x \le y \tag{1.1}$$

and the conditional *pdf* of s^{th} order statistic given r^{th} order statistic is

$$f(y \mid x) = \frac{(n-r)!}{(s-r-1)!(n-s)!} \frac{\left(F(y)-F(x)\right)^{s-r-1}\left(1-F(y)\right)^{n-s}}{\left(1-F(x)\right)^{n-r}} f(y), x \le y . \tag{1.2}$$

Also note that for any monotonic and differentiable function $\xi(x)$ of X over the support (α, β). Here α may be $-\infty$ and β may be $+\infty$.

$$E[\xi(X_{n:n}) \mid X_{n-1:n} = x] = E[\xi(X) \mid X \ge x] \tag{1.3}$$

and

$$E[\xi(X_{1:n}) \mid X_{2:n} = x] = E[\xi(X) \mid X \le x] . \tag{1.4}$$

In this paper, first we have characterized two general form of distributions $F(x)=e^{-a\xi(x)}$, $a\neq 0$ and $F(x)=1-e^{-a\xi(x)}$, $a\neq 0$ through the conditional expectation of order statistics $E\left[\xi(X)\mid X_{r:n}=x\right]$, where $X_{r:n}$ is r^{th} order statistics. Further, these results are applied to some well-known continuous distributions.

II. Characterization Theorems

Before presenting the main result, we will discuss the following propositions that were established by Athar *et al*. [10]:

Proposition 2.1: *Let* X *be a random variable and* $E\left[\xi(X)\mid X_{r:n}=x\right]=\mu_r(x)$*, then for* $1\le r<s\le n$,

$$\frac{f(x)}{1-F(x)}=\frac{n(s-1)\mu_r'(x)-n(r-1)\mu_s'(x)-(s-r)\xi'(x)}{n(s-1)\mu_r(x)-n(r-1)\mu_s(x)-n(s-r)\xi(x)}, \tag{2.1}$$

where $\xi(x)$ is a monotonic and differentiable of $x\in(\alpha,\beta)$.

Proposition 2.2: *Under the conditions as stated in Proposition 2.1,*

$$\frac{f(x)}{F(x)}=-\frac{n(n-r)\mu_s'(x)-n(n-s)\mu_r'(x)-(s-r)\xi'(x)}{n(n-r)\mu_s(x)-n(n-s)\mu_r(x)-n(s-r)\xi(x)}. \tag{2.2}$$

Theorem 2.1: *Let* X *be a absolutely continuous (w.r.t Lebesgue measure) random variable with* CDF $F(x)$ *and* PDF $f(x)$ *on the support* (α,β)*, where* α *and* β *may be finite or infinite. Then for* $1\le r<s\le n$

$$E\left[\xi(X)\mid X_{r:n}=x\right]=\mu_r(x)=T_0(x)+\frac{r-1}{n}T_1(x)+\frac{n-r}{n}T_2(x), \tag{2.3}$$

if and only if

$$\bar{F}(x)=e^{-a\xi(x)},\ x\in(\alpha,\beta),\ a\neq 0, \tag{2.4}$$

where,

$$\bar{F}(x)=1-F(x)$$

$$T_0(x)=\frac{\xi(x)}{n},$$

$$T_1(x)=E\left[\xi(X)\mid X\le x\right]=T_2(x)-\frac{\xi(x)}{1-e^{-a\xi(x)}},$$

$$T_2(x)=E\left[\xi(X)\mid X\ge x\right]=\xi(x)+\frac{1}{a},$$

and $\xi(x)$ *is a monotonic and differentiable function of x, such that* $\xi(x)\to 0$ *as* $x\to\alpha$ *and* $\xi(x)\bar{F}(x)\to 0$ *as* $x\to\beta$.

Proof. To prove necessary part, the value of $T_1(x)$ and $T_2(x)$ can be obtained easily by integrating by parts and noting that

$$1-F(x)=\frac{1}{a\xi'(x)}f(x).$$

To prove the sufficiency part, from (2.1) we have

$$\frac{f(x)}{1-F(x)}=\frac{n(s-1)\mu_r'(x)-n(r-1)\mu_s'(x)-(s-r)\xi'(x)}{n(s-1)\mu_r(x)-n(r-1)\mu_s(x)-n(s-r)\xi(x)}=\frac{N}{D}.$$

Now,

$$n(s-1)\mu_r(x)-n(r-1)\mu_s(x)=(s-r)\xi(x)+(n-1)(s-r)T_2(x).$$

Thus,

$$D=\frac{(n-1)(s-r)}{a}$$

and $N=(n-1)(s-r)\xi'(x)$.

Therefore,

$$\frac{f(x)}{1-F(x)}=a\xi'(x).$$

Integrating both the sides with respect to *x*, we get

$$1-F(x)=e^{-a\xi(x)},$$

and hence the sufficiency part.

Theorem 2.2: *Under the conditions as stated in Theorem 2.1.*

$$E\left[\xi(X)\mid X_{r:n}=x\right]=\lambda_r(x)=T_0(x)+\frac{r-1}{n}T_1(x)+\frac{n-r}{n}T_2(x), \tag{2.5}$$

if and only if

$$F(x)=e^{-a\xi(x)};\ \ x\in(\alpha,\beta),\ \ a\neq 0, \tag{2.6}$$

where,

$$T_1(x)=E\left[\xi(X)\mid X\le x\right]=\xi(x)+\frac{1}{a},$$

and

$$T_2(x)=E\left[\xi(X)\mid X\ge x\right]=T_1(x)-\frac{\xi(x)}{1-e^{-a\xi(x)}},$$

such that $\xi(x)F(x)\to 0$ *when* $x\to\alpha$ *and* $\xi(x)\to 0$ *when* $x\to\beta$.

Proof. Necessary part can be proved on the lines of Theorem 2.1 after noting the relation

$$F(x)=e^{-a\xi(x)}=-\frac{f(x)}{a\xi'(x)}$$

To prove sufficiency part, in view of (2.2), we have

$$\frac{f(x)}{F(x)}=-\frac{N}{D}.$$

Now,

$$n\left[(n-r)\lambda_s(x)-(n-s)\lambda_r(x)\right]=(s-r)\xi(x)+(n-1)(s-r)E\left[\xi(x)\mid X\le x\right].$$

Differentiating the above equation *w.r.t.* x, and after rearranging, we get,

$$N=(n-1)(s-r)\xi'(x)$$

$$D=(n-1)(s-r)\left\{\frac{1}{a}\right\}$$

$$\frac{f(x)}{F(x)}=-\frac{N}{D}=-a\xi'(x).$$

Now, integrating both the sides *w.r.t* x, we get

$$F(x)=e^{-a\xi(x)},\ a>0.$$

Hence the result is proved.

III. Examples and Applications

I. Examples Based on Theorem 2.1

1. Weibull distribution

Let the CDF of Weibull distribution is given as

$$F(x)=1-e^{-\lambda x^p}\ ,\ \ x>0,\lambda>0,p>0. \tag{2.7}$$

On comparison of (2.7) with (2.4), we get

$$a=\lambda \text{ and } \xi(x)=x^p.$$

Thus,

$$T_1(x)=E\left[X^p\mid X\le x\right]=\frac{1}{\lambda}-\frac{e^{-\lambda x^p}}{1-e^{-\lambda x^p}}x^p.$$

$$T_2(x)=E\left[X^p\mid X\ge x\right]=x^p+\frac{1}{\lambda}.$$

Therefore,

$$E\left[X^p\mid X_{r:n}=x\right]=\mu_r(x)=\frac{x^p}{n}+\frac{r-1}{n}\left\{\frac{1}{\lambda}-\frac{e^{-\lambda x^p}}{1-e^{-\lambda x^p}}x^p\right\}+\frac{n-r}{n}\left\{x^p+\frac{1}{\lambda}\right\}.$$

$$=x^p+\frac{n-1}{n\lambda}-\frac{r-1}{n}\frac{x^p}{1-e^{-\lambda x^p}}\ ,$$

if and only if

$$F(x)=1-e^{-\lambda x^p},\quad 0\le x<\infty;\ \lambda,p>0.$$

2. Pareto Distribution

Suppose random variable X follows Pareto distribution with CDF given as

$$F(x) = 1 - \nu^p x^{-p}, \; \nu \le x < \infty, \; p > 0. \tag{2.8}$$

By comparing (2.8) with (2.4), we get

$$a = -p, \xi(x) = \log\left(\frac{\nu}{x}\right).$$

Therefore,

$$T_2(x) = E\left[\log\left(\frac{\nu}{X}\right) \mid X \ge x\right] = -\frac{1}{p} + \log\left(\frac{\nu}{x}\right),$$

or $$T_2(x) = E\left[\log X \mid X \ge x\right] = \log x + p,$$

$$T_1(x) = E\left[\log\left(\frac{\nu}{X}\right) \mid X \le x\right] = T_2(x) - \frac{\log\left(\frac{\nu}{x}\right)}{1 - \nu^p x^{-p}},$$

or $$T_1(x) = E\left[\log X \mid X \le x\right] = \log\left(\frac{\nu}{x}\right)\left[\frac{2 - \nu^p x^{-p}}{1 - \nu^p x^{-p}}\right] - p,$$

and $$\mu_r(x) = \frac{1}{n}\log\left(\frac{\nu}{x}\right) + \frac{r-1}{n}\left\{\log\left(\frac{\nu}{x}\right)\left[\frac{2 - \nu^p x^{-p}}{1 - \nu^p x^{-p}}\right] - p\right\} + \frac{n-r}{n}\left(\log x + p\right).$$

Hence,

$$E\left[\log\left(\frac{\nu}{X}\right) \mid X_{r:n} = x\right] = \frac{r}{n}\log\left(\frac{\nu}{x}\right)\left(\frac{2 - \nu^p x^{-p}}{1 - \nu^p x^{-p}}\right) + \frac{n-1}{n}p + \frac{n-r}{n}\log x,$$

or $$E\left[\log X \mid X_{r:n} = x\right] = \log \nu - \frac{r}{n}\log\left(\frac{\nu}{x}\right)\left(\frac{2 - \nu^p x^{-p}}{1 - \nu^p x^{-p}}\right) - \frac{n-1}{n}p - \frac{n-r}{n}\log x,$$

if and only if

$$F(x) = 1 - \nu^p x^{-p}, \quad \nu \le x < \infty.$$

3. Gumbel Extreme Value I

Let the CDF of Gumbel extreme value I distribution is given as

$$F(x) = 1 - e^{-e^x}, \; -\infty < x < \infty. \tag{2.9}$$

Now on comparison of (2.9) with (2.4), we get

$$a = 1, \xi(x) = e^x.$$

Therefore,

$$E\left[e^X \mid X \ge x\right] = T_2(x) = e^x + 1$$

$$E\left[e^X \mid X \le x\right] = T_1(x) = 1 - \frac{e^x e^{-e^x}}{1 - e^{-e^x}},$$

and $$E\left[e^X \mid X_{r:n}=x\right]=\mu_r(x)=\frac{n-r+1}{n}e^x+\frac{r-1}{n}\left(1-\frac{e^x e^{-e^x}}{1-e^{-e^x}}\right)+\frac{n-r}{n}.$$

if and only if

$$F(x)=1-e^{-e^x},\ x>0.$$

Similarly, with proper choice of a and $\xi(x)$ several other distributions can be characterized using Theorem 2.1. For more distribution belonging to this class, one may refer to Khan and Abu-Salih [18] and Noor and Athar [28].

II. Examples Based on Theorem 2.2

1. Inverse Weibull Distribution

Let random variable X follows inverse Weibull distribution with CDF

$$F(x)=e^{-\lambda x^{-p}},0\le x<\infty. \tag{2.10}$$

On comparison of (2.10) with (2.6), we get

Here, $a=\lambda$ and $\xi(x)=x^{-p}$.

Therefore,

$$E\left[X^{-p} \mid X\le x\right]=T_1(x)=x^{-p}+\frac{1}{\lambda},$$

$$E\left[X^{-p} \mid X\ge x\right]=T_2(x)=\frac{1}{\lambda}-\frac{x^{-p}e^{-\lambda x^{-p}}}{1-e^{-\lambda x^{-p}}},$$

and $$E\left(X^{-p} \mid X_{r:n}=x\right)=\lambda_r(x)=\frac{n-1}{n\lambda}+\left(\frac{r}{n}-e^{-\lambda x^{-p}}\right)\left(\frac{x^{-p}}{1-e^{-\lambda x^{-p}}}\right),$$

if and only if

$$F(x)=e^{-\lambda x^{-p}},0\le x<\infty.$$

2. Power Function Distribution

Let the CDF of power function distribution is

$$F(x)=\nu^{-p}x^p,0<x<\nu;\nu,p>0. \tag{2.11}$$

Now on comparing (2.11) with (2.6), we get

$$a=-p \text{ and } \xi(x)=\log\left(\frac{x}{\nu}\right).$$

Therefore,

$$E\left[\log\left(\frac{X}{\nu}\right) \mid X\le x\right]=T_1(x)=\log x-\log\nu-\frac{1}{p}$$

or $$E\left[\log X \mid X\le x\right]=T_1(x)=\log x-\frac{1}{p}.$$

$$E\left[\log\left(\frac{X}{\nu}\right) \mid X\ge x\right]=T_2(x)=\log x-\frac{1}{p}-\frac{\log(x/\nu)}{1-(x/\nu)^p}$$

or $$E\left[\log X \mid X\ge x\right]=T_2(x)=\log\nu+\log x-\frac{1}{p}-\frac{\log(x/\nu)}{1-(x/\nu)^p},$$

and hence

$$E\left[\log\left(\frac{X}{\nu}\right) \mid X_{r:n}=x\right]=\lambda_r(x)=\frac{\log(x/\nu)}{n}+\frac{r-1}{n}\left(\log x-\frac{1}{p}\right)+\frac{n-r}{n}\left(\log \nu x-\frac{1}{p}-\frac{\nu^p\log(x/\nu)}{(\nu^p-x^p)}\right)$$

or $$E\left[\log X \mid X_{r:n}=x\right]=\log x+\frac{2n-r-1}{n}\log\nu-\frac{n-r}{n}\frac{\nu^p}{(\nu^p-x^p)}\log(x/\nu)-\frac{n-1}{np}.$$

if and only if

$$F(x)=\nu^{-p}x^p, 0<x<\nu;\ \nu,p>0.$$

3. Burr type II

Let the CDF of Burr type II distribution is given as

$$F(x)=(1+e^{-x})^{-k} \qquad -\infty<x<\infty. \tag{2.12}$$

Now on comparing (2.12) with (2.6), we get

$$a=k \text{ and } \xi(x)=\log(1+e^{-x}).$$

Therefore,

$$E\left[\log(1+e^{-X}) \mid X\le x\right]=T_1(x)=\log\left(1+e^{-x}\right)+\frac{1}{k},$$

$$E\left[\log(1+e^{-X}) \mid X\ge x\right]=T_2(x)=\log(1+e^{-x})+\frac{1}{k}-\frac{\log(1+e^{-x})}{1-(1+e^{-x})^{-k}},$$

and

$$E\left[\log(1+e^{-X}) \mid X_{r:n}=x\right]=\lambda_r(x)=\frac{n-1}{nk}+\log(1+e^{-x})-\frac{n-r}{n}\frac{\log(1+e^{-x})}{1-(1+e^{-x})^{-k}}.$$

IV. Numerical computation

In this section, we have carried out some numerical computation. In Table 1 estimated value of X using Theorem 2.1 and based on Weibull, Pareto, and Gumbel distributions for different randomly chosen truncation points are listed while Table 2 is based on Theorem 2.2 for inverse Weibull, power function and Burr type II. A random number is used to choose the various random truncation points. For the purpose of computation work, we have taken the real data set represents the failure times of 50 components (per 1000 hours) [Merovei et al. [26]].

0.036	0.058	0.061	0.074	0.078	0.086	0.102	0.103	0.114	0.116
0.148	0.183	0.192	0.254	0.262	0.379	0.381	0.538	0.570	0.574
0.590	0.618	0.645	0.961	1.228	1.600	2.006	2.054	2.804	3.058
3.076	3.147	3.625	3.704	3.931	4.073	4.393	4.534	4.893	6.274
6.816	7.896	7.904	8.022	9.337	10.940	11.020	13.880	14.730	15.080

For the power function distribution, the values in the given data set are not within an interval of [0, 1]. Thus, the original values are divided by the maximum value (15.080) and changed into the interval [0, 1]. Similarly, for the Pareto distribution, the original values are divided by the minimum value (0.036) of the data set and shifted them in the interval of (1, ∞).

Table 1. *Estimated values of X for different parameters*

Distribution	Parameters	r	x	$T_0(x)$	$T_1(x)$	$T_2(x)$	$\mu_r(x)$	$\hat{X}$
Weibull	$p=0.5, \lambda=0.5$	3	0.061	0.005	0.048	1.292	1.221	1.492
		8	0.103	0.006	0.099	1.330	1.138	1.294
		13	0.192	0.009	0.168	1.387	1.076	1.157
		18	0.538	0.015	0.317	1.515	1.092	1.193
		22	0.618	0.016	0.341	1.536	1.019	1.038
	$p=0.5, \lambda=1.0$	3	0.061	0.005	0.046	1.118	1.058	1.119
		8	0.103	0.006	0.094	1.182	1.013	1.026
		13	0.192	0.009	0.158	1.282	0.996	0.992
		18	0.538	0.015	0.292	1.524	1.089	1.187
		22	0.618	0.016	0.313	1.566	1.024	1.048
	$p=1.0, \lambda=0.5$	3	0.061	0.001	0.020	2.052	1.931	1.931
		8	0.103	0.002	0.045	2.093	1.767	1.767
		13	0.192	0.004	0.091	2.182	1.640	1.640
		18	0.538	0.011	0.256	2.523	1.714	1.714
		22	0.618	0.012	0.292	2.606	1.594	1.594
	$p=1.0, \lambda=1.0$	3	0.061	0.001	0.019	1.061	0.999	0.999
		8	0.103	0.002	0.044	1.103	0.935	0.935
		13	0.192	0.004	0.089	1.192	0.907	0.907
		18	0.538	0.011	0.243	1.538	1.078	1.078
		22	0.618	0.012	0.276	1.618	1.034	1.034
Pareto	$p=1.5$	5	2.167	0.016	0.314	1.438	1.334	0.137
		10	3.222	0.023	0.422	1.832	1.565	0.172
		15	7.278	0.040	0.560	2.636	2.042	0.277
		25	34.111	0.071	0.649	4.041	2.402	0.398
		45	259.361	0.111	0.665	2.959	0.993	0.097
	$p=4.5$	5	2.167	0.015	0.198	0.996	0.927	0.091
		10	3.222	0.023	0.216	1.392	1.176	0.117
		15	7.278	0.040	0.222	2.207	1.647	0.187
		25	34.111	0.071	0.222	3.752	2.053	0.281
		45	259.361	0.111	0.222	5.057	0.812	0.081
	$p=7.5$	5	2.167	0.016	0.131	0.907	0.842	0.084
		10	3.222	0.023	0.133	1.303	1.090	0.107
		15	7.278	0.040	0.133	2.126	1.565	0.172
		25	34.111	0.071	0.133	3.663	1.966	0.257
		45	259.361	0.111	0.133	5.522	0.781	0.079
Gumbel		4	0.074	0.022	0.022	2.077	1.934	0.659
		8	0.103	0.022	0.034	2.108	1.798	0.587
		13	0.192	0.024	0.091	2.212	1.683	0.520
		21	0.590	0.036	0.312	2.804	1.787	0.581
		29	2.804	0.330	0.722	17.511	8.089	2.091
		33	3.625	0.751	0.722	38.602	14.338	2.663
		47	11.020	1221.674	0.722	0.000	1222.338	7.108

Table 2. *Estimated values of X for different parameters*

Distribution	Parameters	r	x	$T_0(x)$	$T_1(x)$	$T_2(x)$	$\lambda_r(x)$	$\hat{X}$
Inverse Weibull	$p=1.0$, $\lambda=1.0$	4	0.074	0.270	14.513	0.998	2.059	0.486
		15	0.262	0.076	4.817	0.912	2.063	0.485
		26	1.600	0.013	1.625	0.276	0.957	1.045
		37	4.393	0.005	1.228	0.099	0.914	1.094
		48	13.880	0.002	1.072	0.005	1.009	0.991
	$p=1.0$, $\lambda=1.5$	4	0.074	0.001	14.180	0.664	1.463	0.684
		15	0.262	0.001	4.484	0.651	1.713	0.584
		26	1.600	0.013	1.292	0.259	0.783	1.278
		37	4.393	0.005	0.894	0.097	0.674	1.485
		48	13.880	0.001	0.739	0.005	0.696	1.437
	$p=1.5$, $\lambda=1.0$	4	0.074	0.001	50.677	0.999	3.962	0.399
		15	0.262	0.149	8.457	0.996	3.214	0.459
		26	1.600	0.010	1.494	0.226	0.867	1.101
		37	4.393	0.002	1.109	0.052	0.814	1.147
		48	13.880	0.001	1.0193	0.002	0.960	1.028
Power Function	$p=1.0$	7	0.008	-0.097	-4.025	-0.961	-1.406	3.696
		19	0.038	-0.065	-3.891	-0.871	-2.006	2.029
		28	0.136	-0.040	-2.889	-0.686	-1.902	2.251
		39	0.325	-0.023	-2.080	-0.459	-1.704	2.744
		46	0.726	-0.006	-1.296	-0.150	-1.185	4.611
	$p=1.5$	7	0.008	-0.097	-4.635	-0.663	-1.223	4.438
		19	0.038	-0.065	-3.854	-0.642	-1.851	2.369
		28	0.136	-0.040	-2.649	-0.561	-1.718	2.707
		39	0.325	-0.023	-1.787	-0.411	-1.471	3.463
		46	0.726	-0.006	-0.986	-0.147	-0.905	6.098
	$p=3.5$	7	0.008	-0.097	-5.022	-0.400	-1.043	5.313
		19	0.038	-0.065	-3.666	-0.399	-1.633	2.947
		28	0.136	-0.040	-2.395	-0.386	-1.503	3.354
		39	0.325	-0.023	-1.524	-0.328	-1.253	4.308
		46	0.726	-0.006	-0.720	-0.139	-0.666	7.750
Burr Type II	$k=0.5$	8	0.103	0.013	0.011	0.304	0.270	1.172
		16	0.379	0.010	0.044	0.249	0.193	1.545
		27	2.006	0.003	0.093	0.062	0.080	2.489
		38	4.534	0.000	0.092	0.005	0.069	2.633
	$k=2.0$	8	0.103	0.013	0.041	0.254	0.232	1.341
		16	0.379	0.010	0.157	0.216	0.205	1.482
		27	2.006	0.003	0.234	0.061	0.152	1.806
		38	4.534	0.000	0.201	0.005	0.149	1.827
	$k=3.5$	8	0.103	0.013	0.071	0.210	0.199	1.513
		16	0.379	0.010	0.246	0.186	0.211	1.451
		27	2.006	0.003	0.271	0.058	0.171	1.683
		38	4.534	0.000	0.203	0.005	0.151	1.811

Table 1 presents the estimated values of X for various parameters under three different distributions: Weibull, Pareto, and Gumbel. These distributions exhibit different behaviors as their parameters change. For the Weibull distribution, increasing the shape parameters $(p=0.5,1.0)$ leads to an increase in the estimated values of X. Conversely, increasing the scale parameters $(\lambda=0.5,1.0)$ and the $r-th$ order statistics results in a decrease in the estimated values of X. In the

Pareto distribution, increasing the parameters causes a decrease in the estimated values of X, while an increase in the $r-th$ order statistics results in higher estimated values of X. For the Gumbel distribution, no specific pattern is observed in the behavior of the estimated values of X. In Table 2, the estimated values X for the inverse Weibull, power function, and Burr type II distributions increase as the parameter values increase.

V. Summary

The paper contributes to the field of probability and statistics by proposing a method to characterize continuous distributions based on the conditional expectation of order statistics. This approach not only enhances theoretical understanding but also provides practical insights into modelling and analyzing data using well-known distributional forms. Further, the results are applied to some well-known continuous distributions, like Weibull, Pareto, extreme value I, inverse Weibull, power function, Burr type II and Lindley distribution. One may utilize our results to characterize more distributions belonging to these classes. The numerical computations serve to support the theoretical findings and demonstrate the applicability of the method.

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