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TRIANGULAR AND SKEW-SYMMETRIC SPLITTING METHOD FOR SOLVING FUZZY STOCHASTIC LINEAR SYSTEM

A. Shivaji [1], B. Harika [2], D. Rajaiah [3], L.P. Rajkumar [4]

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Department of Mathematics, Kakatiya University, Telangana, India [1,2,4]
Department of Mathematics, KITS Warangal, Telangana, India [3]
sivajiarepalli@gmail.com, hbolledla@gmail.com , dr.mh@kitsw.ac.in , ladalla.raj@gmail.com

Abstract

Based on the Triangular and Skew Symmetric (TSS) splitting method, a novel iterative approach is proposed to solve a class of fuzzy regularized linear system of equations with fuzzy coefficient stochastic rate matrix. The non-homogeneous fully fuzzy linear system is same as the non-homogeneous linear system which is derived from the homogeneous linear system with stochastic rate matrix and steady state vector. An iterative procedure is developed for finding a unique non-trivial solution. Numerical results shown that the proposed method is effective and efficient when compared with the existing classical methods.

Keywords: Fuzzy Stochastic Rate Matrix; Triangular and Symmetric Splitting Method; Parametric Form of Fuzzy Number; Fully Fuzzy System of Liner Equations; Error Analysis.

1. Introduction

The system of fuzzy linear equations has a variety of applications in the areas of information, engineering, statistics, mathematics, etc. In several applications, the system's parameters and measurements are performed by fuzzy values rather than crisp ones. Thus, it is key to expand mathematical models and numerical mechanisms that would be treated as an ordinary fuzzy linear system (FLS) and solve them using different techniques. The mathematical modelling of the problem is considered as fuzzy system of linear equations. The homogeneous system $\pi Q = 0$, where π is the stationary probability vector and Q is the stochastic rate matrix is transformed into a non-homogeneous system $Ax = b$, where $A = Q^T + \epsilon I$ with small perturbation $\epsilon > 0$. The regularized linear system $Ax = b$ where the parameters are uncertain and vague is transformed into a fully fuzzy linear system (FFLS) $\Phi X = \Psi$, with Φ and Ψ are fuzzy matrices and X is an unknown fuzzy vector to be determined for a unique non-zero solution.

Friedman [1] et al designed a model with an embedding technique for computing a class of $n \times n$ (FLS). LU decomposition method was developed by Abbasbandy [2] et al, Steepest descent method by Abbasbandy and Jafarian [3], The Jacobi, Gauss-Seidel and SOR iterative methods are used by Allahviranloo [4]. Adomian decomposition method was suggested by Allahviranloo [5], inherited LU factorization method by M. A. Fariborzi Araghi and A. Fallahzadeh [6] for solving a fuzzy system of linear equations. A few numerical methods were developed and discussed by the general model [7, 8]. A.N.A. Koam [9] et al LU decomposition scheme is used for solving m-polar fuzzy system of linear equations. Block SOR method was proposed by S. X. Miao [10] et al, the QR-decomposition method was developed by S.H. Nasseri [11] et al, K. Wang and Y. Wu introduced the Uzawa-SOR method [12], Symmetric Successive Over Relaxation method, Block iterative method, and Splitting iterative methods were established by K. Wang B. Zheng and J. F. Yin[13, 14, 15]. Y.R. Wang and Y. L. Chen suggested a modified Jacobi iterative method for large-size linear systems [16], and a new method based on Jacobi iteration was proposed for solving the fuzzy linear systems by Zhen Huang [17] et al. The traditional TSS method is easy to execute and applicable to compute stationary probability vector and the performance measures in understanding many real-time systems. In this research work, an advanced iterative method is deployed based on TSS iteration method which brings the solution for fuzzy linear systems [18].

The rest of the paper is organized as follows. Section 2 gives some fundamentals of FLS. In section 3, the new method is established. Numerical examples are presented in section 4 and the conclusions are drawn in section 5.

2. Fundamentals

Fuzzy number : A pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$, which satisfies the conditions,

- $\underline{u}(r)$ is a bounded left continuous nondecreasing function over [0, 1]
- $\overline{u}(r)$ is a bounded left continuous nonincreasing function over [0, 1]
- $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$

is known as a fuzzy number.

Arithmetic Operations : The arithmetic operations involving in arbitrary fuzzy numbers $x = (\underline{x}(r), \overline{x}(r))$, $y = (\underline{y}(r), \overline{y}(r))$, $0 \leq r \leq 1$, and for $k \in R$, are defined by:

- $x = y$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$
- $x + y = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$ and
- $kx = \begin{cases} (k\underline{x}(r), k\overline{x}(r)), k > 0, \\ (k\overline{x}(r), k\underline{x}(r)), k < 0. \end{cases}$

Fuzzy System of Linear Equations : The $n \times n$ fuzzy linear system (FLS) may be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdot + a_{nn}x_n &= b_n \end{aligned}$$

the matrix form of the above linear system is

$$Ax = b \tag{1}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{11} & a_{nn} & \cdots & a_{nn} \end{bmatrix} \text{ is a crisp matrix}$$

$b = [b_1, b_2, \cdots, b_n]^T$ is a fuzzy vector and $x = [x_1, x_2, \cdots, x_n]^T$ is unknown.

Solution of Fuzzy Linear System : The solution of the fuzzy linear system is a fuzzy vector $x = (x_1, x_2, \ldots, x_n)^T$
given by

$$x_i = (\underline{x}_i(r), \overline{x}_i(r)), 1 \le i \le n, 0 \le r \le 1$$

if

$$\begin{cases} \underline{\sum_{j=1}^{n} a_{ij} x_j} = \sum_{j=1}^{n} a_{ij} \underline{x_j} = \underline{b}_i \\ \overline{\sum_{j=1}^{n} a_{ij} x_j} = \sum_{j=1}^{n} \overline{a_{ij} x_j} = \overline{b}_i \end{cases} \tag{2}$$

The extended FLS (1) into the $2n \times 2n$ crisp linear system is defined as,

$$\Phi X = \Psi \tag{3}$$

where, $\Phi = (\phi_{kl})$, ϕ_{kl} are determined as follows

$$a_{ij} > 0 \Rightarrow \phi_{kj} = a_{ij}, \phi_{n+i,n+j} = a_{ij}$$
$$a_{ij} < 0 \Rightarrow \phi_{i,n+j} = a_{ij}, \phi_{n+i,j} = a_{ij}, 1 \le i, j \le n,$$

and any ϕ_{kl} which is not determined by the above items is zero, $1 \le i, j \le 2n$, and

$$X = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ \overline{x}_1 \\ \vdots \\ \overline{x}_n \end{bmatrix}, \Psi \begin{bmatrix} & \underline{b}_1 \\ & \vdots \\ = & \underline{b}_n \\ & \overline{b}_1 \\ & \vdots \\ & \overline{b}_n \end{bmatrix}$$

Moreover, the matrix Φ has the form $\begin{bmatrix} \Phi_1 & \Phi_2 \\ \Phi_2 & \Phi_1 \end{bmatrix}$, $A = \Phi_1 + \Phi_2$, and (2) can be written as

$$\begin{cases} \Phi_1 \underline{X} + \Phi_2 \overline{X} = \underline{\Psi} \\ \Phi_2 \underline{X} + \Phi_1 \overline{X} = \overline{\Psi} \end{cases}$$

where

$$\underline{X} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \vdots \\ \underline{x}_n \end{bmatrix} \quad \overline{X} = \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \vdots \\ \overline{x}_n \end{bmatrix}$$

$$\underline{\Psi} = \begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \vdots \\ \underline{b}_n \end{bmatrix} \quad \overline{\Psi} = \begin{bmatrix} \overline{b}_1 \\ \overline{b}_2 \\ \vdots \\ \overline{b}_n \end{bmatrix}$$

Stochastic Rate Matrix : A matrix $Q = [q_{ij}]$ is known as stochastic rate matrix, if is satisfies the following conditions

- $q_{ij} \ge 0, 1 \le i, j \le n$
- $q_{ii} = -\sum_{j \ne i} q_{ij}$

3. Triangular and Skew Symmetric Splitting Iterative Method for Regularized Linear System

In this section, the stationary probability vector π of $\pi Q = 0$ can be found using the Fuzzy Triangular and Skew-Symmetric (FTSS) iterative method of a regularized linear system (3). We establish the TSS splitting method for the stochastic rate matrix as follows,
The matrix A of the system (1) is split into the form,

$$\begin{aligned} A =& (L + D + U^T) + (U - U^T) \\ =& T + S, \end{aligned}$$

where,
$T = L + D + U^T$, $S = (U - U^T)$ are triangular and skew symmetric matrices.

Thus the regularized system (2) can take the form $(T + S)X = Y$.

Consider $D = \begin{bmatrix} D_1 & 0 \\ 0 & D_1 \end{bmatrix}$, $L = \begin{bmatrix} L_1 & 0 \\ -S_2 & L_1 \end{bmatrix}$, $U = \begin{bmatrix} U_1 & -S_2 \\ 0 & U_1 \end{bmatrix}$

$$\Rightarrow U^T = \begin{bmatrix} U_1 & 0 \\ -S_2 & U_1 \end{bmatrix}$$

Then

$$\begin{aligned} L + D + U^T &= \begin{bmatrix} L_1 & 0 \\ -S_2 & L_1 \end{bmatrix} + \begin{bmatrix} D_1 & 0 \\ 0 & D_1 \end{bmatrix} + \begin{bmatrix} U_1 & -S_2 \\ 0 & U_1 \end{bmatrix} \\ &= \begin{bmatrix} L_1 + D_1 + U_1 & 0 \\ -2S_2 & L_1 + D_1 + U_1 \end{bmatrix} \end{aligned}$$

and

$$U - U^T = \begin{bmatrix} 0 & -S_2 \\ S_2 & 0 \end{bmatrix}$$

where D_1, L_1, and U_1 are diagonal, lower, and upper triangular matrices respectively.
The method of triangular and skew-symmetric splitting iterative is as follows,

$$X^{(k+1)} = H(\alpha)X^{(k)} + G(\alpha)b, \quad \text{for} \quad k = 0, 1, 2, \ldots,$$

where,

$$X^{(k+1)} = \begin{bmatrix} \underline{x}^{k+1} \\ \overline{x^{k+1}} \end{bmatrix}$$

$$H(\alpha) = (\alpha I_n + S)^{-1}(\alpha I_n - T)(\alpha I_n + T)^{-1}(\alpha I_n - S)$$

and

$$G(\alpha) = 2\alpha(\alpha I_n + S)^{-1}(\alpha I_n + T)^{-1}$$

We have

$$\alpha I_n + S = \begin{bmatrix} \alpha I_n & -S_2 \\ S_2 & \alpha I_n \end{bmatrix}$$
$$\alpha I_n - S = \begin{bmatrix} \alpha I_n & S_2 \\ -S_2 & \alpha I_n \end{bmatrix}$$
$$\alpha I_n + T = \begin{bmatrix} \alpha I_n + T_1 & 0 \\ -2S_2 & \alpha I_n + T_1 \end{bmatrix}$$
$$\alpha I_n - T = \begin{bmatrix} \alpha I_n - T_1 & 0 \\ 2S_2 & \alpha I_n - T_1 \end{bmatrix}$$

$$\Rightarrow H(\alpha) = \frac{1}{[(\alpha I_n)^2 + S_2^2][\alpha I_n + T_1]^2} \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}$$

where

$$\begin{aligned} H_1 =& (\alpha I_n)^4 - (\alpha I_n)^2 T_1^2 + 3(\alpha I_n)^2 S_2^2 + T_1^2 S_2^2 \\ H_2 =& 2(\alpha I_n)^3 S_2 - 2\alpha I_n T_1^2 S_2 + 4\alpha I_n S_2^3 \\ H_3 =& 2(\alpha I_n)^3 S_2 + 2\alpha I_n T_1^2 S_2 \\ H_4 =& (\alpha I_n)^4 - (\alpha I_n)^2 T_1^2 + 3(\alpha I_n)^2 S_2^2 + T_1^2 S_2^2 \end{aligned}$$

and

$$G(\alpha) = \frac{2\alpha}{[(\alpha I_n)^2 + S_2^2][\alpha I_n + T_1]^2} \begin{bmatrix} (\alpha I_n)^2 + \alpha I_n T_1 + 2S_2^2 + T_1^2 S_2^2 & \alpha I_n S_2 + T_1 S_2 \\ \alpha I_n S_2 - T_1 S_2 & (\alpha I_n)^2 + \alpha I_n T_1 \end{bmatrix}$$

The next section involves the numerical solution of the non-homogeneous regularized fuzzy linear system (1). It is clear that the solution to the system (1) may give a small error due to the membership value r.

Theorem: The iterative solution of the FLS (1) is convergent if $\rho(H(\alpha)) < 1$.

4. Numerical Results

In this section, the adequacy of the FTSS iterative method for the numerical solution of the stochastic rate matrices in a fuzzy nature is inspected. The convergence analysis of the stationary probability vector of FTSS method is compared with traditional Jacobi and TSS iterative methods. For validation, we take the following 3×3 stochastic rate matrix

$$Q = \begin{bmatrix} 0.7 & -0.55 & -0.15 \\ -0.15 & 0.7 & -0.55 \\ -0.55 & -0.15 & 0.7 \end{bmatrix}$$

The above system is transformed into a regularized linear system (1). The regularized linear system is remodelled into 6×6 fully fuzzy linear system

$$\Phi X = \Psi$$

where

$$\Phi = \begin{bmatrix} 0.7+r & 0 & 0 & 0 & -0.55 & -0.15 \\ 0 & 0.7+r & 0 & -0.15 & 0 & -0.55 \\ 0 & 0 & 0.7+r & -0.55 & -0.15 & 0 \\ 0 & -0.55 & -0.15 & 0.7+r & 0 & 0 \\ -0.15 & 0 & -0.55 & 0 & 0.7+r & 0 \\ -0.55 & -0.15 & 0 & 0 & 0 & 0.7+r \end{bmatrix}$$

The initial distribution $x^{(0)}$ and Ψ are taken as $x^{(0)} = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1]^T$ and $\Psi = [0 \;\; 0 \;\; r \;\; 0 \;\; 0 \;\; -r]$, where, r is the membership value which varies from 0 to 1. In this work, only one case $A = (L + D + U^T) + (U - U^T) = T_1 + S_1$ of TSS splitting iterative method is taken and the remaining methods would obey the same. The result for the incident of contraction factor $\alpha = 0.7 + r$ is numerically the same as the diagonal elements of the matrix Q for the diverse values of r. The absolute error and relative errors of the fuzzy linear system are computed and the same are compared with the traditional Jacobi and TSS splitting iterative method which is depicted in Figures (1-4).

Figure 1 depicts the convergence of iterative solutions for the classical Jacobi, TSS and FTSS splitting methods. Figure 2 shows the absolute error in different cases of r for the Fuzzy TSS iterative method. Figure 3 displays the relative error for various values of r for the newly established method, and Figure 4 illustrates the comparison of absolute and relative errors of the TSS splitting iterative model in a fuzzy environment.

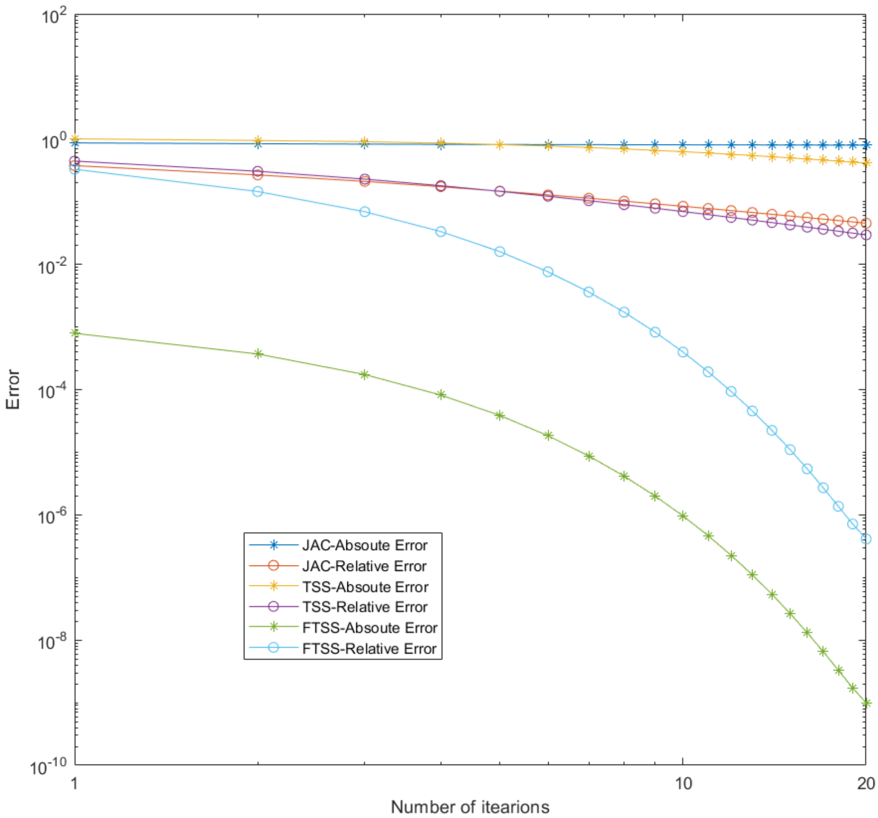

Figure 1: *Absolute and Relative Errors of the Jacobi, TSS and FTSS iterative methods*

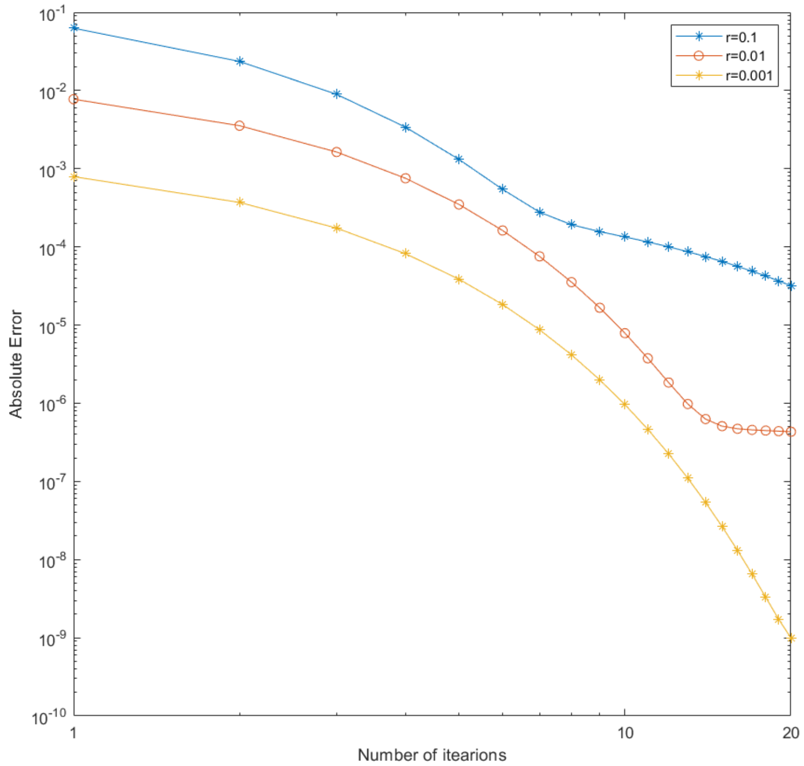

Figure 2: *Absolute Error of FTSS for different values of r*

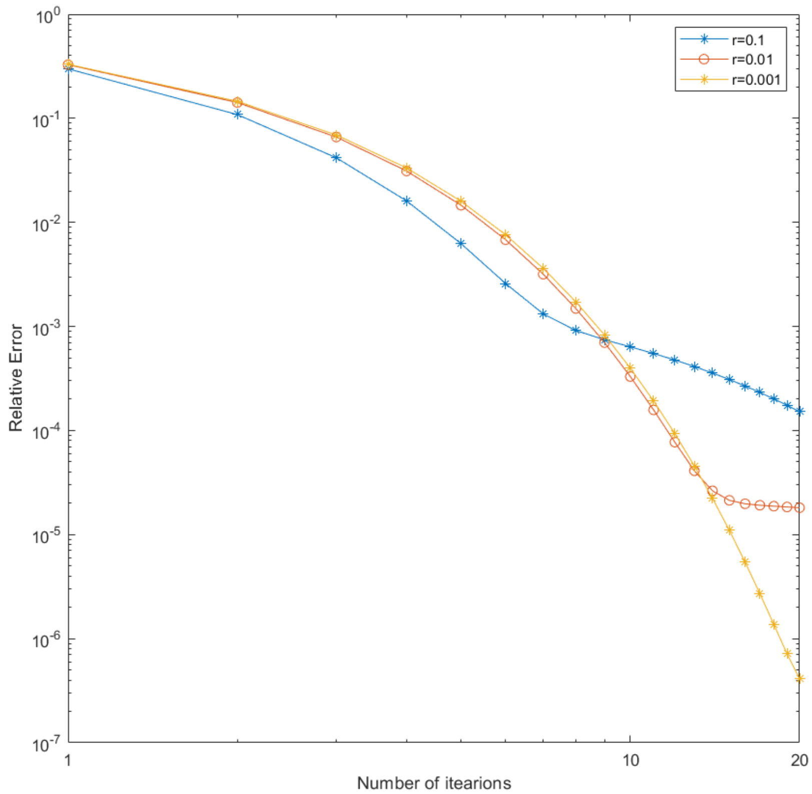

Figure 3: *Relative Error of FTSS for different values of r*

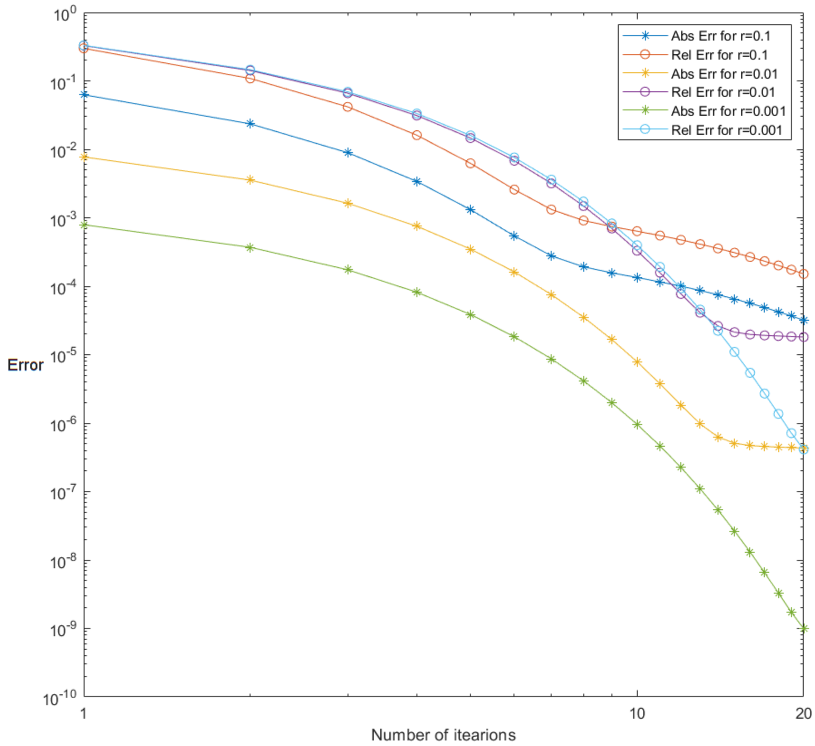

Figure 4: *Absolute and Relative Errors of FTSS for different values of r*

5. Conclusions

In this research work, a new iterative method is formulated on Triangular and Skew-Symmetric iteration for solving a class of fuzzy linear systems of equations with crisp valued stochastic rate matrix. The iterative method is presented and the solution is compared with the traditional methods. The numerical example portrays that the proposed method is effective and competent when compared with traditional iterative methods. We conclude that the suggested method converges to a unique solution and the rate of convergence is faster than the existing traditional methods.

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