THE INVERSE LOMAX ODD-EXPONENTIATED EXPONENTIAL DISTRIBUTION WITH INDUSTRIAL APPLICATIONS

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Abstract

Based on the limitations of the Inverse Lomax distribution and exponential distribution as outlined in the literature, a new extension of the exponential distribution is introduced in this paper. Some statistical properties of the ILOEED such as mean, variance, skewness, quantile function, moment, moment generating function, as well as kurtosis were demonstrated. The shapes of the hazard function of the proposed distribution suggest that it can be used to fit a dataset with increasing and bath-tube shapes. A simulation study for three different cases was also presented. The result of the simulation for three different cases (I, II, and III) indicated that ILOEED's estimates are consistent. Lastly, an application to Industry datasets was demonstrated based on the ILOEED. Having minimum values of the Goodness-of-fit criteria and Goodness-of-fit statistics, the ILOEED can be recommended to fit these three datasets, in preference to other distributions considered in this paper.

Keywords: Inverse Lomax-G family, Exponentiated Exponential Distribution, Weibull Distribution, Heavy Tail Distribution.

I. Introduction

In their ongoing pursuit of adaptive and flexible statistical models, scientists and researchers have been investigating new distributions that may accurately represent a wide range of real-world data patterns. In this quest, the proposed distribution shows great promise as a more versatile model for a range of phenomena. Interestingly, it leverages the advantages of its parent distributions, the Odd-Exponentiated and the Inverse Lomax distribution (ILD), to produce a distribution that can describe a wide range of datasets, especially those with heavy tails and non-monotone failure rates ([1],[2],[3]). Extreme occurrences or outliers are more likely to occur in a heavy-tail distribution because its tails decay more slowly than those of a normal distribution. This suggests that there is more risk or variability in the data. The Pareto II (ILD), Cauchy, and Student's t-distributions are a few instances of heavy-tail distributions.

Inverse Exponentiated Odd Lomax Exponential distribution was proposed by [4], offering a fresh outlook on statistical modeling and analysis. Their research investigates the statistical properties of this distribution, contributing to a better comprehension of its practical applicability.

A four-parameter Exponentiated Odd Lomax Exponential (EOLE) distribution was proposed by [3], combining an exponentiated odd function with Lomax and exponential elements, thereby enhancing the distribution's versatility and providing a more intricate parameterization for modeling. To give researchers a wider range of tools for a variety of applications, Inverse Exponentiated Lomax Power Series distribution was proposed by [2], which expanded the family of distributions that combines Lomax and exponential components. Inverse Lomax distribution has two major drawbacks. These are: Limited Flexibility in Shape i.e. ILD has some difficulty adjusting its probability density function, especially in the peak and tail areas. This can hinder its ability to precisely model data with certain patterns [5]. A potential constraint in modeling hazard rates i.e. while often used to model non-monotone hazard rates (failure rates that vary over time), the Inverse Lomax distribution may not be able to capture all possible hazard rate shapes that can arise in real-world situations [6]. However, exponential distribution has limitations of constant failure rate and memory-less property. Hence, the need to study the proposed distribution to remedy some of the drawbacks of the ILD and exponential distribution.

To tackle the challenge of modeling the lifespans of electronic devices, which is essential for predicting future failures and achieving energy savings, a reliability model that is based on the inverse power law and generalized inverse Weibull distribution [7]. It demonstrates how successful the proposed distribution is in influencing the average time to failure of the examined capacitor, as opposed to standard models such as the inverse Weibull, using an empirical analysis that focuses on the life cycle of a surface-mounted electrolytic capacitor. To analyze COVID-19 death cases in Europe and China, the Exponentiated Transformation of Gumbel Type-II (ETGT-II) model [8]. This model provides a thorough analysis of statistical features and estimates model parameters using maximum likelihood and Bayesian approaches. The ETGT-II model is shown to be efficient through simulation analysis. It exhibits a promising adaptation to the COVID-19 death data sets, perhaps providing a better fit than other models. The new exponential inverted Topp-Leone (NEITL) distribution is presented by [9]. It is an extension of the inverted Topp-Leone distribution with an extra shape parameter. Its features, estimation methods, and application to actual datasets in the engineering and medical domains are all explored. The generalized log-exponential transformation of Gumbel Type-II (GLET-GTII), which was proposed by [10] as a generator for a generalized version of the Gumbel type-II model, increases modeling flexibility by adding a new parameter. Quantiles, survival function, and reliability are among the statistical attributes that are examined. Maximum likelihood and Bayesian approaches are used as parametric estimation methods, and they show consistency through Monte Carlo simulations and outperform other models in practical implementations, especially when it comes to infectious diseases like COVID-19. Using a new power function and a modified Kies generalized transformation, a novel statistical model and discusses its theoretical characteristics, including the density function, quantile function, and stochastic ordering [11]. The moment exponential distribution is extended by the two-parameter alpha power-transformed moment exponential (APTME) distribution, which shows excellent fit and performance through a variety of estimators and simulation studies. Its practical significance is demonstrated by its application to real-world datasets [12].

Moreover, the flexible four-parameter Kumaraswamy extended exponential (KwEE) distribution is presented by [13]. This model shows that the novel distribution may provide a better fit than current models in several COVID-19 spread analysis situations by evaluating COVID-19 mortality rates in nations such as Italy and the United Kingdom. To represent the dependability metrics of a generalized exponential model based on the inverse power law (IPL), was suggested using a multilayer ANN with Bayesian regularization by [14]. The outcomes show how well ANNs operate as a reliable mathematical tool for evaluating lifetime model reliability, and they are backed by a real-world application. Under the generalized type-I progressive hybrid censoring sample (GTI-PHCS), statistical inference for the Kavya-Manoharan generalized exponential distribution was proposed by [15]. It does this by examining different estimation techniques, such as maximum likelihood and Bayesian approaches, and using real-world data analysis and simulations to illustrate how well the techniques work. Through simulation studies and application to engineering datasets, the half-logistic modified Kies exponential (HLMKEx) distribution as a flexible three-parameter model for modeling real-world data was presented [16]. It provides detailed mathematical features, such as density function forms and estimation methods, and shows its superior fit over competing distributions. Generalized exponentiated unit Gompertz (GEUG), a unique four-parametric model, was introduced by [17] to represent clinical trial data of patients with arthritis. By adding new tuning factors to the unit Gompertz (UG) model, the GEUG model aims to improve the estimate of distribution parameters, hence increasing the model's adaptability.

Motivation: The inability of Inverse Lomax distribution and exponential distribution to adequately capture some intricate data patterns led to the development of the proposed distribution. To introduce a more adaptable model, that expands on the advantages of its parent distributions; the Odd-Exponentiated and the Inverse Lomax. **Key features:** More flexibility in defining its density and hazard functions, Non-monotone hazard rates capable of simulating phenomena with fluctuating risk characteristics over time, as well as heavy tails that capture extremes and outliers well. In this article, a new extension of Exponential distribution is introduced. The proposed distribution is formulated based on the Inverse Lomax Odd Exponentiated-G family of distributions. The most important feature of the proposed distribution, with two shape parameters, a scale, and rate parameters represents its ability to provide different density shapes. This means that the proposed distribution can fit various datasets adequately. The proposed distribution has the following desirable properties. (i) The probability density function (pdf) of the ILOEED proposed distribution has a simple closed form. Then, ILOEED can be used for modeling and analyzing real-life data in Industries; (ii) The shape parameters of the proposed distribution make it very flexible to exhibit increasing and bath-tube failure rate shapes; (iii) Additionally, the density of the proposed distribution can also provide more flexible shapes. The paper is organized into six sections. The proposed distribution is defined in Section 2. The Statistical properties of the proposed distribution are presented in section 3. The estimation of the parameters of the proposed distribution using the method of Maximum Likelihood Estimates (MLEs) is introduced in Section 4. In Section 5, a simulation study based on the properties of the MLEs of the proposed distribution is presented. Applications of the proposed distribution to industry datasets are presented In Section 6. Section 7 concludes the paper.

The Inverse Lomax-Odd Exponentiated G (IL-OEG) family was proposed by [1] based on the T-X generator of [18].The cumulative density function (CDF) and probability density function (PDF) of IL-OEG are given as

$$F(x;\lambda,\gamma,\theta,\Delta)=\left[1+\lambda\left\{\frac{(1-G(x;\boldsymbol{\Delta}))}{G(x;\boldsymbol{\Delta})}\right\}^{\theta}\right]^{-\gamma};x>0,\lambda,\gamma,\theta,\boldsymbol{\Delta}>0 \quad (1)$$

And

$$f(x;\lambda,\gamma,\theta,\Delta)=\frac{\theta\gamma\lambda g(x;\Delta)[1-G(x;\boldsymbol{\Delta})]^{\theta-1}}{[G(x;\boldsymbol{\Delta})]^{\theta+1}}\left[1+\lambda\left[\frac{(1-G(x;\boldsymbol{\Delta}))}{G(x;\boldsymbol{\Delta})}\right]^{\theta}\right]^{-(1+\gamma)};x>0,\lambda,\gamma,\theta,\boldsymbol{\Delta}>0 \quad (2)$$

Where $\boldsymbol{\Delta}$ is a vector of parameter(s) for the baseline distribution, $G(x;\boldsymbol{\Delta})=1-G(x;\boldsymbol{\Delta})$. Based on equations (1) and (2), Exponential distribution is considered to be the baseline distribution. So, the $G(x;\Delta)$ is equivalent to an exponential density.

II. Inverse Lomax Odd Exponentiated Exponential distribution (ILOEED)

The exponential distribution is the probability distribution of the time between events in a Poisson point process, that is, a process in which events occur continuously and independently at a constant average rate. It is a subset of the gamma distribution. The CDF and PDF of the exponential distribution are presented in equations (3) and (4).

$$G(x;\upsilon)=1-e^{-\upsilon x},x,\upsilon>0 \tag{3}$$

And

$$g(x;\upsilon)=\upsilon e^{-\upsilon x};x,\upsilon>0 \tag{4}$$

Then, the CDF and PDF of the ILOEED can be given as:

$$F(x;\theta,\lambda,\gamma,\upsilon)=\left[1+\lambda\left\{\frac{e^{-\upsilon x}}{(1-e^{-\upsilon x})}\right\}^{\theta}\right]^{-\gamma};x,\upsilon,\gamma,\lambda,\theta>0 \tag{5}$$

And

$$f(x;\theta,\lambda,\gamma,\upsilon)=\frac{\upsilon\theta\gamma\lambda e^{-\upsilon\theta x}\left[1+\lambda\left\{\frac{e^{-\upsilon x}}{(1-e^{-\upsilon x})}\right\}^{\theta}\right]^{-(1+\gamma)}}{[1-e^{-\upsilon x}]^{\theta+1}};x,\upsilon,\gamma,\lambda,\theta>0 \tag{6}$$

Where υ is the rate, λ is the scale, and θ and γ are the shape parameters, respectively. Having this combination of parameters, we hope that ILOEED will fit datasets of different shapes. The reliability, hazard, and cumulative hazard functions of the ILOEED are presented in equations (7), (8), and (9).

$$\begin{aligned}R(x;\lambda,\gamma,\theta,\upsilon)&=\int_t^{\infty}f(x;\lambda,\gamma,\theta,\upsilon)dx\\&=1-\left[1+\lambda\left\{\frac{e^{-\upsilon x}}{(1-e^{-\upsilon x})}\right\}^{\theta}\right]^{-\gamma};x,\lambda,\gamma,\theta,\upsilon>0,\end{aligned} \tag{7}$$

And

$$\begin{aligned}H(x;\lambda,\gamma,\theta,\upsilon)&=\int_{\infty}^{x}h(v)dv=-log(R(x;\lambda,\gamma,\theta,\upsilon))\\&=-log\left\{1-\left[1+\lambda\left\{\frac{e^{-\upsilon x}}{(1-e^{-\upsilon x})}\right\}^{\theta}\right]^{-\gamma}\right\}.\end{aligned} \tag{8}$$

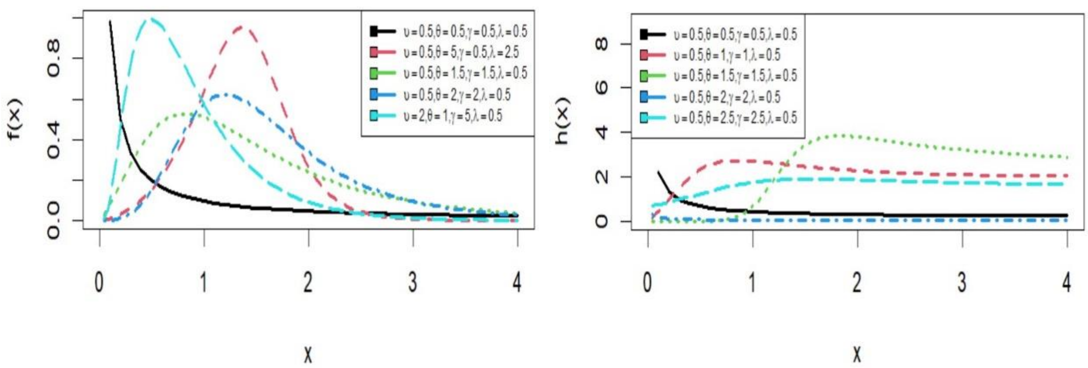

Figure 1: *PDF and CDF plots of ILOEED at various parameter values*

Figure (1) shows the various shapes of the ILOEED's PDF. This includes skewed and symmetry. Figure (1) also indicates the various shapes that the ILOEED can take which include constant, Bathtub, and monotone-increasing hazard shapes.

III. The Statistical Properties of the ILOEED

I. The quantile function of the ILOEED

The quantile function of ILOEED can be derived by inverting the CDF of the ILOEED given in equation (5) as follows:

$$F(x;\lambda,\gamma,\theta,\upsilon)=U=\left[1+\lambda\left\{\frac{e^{-\upsilon x}}{(1-e^{-\upsilon x})}\right\}^{\theta}\right]^{-\gamma} \tag{9}$$

Then,

$$U^{\frac{-1}{\gamma}}-1=\lambda\left\{\frac{e^{-\upsilon x}}{(1-e^{-\upsilon x})}\right\}^{\theta}$$

After simplifying and by making x the subject of the formula, we have

$$x=\frac{-log\left(\frac{k}{1+k}\right)}{\upsilon} \tag{10}$$

Where $k=\left(\frac{U^{\frac{-1}{\gamma}}-1}{\lambda}\right)^{\frac{1}{\theta}}$ and U is uniformly distributed between 0 and 1.

The median of the ILOEED family can be derived by setting U=0.5 in equation (10).

II. The moments of the ILOEED

Let X be a random variable that follows ILOEED with parameters $(\lambda,\gamma,\theta,\upsilon)$, then the C^{th} moment about the origin is given by:

$$\mu_C^{'}=E(X^C)=\int_0^{\infty}x^c f(x;\lambda,\gamma,\theta,\upsilon)dx$$

Using some linear representation, we have the moment of the ILOOED as

$$\mu_C^{'}=\sum_{k_1,k_2=0}^{\infty}\psi\,[k_2-\theta k_1]\int_0^{\infty}x^c\upsilon e^{-\upsilon x}(1-e^{-\upsilon x})^{[k_2-\theta k_1-1]}dx \tag{11}$$

Where $\psi=\frac{(-1)^{k_1+k_2}\Gamma(\gamma+k_1)\Gamma(\theta k_1+k_2)\lambda^{k_1}}{k_1\Gamma(\gamma)k_2!\Gamma(\theta k_1)}$. By considering the binomial expansion of the term $(1-e^{-\upsilon x})^{[k_2-\theta k_1-1]}=\sum_{j=0}^{\infty}(-1)^j\binom{[k_2-\theta k_1-1]}{j}e^{-j\upsilon x}$. Then, equation (11) becomes

$$\mu_C^{'}=\Upsilon_{k,j}\int_0^{\infty}x^c e^{-\upsilon(1+j)x}dx=\frac{\Upsilon_{k,j}c!}{\left[\upsilon(1+j)\right]^{c+1}}. \tag{12}$$

Where $\Upsilon_{k,j} = \sum_{j,k_1,k_2}^{\infty} \psi\upsilon[k_2-\theta k_1-1](-1)^j \binom{[k_2-\theta k_1-1]}{j}$. The mean of the ILOEED can be derived by setting c=1 in equation (12). Moreover, the second moment can also be derived by setting c=2, and then using the relation $Var(X)=\mu_2'-[\mu_1']^2$ to find the variance.

III. The Characteristic and Moment Generating Functions of the ILOEED

The characteristic function of the ILOEED can be given as

$$L(x;\upsilon,\theta,\gamma,\lambda)=nlog(\lambda\gamma\theta\upsilon)-\upsilon\theta\sum_{i=1}^{n}x_i-(1+\gamma)\sum_{i=1}^{n}log\left[1+\lambda\left(\frac{e^{-\upsilon x_i}}{1-e^{-\upsilon x_i}}\right)^{\theta}\right]-(1+\theta)\sum_{i=1}^{n}log\left[1-e^{-\upsilon x_i}\right] \quad (13)$$

And the moment generating function of the ILOEED can be given as

$$\begin{aligned} M_X(t) &= \int_0^{\infty} e^{tx} f(x;\lambda,\gamma,\theta,\upsilon)dx \\ &= \sum_{k_1,k_2}^{\infty}\psi[k_2-\theta k_1]\int_0^{\infty} e^{tx}\upsilon e^{-\upsilon x}(1-e^{-\upsilon x})^{[k_2-\theta k_1-1]}dx \\ &= \Upsilon_{k,j}\int_0^{\infty} e^{-(t-\upsilon(1+j))x}dx = \frac{\Upsilon_{k,j}}{\upsilon(1+j)-t} \end{aligned} \quad (14)$$

Table 1: *Mean, Variance, Skewness, and Kurtosis of ILOEED at various parameter values*

Parameter	E(X)	Var(X)	SK(X)	KUR(X)
$\upsilon=\theta=\gamma=\lambda=0.5$	1.0694	6.6518	4.1394	26.2247
$\upsilon=1,\theta=0.5,\gamma=1,\lambda=0.5$	0.9794	2.8142	3.325	14.6679
$\upsilon=1.5,\theta=0.5,\gamma=1.5,\lambda=0.5$	0.9061	1.6187	2.4044	10.900
$\upsilon=2,\theta=0.5,\gamma=2,\lambda=0.5$	0.8449	1.0637	2.0973	9.0716
$\upsilon=2.5,\theta=0.5,\gamma=2.5,\lambda=0.5$	0.7927	0.7549	1.8992	8.0158
$\upsilon=3,\theta=0.5,\gamma=3,\lambda=0.5$	0.7476	0.5639	1.7499	7.3406
$\upsilon=3.5,\theta=0.5,\gamma=3.5,\lambda=0.5$	0.7081	0.4372	1.6408	6.8795
$\upsilon=4,\theta=0.5,\gamma=4,\lambda=0.5$	0.6320	0.3486	1.5570	6.5505

Table (1) presents some basic statistics based on the moments of the ILOEED at varying parameter values. It's evident from the table that as the shape parameters increase, all the values of the statistics decrease. The skewness is positive and the tails tend towards the right. Positive kurtosis also indicates the tails are heavy.

IV. Maximum Likelihood Estimates (MLE)

In this section, we used the maximum likelihood method to estimate the parameters of the ILOEED. Let $x_1, x_2, x_3, x_4, ..., x_n$ be a random sample independently drawn from ILOEED family. Then, the log-likelihood function $L(\upsilon,\theta,\gamma,\lambda)$ of equation (6) is given as

$$L(x;\upsilon,\theta,\gamma,\lambda)=nlog(\lambda\gamma\theta\upsilon)-\upsilon\theta\sum_{i=1}^{n}x_i-(1+\gamma)\sum_{i=1}^{n}log\left[1+\lambda\left(\frac{e^{-\upsilon x_i}}{1-e^{-\upsilon x_i}}\right)^{\theta}\right]-(1+\theta)\sum_{i=1}^{n}log\left[1-e^{-\upsilon x_i}\right] \quad (15)$$

Taking the partial derivatives of equation (15) with respect to υ, θ, γ, and λ, yields:

$$\frac{\partial L}{\partial \upsilon}=\frac{n}{\upsilon}-\theta\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}\frac{(1+\gamma)\theta\lambda\left(\frac{e^{-\upsilon x_i}}{1-e^{-\upsilon x_i}}\right)^{\theta-1}xe^{\upsilon x}}{\left(1-e^{-\upsilon x}\right)^2\left[1+\lambda\left(\frac{e^{-\upsilon x_i}}{1-e^{-\upsilon x_i}}\right)^{\theta}\right]}. \tag{16}$$

$$\frac{\partial L}{\partial \theta}=\frac{n}{\theta}-\upsilon\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}\frac{(1+\gamma)\lambda\left(\frac{e^{-\upsilon x_i}}{1-e^{-\upsilon x_i}}\right)^{\theta}log\left(\frac{e^{-\upsilon x_i}}{1-e^{-\upsilon x_i}}\right)}{\left[1+\lambda\left(\frac{e^{-\upsilon x_i}}{1-e^{-\upsilon x_i}}\right)^{\theta}\right]}-\sum_{i=1}^{n}\log\left[1-e^{-\upsilon x_i}\right]. \tag{17}$$

$$\frac{\partial L}{\partial \gamma}=\frac{n}{\gamma}-\sum_{i=1}^{n}log\left[1+\lambda\left(\frac{e^{-\upsilon x_i}}{1-e^{-\upsilon x_i}}\right)^{\theta}\right]. \tag{18}$$

$$\frac{\partial L}{\partial \lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}\frac{(1+\gamma)\left(\frac{e^{-\upsilon x_i}}{1-e^{-\upsilon x_i}}\right)^{\theta}}{\left[1+\lambda\left(\frac{e^{-\upsilon x_i}}{1-e^{-\upsilon x_i}}\right)^{\theta}\right]}. \tag{19}$$

The MLEs of the υ, θ, γ, and λ can be determine by solving the following non-linear equations (16), (17), (18), and (19) with respect to each parameter.

V. The Simulation Studies of the ILOEED

Here, we highlighted five (5) steps on how to do a Monte Carlo simulation study as follows:
Step 1: Clearly state the pseudo-population that can be used in generating random samples usually by writing code in a specific method. In this study, the pseudo-population is the quantile function of ILOEED given in equation (10).
Step 2: Sample from the population of interest (depending on your objective).
Step 3: Estimate the parameter of interest from the sample and keep it in a vector.
Step 4: Repeat the previous steps i.e 2 and 3 n-times (n is the number of trials).
Step 5: Create a relative frequency distribution of resulting values that is a Monte Carlo approximation of the distribution of samples under the conditions defined by the pseudo-population and the procedures of sampling. Based on the above procedure, we carry out a simulation studies as explained below:
i). For known parameter values i.e $\Theta=(\theta,\upsilon,\gamma,\lambda)^T$, we simulated a random sample of size n from the ILOEED using equation (10).
ii). We then estimated the parameters of the ILOEED by the method of maximum likelihood. iii). Perform 1,000 replications of steps i through ii.
iv). For each of the four (4) parameters of the ILOEED, we compute the mean, bias, and Root mean squared error (RMSE) from the 1,000 parameter estimates. The statistics are given by

$$\hat{\Theta}=\frac{1}{1,000}\sum_{i=1}^{1,000}\Theta_i, Bias(\hat{\Theta})=\frac{1}{1,000}\sum_{i=1}^{1,000}(\Theta_i-\Theta), RMSE(\hat{\Theta})=\sqrt{\frac{1}{1,000}\sum_{i=1}^{1,000}(\Theta_i-\Theta)^2} \tag{20}$$

Where $\Theta_i = (\hat{\theta}, \hat{\upsilon}, \hat{\gamma}, \hat{\lambda})$ is the MLE for each iteration (n=10, 20, 30, 50, 70, 90, 150, 170). Table (2) reports Case I. R software by [19] was used for the simulation. 123 was set as the seed for reproducibility. Three cases were considered for the simulation. Case I:(υ = 0.3, θ = 1, γ = 0.9 and λ = 0.4), Case II:(υ= 0.6, θ = 0.5, γ = 0.9 and λ = 0.5), and Case III: (υ= 2, θ = 5, γ = 0.5 and λ = 0.3). Tables (3) and (4) are for Case II and Case III, respectively.

Table 2: *Simulation Results for Case I*

N	Estimates	Bias	RMSE	n	Estimates	Bias	RMSE
10	0.4219	0.1219	0.2651	70	0.3335	0.0335	0.1254
	1.1481	0.1481	0.4772		1.0131	0.0131	0.1942
	1.0841	0.1841	0.4819		0.9849	0.0849	0.305
	0.5721	0.1721	0.5426		0.4487	0.0487	0.2497
20	0.3735	0.0735	0.2062	90	0.3275	0.0275	0.1124
	1.0835	0.0835	0.348		1.0048	0.0048	0.1701
	1.0434	0.1434	0.4297		0.9808	0.0808	0.2909
	0.5046	0.1046	0.4076		0.4373	0.3730	0.2225
30	0.3540	0.0540	0.1713	150	0.3204	0.0204	0.0853
	1.0533	0.0533	0.2797		0.0998	-0.0021	0.1396
	1.0039	0.1039	0.3742		0.9600	0.0600	0.2393
	0.4951	0.0951	0.3606		0.4293	0.0293	0.1763
50	0.3378	0.0378	0.1405	170	0.3174	0.0174	0.0823
	1.0324	0.0324	0.2324		0.9969	-0.0031	0.1315
	0.9905	0.0905	0.3365		0.9549	0.0549	0.2261
	0.4550	0.0550	0.2725		0.4219	0.0219	0.1635

Table 3: *Simulation Results for Case II*

n	Estimates	Bias	RMSE	N	Estimates	Bias	RMSE
10	0.7954	0.1954	0.4988	70	0.6469	0.0469	0.2290
	0.6110	0.1110	0.3080		0.5111	0.0111	0.1061
	1.0250	0.1250	0.4286		0.9809	0.0809	0.2904
	0.6217	0.1217	0.5800		0.5105	0.0105	0.2252
20	0.7301	0.1301	0.3991	90	0.6320	0.0320	0.1907
	0.5496	0.0496	0.1884		0.5084	0.0084	0.0931
	1.0232	0.1232	0.3936		0.9665	0.0665	0.2571
	0.5751	0.0751	0.4643		0.5096	0.0096	0.2034
30	0.7013	0.1013	0.3505	150	0.6243	0.0243	0.1497
	0.5289	0.0289	0.1521		0.5032	0.0032	0.0724
	0.9932	0.0932	0.3403		0.9512	0.0512	0.2258
	0.5607	0.0607	0.3573		0.5070	0.0070	0.1680
50	0.6588	0.0588	0.2572	170	0.6192	0.0192	0.1462
	0.5177	0.0177	0.1177		0.5028	0.0028	0.0718
	0.9869	0.0869	0.3146		0.9469	0.0469	0.2102
	0.5268	0.0268	0.2862		0.5025	0.0025	0.1513

Table 4: *Simulation Results for Case III*

n	Estimates	Bias	RMSE	N	Estimates	Bias	RMSE
10	2.2708	0.2708	0.5049	70	2.1298	0.1298	0.2559
	5.4425	0.4425	1.7522		5.0473	0.0473	0.8753
	0.7649	0.2649	0.6554		0.5417	0.0417	0.1907
	0.5027	0.2027	0.4227		0.4346	0.1346	0.2376
20	2.2121	0.2121	0.3836	90	2.1104	0.1104	0.2116
	5.2441	0.2441	1.3917		5.0473	0.0473	0.7814
	0.6323	0.1323	0.3960		0.5322	0.0322	0.1673
	0.4837	0.1837	0.3432		0.4188	0.1188	0.2100
30	2.1779	0.1779	0.3433	150	2.0906	0.0906	0.1679
	5.2282	0.2282	1.2953		5.0259	0.0259	0.6080
	0.5810	0.0810	0.3174		0.5159	0.0159	0.1181
	0.4822	0.1822	0.3395		0.4053	0.1053	0.1776
50	2.1505	0.1505	0.2937	170	2.0785	0.0785	0.1473
	5.1003	0.1003	1.0516		5.0160	0.0160	0.5512
	0.5593	0.0593	0.2506		0.5135	0.0135	0.1033
	0.4484	0.1484	0.2640		0.3933	0.0933	0.1581

Tables (2), (3), and (4) presents the simulation results. As the value of the sample size (n) increases, the simulation results of the ILOEED show:

- Stability of the MLES,
- The bias of the MLEs approach zero, and
- Decrease in the RMSEs of the MLEs.

VI. Applications of the ILOEED to Industry Datasets

Odd Exponentiated Inverse Lomax Distribution (ILOEED) was fitted to three datasets. This includes datasets with increasing and bathtub hazard shapes. The Goodness-of-fit criteria used are the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). A statistical metric called the Akaike Information Criterion (AIC) by [20] was employed in the selection of models. AIC offers a quantitative method for weighing a model's complexity concerning it's goodness of fit, intending to choose the model that most accurately describes the data while preventing overfitting. Bayesian Information Criterion (BIC), a statistical metric for model selection, was developed by [21]. Similar to AIC, BIC takes a Bayesian approach to balancing model fit and complexity. Moreover, three Goodness-of-fir statistics were used. These are the Kolmogorov-Smirnov (K-s), Anderson-Darling (A-D), and Cramer-Von Mises (C-vM). By comparing the cumulative distribution functions (PDFs) of two datasets, the Kolmogorov-Smirnov test determines whether they have the same continuous distribution. It is a widely applicable test that helps to compare theoretical and empirical distributions and is especially helpful when parametric assumptions are not met. The A-D test is useful for determining fit, especially in situations with extreme values or interesting tail behavior. It does this by calculating the goodness-of-fit between the sample's empirical distribution function and the specified distribution's cumulative distribution function. This test is sensitive to deviations in the distribution's tails. The C-vM test, named for

Carl von Mises and Harald Cramer, measures the difference between the sample's empirical distribution function and the specified distribution's cumulative distribution function to determine whether the sample fits the distribution. This test is favored in some circumstances because it is easy to compute and provides a good measure of goodness-of-fit. Finally, The negative log-likelihood (-ll), which is frequently minimized in maximum likelihood estimation, measures how well a model fits observed data by esti-mating the probability of observing the data given the model parameters. It is favored for its stability and ease of use in parameter estimation and is essential to many different disciplines, including biology, econometrics, and machine learning. Fitdistrplus package by [22] in R was used in fitting the three datasets in this section.

I. Application to the Breaking Stress of Carbon Fibres Dataset

These data as reported by the [23], represent the breaking stress of carbon fibres of 50mm length in Gpa. The data is symmetry and has an increasing hazard shape. The dataset is as follows:
0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.
ILOEED was fitted alongside the Inverse Exponentiated Odd Lomax Exponential Distribution (IEOLED) by [4], Alpa Power Exponential Distribution (APED) by [24], as well as the Exponential Distribution (ED).

Table 5*: MLEs and Goodness-of-fit Criteria for the fitted ILOEED and other comparators for the comparators for the Breaking Strengths of Carbon Fibres Dataset*

Distributions	Estimates	Standard Error	-ll	AIC	BIC
ILOEED$(\upsilon,\theta,\gamma,\lambda)$	0.6586	1.3019	84.6124	177.2247	185.9833
	3.4545	5.7285			
	0.4797	0.3904			
	1047.5537	4727.8614			
IEOLED$(\upsilon,\theta,\gamma,\lambda)$	0.0854	0.0274	86.1725	180.3251	189.1037
	3.369	0.2963			
	1.03E+07	1.67E+04			
	1.69E+12	2.37E+04			
APED(υ,θ)	2.99E+05	1.19E+04	92.3964	188.7927	193.172
	1.10E+00	5.46E-02			
ED(υ)	3.63E-01	4.46E-02	132.9944	267.9887	270.1784

Table (5) presents the MLEs, log-likelihoods, AICs, and BICs of the ILOEED and others. The Table indicates that ILOEED is the best with minimum values of AIC and BIC. Furthermore, Table (6) indicated that the ILOEED fitted the data well with small values of the Goodness-of-fit statistics. These are the K-S, C-vM, and A-D.

Table 6: *The Goodness-of-fit statistics of the ILOEED and others for the Breaking Strengths of Carbon Fibres Dataset*

Distributions	K-S	C-vM	A-D
ILOEED	0.0565	0.033	0.2298
IEOLED	0.085	0.0884	0.5055
APED	0.1353	0.2826	1.6089
ED	0.3581	2.871	14.0343

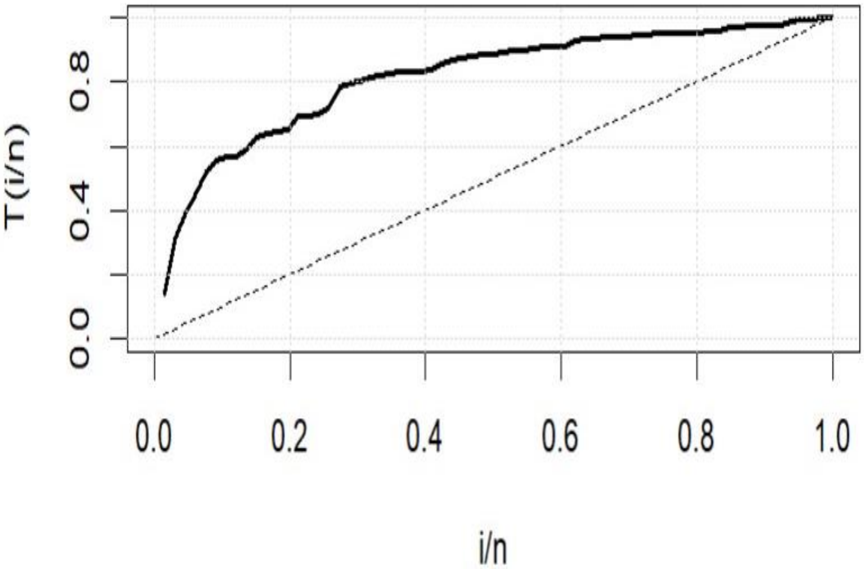

Figure 3: *The TTT-Plot for the Carbon Fibre Dataset*

The Total Time on Test (TTT) plot for the Carbon Fibre Dataset indicates an increasing hazard rate (concave shape), as seen in Figure (3).

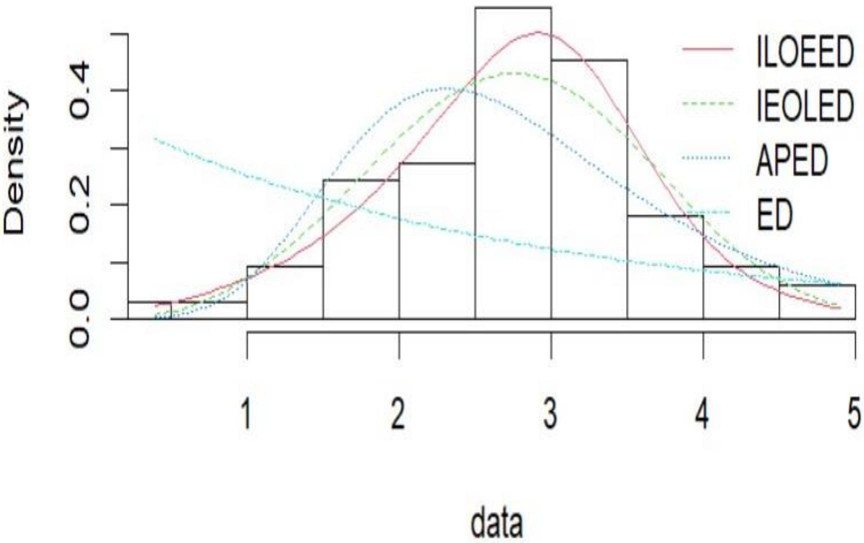

Figure 4: *The Fitted PDFs and CDFs of the ILOEED and others for the Carbon Fibre Dataset*

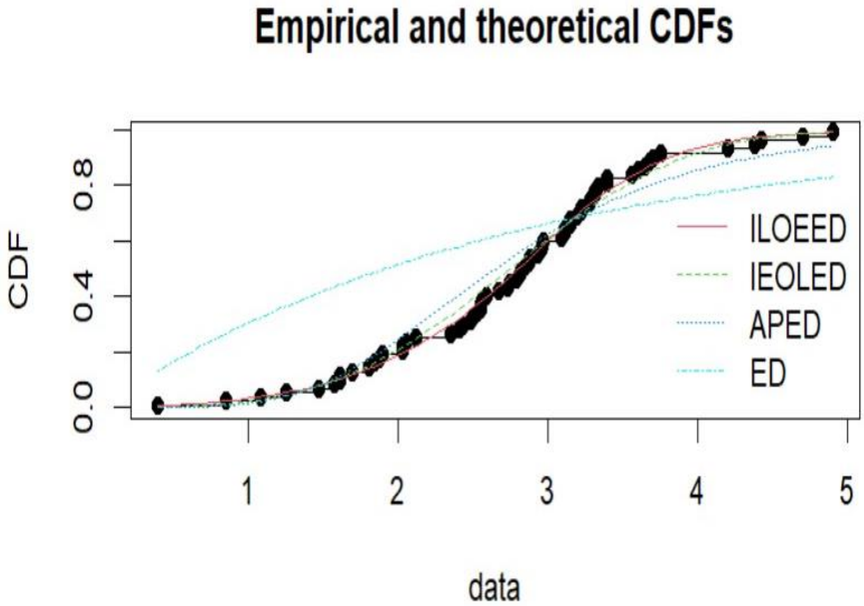

Figure 5: *The Fitted PDFs and CDFs of the ILOEED and others for the Carbon Fibre Dataset*

Figures (4) and (5) indicates that the Carbon Fibre data is symmetrical. ILOEED fitted the data better compared with the other comparators.

II. Application to the Strengths Dataset

The following dataset reported by [25], is the Strengths reading in GPa of individual carbon fibres that were put to stress at 20mm gauges and the values are:
1.359, 1.382, 1.382, 1.426, 1.434, 1.435, 1.478, 1.490, 1.511, 1.514, 1.535, 1.554, 1.566, 1.570, 1.586, 1.629, 1.633, 1.642, 1.648, 1.684, 1.697, 1.726, 1.770, 1.773, 1.800, 1.809, 1.818, 1.821,1.848, 1.880, 1.954, 2.012, 2.067, 2.084, 2.090, 2.096, 2.128, 2.233, 2.433, 2.585, 2.585.

Table 7: *MLEs and Goodness-of-fit Criteria for the fitted ILOEED and other comparators for the comparators for the Strengths Dataset*

Distributions	Estimates	Standard Error	-ll	AIC	BIC
ILOEED$(\upsilon,\theta,\gamma,\lambda)$	0.7639	1.2163	86.0082	180.0163	188.775
	3.1315	4.4416			
	0.4570	0.3044			
	1943.9353	5787.6972			
IEOLED$(\upsilon,\theta,\gamma,\lambda)$	0.1017	0.0311	88.2736	184.5472	193.3058
	3.3417	0.2939			
	786928.5	12792.55			
	6.4E+10	1.79E+04			
APED(υ,θ)	59738.7189	17013.25	95.0118	194.0236	198.4029
	1.0525	0.0556			
ED(υ)	0.5642	0.0881	96.0231	195.0986	197.0178

Table (7) presents the MLEs, log-likelihoods, AICs, and BICs of the ILOEED and others. The

Table indicates that $ILOEED$ is the best with minimum values of AIC and BIC. Furthermore, Table (8) indicated that the $ILOEED$ fitted the data well with small values of the Goodness-of-fit statistics.

Table 8: *The Goodness-of-fit statistics of the ILOEED and others for the Strengths Dataset*

Distributions	K-S	C-vM	A-D
ILOEED	0.0581	0.046	0.2382
IEOLED	0.0876	0.0949	0.6043
APED	0.1461	0.3327	1.9048
ED	0.3447	2.7639	13.5301

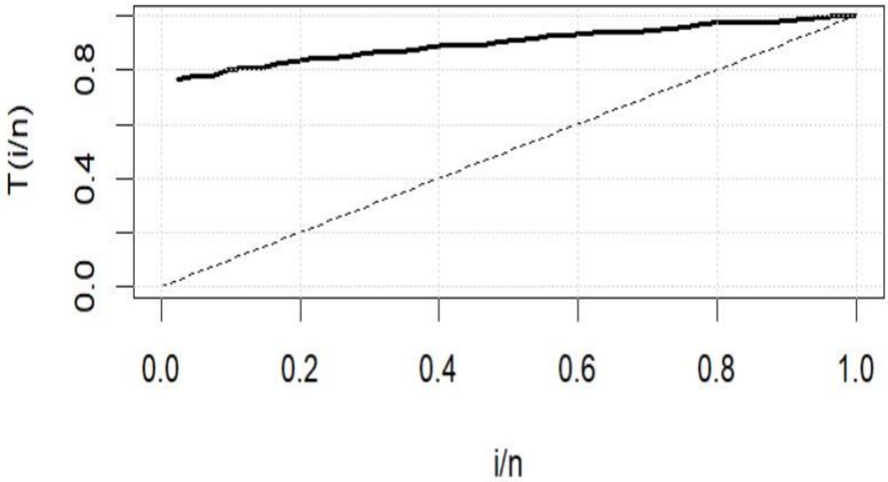

Figure 6: *The TTT-Plot for the Strengths Dataset*

The Total Time on Test (TTT) plot for the Strengths Dataset indicates an increasing hazard rate (concave shape), as seen in Figure (6).

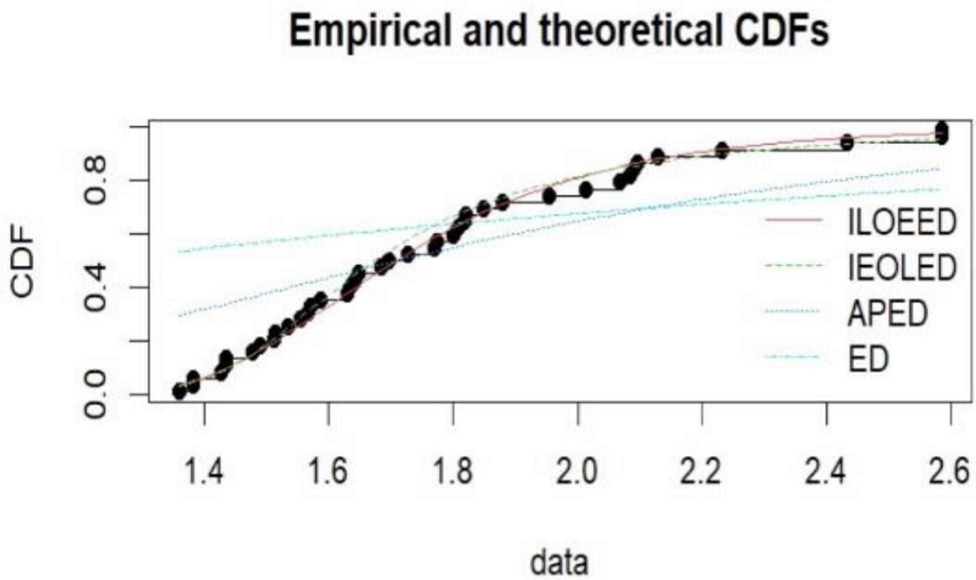

Figure 7: *The Fitted CDFs of the ILOEED and others for the Strengths Dataset*

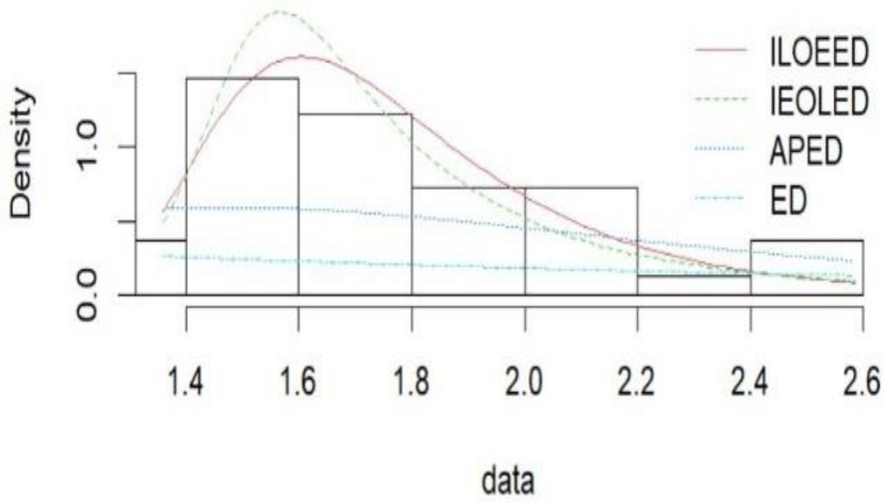

Figure 8: *The Fitted PDFs of the ILOEED and others for the Strengths Dataset*

Figures (7) and (8) indicates that the Strengths data is symmetry. ILOEED fitted the data well compared with the other comparators.

III. Application to the Times to Failure Dataset

This dataset has a Bathtub-shape hazard rate as reported by [26]. The data is about the Times to Failure of 50 Devices that were on life test at time 0. The dataset is as follows:
0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 1, 1, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86.

Table 9: *MLEs and Goodness-of-fit Criteria for the fitted ILOEED and other comparators for the comparators for the Times to Failure Dataset*

Distributions	Estimates	Standard Error	-ll	AIC	BIC
ILOEED$(\upsilon,\theta,\gamma,\lambda)$	0.1025	0.0478	234.3902	476.7803	484.5076
	0.3363	0.2143			
	1.4327	1.7004			
	2.3317	4.4526			
IEOLED$(\upsilon,\theta,\gamma,\lambda)$	0.1104	0.2196	243.9899	495.9799	503.7072
	0.8589	0.1104			
	2968.220	5982.6273			
	490198	1.4053			
APED(υ,θ)	1.9690	1.4053	244.2936	492.5873	496.4509
	0.0259	0.0048			
ED(υ)	0.0224	0.0031	244.7001	491.4002	493.3321

Table (9) presents the MLEs, log-likelihoods, AICs, and BICs of the ILOEED and others. The Table indicates that ILOEED is the best with minimum values of AIC and BIC. Furthermore, Table (10) indicated that the ILOEED fitted the data well with small values of the Goodness-of-fit statistics.

Table 10: *The Goodness-of-fit statistics of the ILOEED and others for the tomes to Failure Dataset*

Distributions	K-S	C-vM	A-D
ILOEED	0.1503	0.2817	1.9053
IEOLED	0.2012	0.5934	3.8944
APED	0.1799	0.5331	4.7647
ED	0.1904	0.5727	4.5609

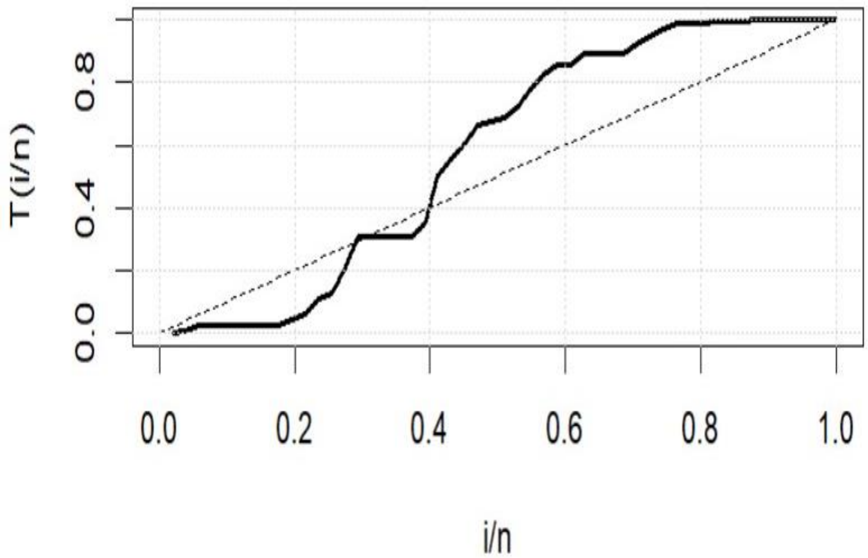

Figure 9: *The TTT-Plot for the Times to Failures Dataset*

The Total Time on Test (TTT) plot for the Fatigue Fracture Dataset indicates a decreasing and then increasing hazard rate (Bath-tub shape), as seen in Figure (9).

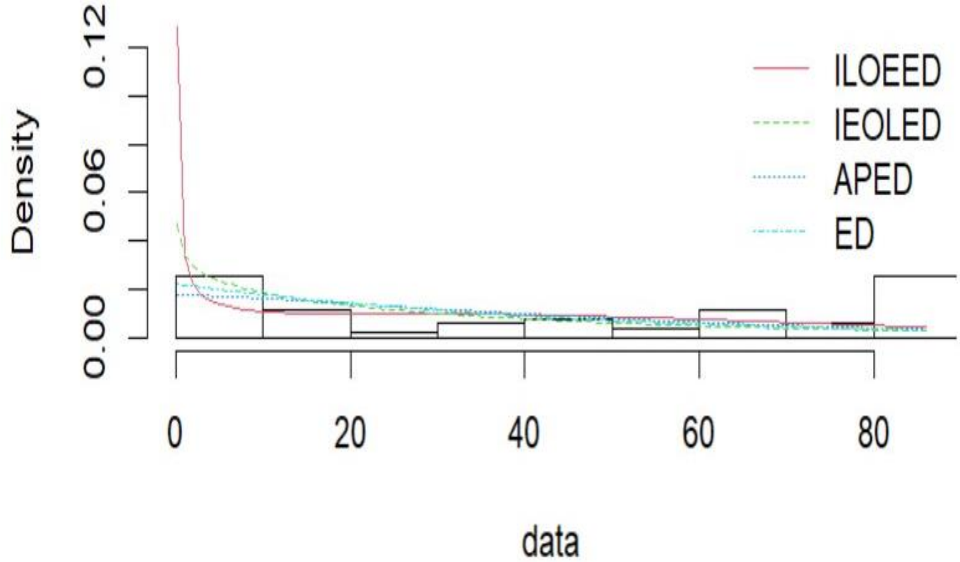

Figure 11: *The Fitted PDFs of the ILOEED and others for the Times to Failure Dataset*

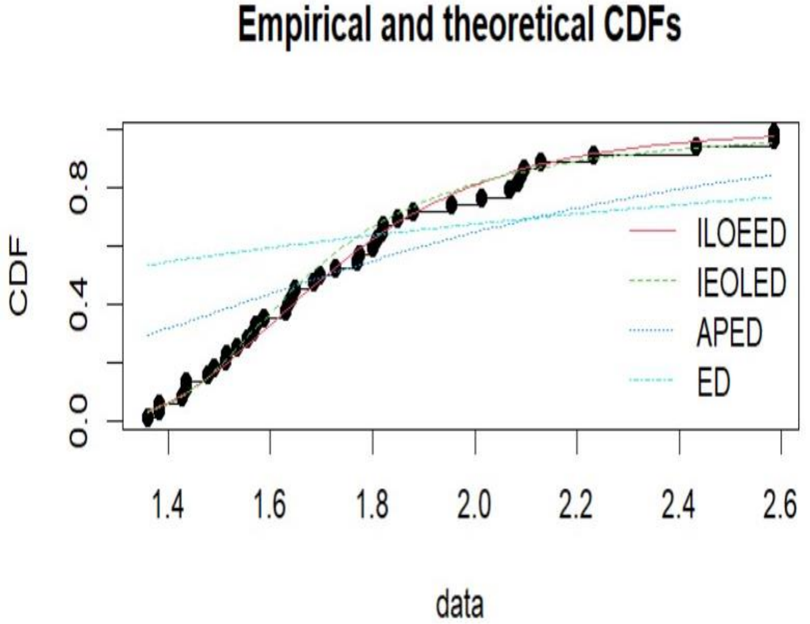

Figure 11: *The Fitted CDFs of the ILOEED and others for the Times to Failure Dataset*

Figures (10) and (11) indicates that the Times to Failure data is skewed to the right. ILOEED fitted the data well compared with the other comparators.

VII. Conclusion

In this research, we suggest and investigate a novel probability distribution that is a combination of the Inverse Lomax and exponential distributions, combining the properties of the two. This merger is required if the data in issue combines both the Inverse Lomax and the exponential distributions' features described in Section 1. We looked into some of its statistical properties, such as moments, the moment generating function, the characteristic function, and the quantile function. The parameters were determined using the maximum likelihood technique. According to the simulation studies, as the sample size grows, the estimations of the Biases and RMSEs approach zero, indicating that the estimates are more accurate. Three cases of parameter combination were considered for the simulation studies. The estimates were stable. Exemplifications of real-world datasets demonstrate the ILOEED's significance. For the three datasets used, the proposed distribution is the best with minimum values of the Goodness-of-fit criteria and Goodness-of-fit statistics. This means ILOEED can be used to fit datasets with increasing and bath-tube hazard rates. Based on these facts, we hope that the ILOEED will be preferred above the other models considered in this study. Only datasets from the industry were considered to fit the proposed distribution. We suggest that other areas should be explored in terms of the application of the proposed distribution. Also, other methods of estimation can be considered in further studies.

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