STOCHASTIC OPTIMIZATION OF PERISHABLE INVENTORY INCORPORATING PRESERVATION, FRESHNESS INDEX, EXPIRY DATE AND OPTIMISING PROMOTIONAL STRATEGIES FOR EFFECTIVE MANAGEMENT

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Abstract

Inventory management is a critical aspect of supply chain efficiency and can be influenced by various factors such as advertising, pricing, and preservation policies. Recent research has proposed a model that considers critical variables such as fluctuations in pricing, advertising tactics, and preservation expenses within uncertain scenarios to improve inventory management. The study provides valuable insights into advertising dynamics, optimal pricing strategies, and the impact of preservation costs on decision-making. Decision-makers can apply these insights to enhance the efficiency of their supply chains in a competitive environment. The study emphasizes the importance of flexibility while aligning inventory practices with corporate sustainability goals. Although the model's applicability may be context-specific, the findings contribute to discussions on inventory management strategies while acknowledging certain assumptions made during the study. Proper advertising, pricing, and preservation policies can increase awareness, attract customers, and maintain quality, influencing product demand. This research proposes a model to improve inventory management, considering variables such as pricing fluctuations, advertising tactics, and preservation expenses in uncertain scenarios. The study provides insights into advertising dynamics, optimal pricing strategies, and how preservation costs influence decision-making. Decision-makers can apply these insights to improve supply chain efficiency. The study stresses the importance of flexibility in a competitive environment and aligning inventory practices with corporate sustainability goals. The findings contribute to discussions on inventory management strategies, but the model's applicability may be context-specific, and the study makes certain assumptions.

Keywords: Inventory, Deterioration, Preservation, Stochastic optimisation, Advertising

1. Introduction

The issue of demand uncertainty is a challenging problem in inventory control that makes it difficult to accurately predict market dynamics and consumer behaviour. This uncertainty arises due to several factors, including evolving consumer preferences influenced by societal changes and trends and market dynamics such as economic conditions and currency fluctuations, which compound the complexity of demand forecasting. For example, during a seasonal sale, a sudden and unpredictable surge in demand for a popular product may occur due to external factors such as economic incentives and promotional activities, in addition to consumer behaviour.

The empirical evaluation section of this paper focuses on revealing the mathematical behaviour and patterns of demand uncertainty by closely examining real-world data to identify trends and deviations. The paper associates these findings with probability distributions, providing businesses with a framework to respond to the uncertainty of market dynamics. It is characterised by Stochastic dynamics, which are characterized by unpredictability and randomness, play a critical role in understanding the dynamics of modern markets. The study aims to fill a gap in existing knowledge by recognizing the stochastic nature of demand and seamlessly incorporating it into the decision-making process.

Traditional models may need to pay more attention to the impact of preservation costs on overall profitability. This research acknowledges preservation costs as a critical factor and efforts to provide decision-makers with insights into optimizing preservation strategies. By recognizing the insubstantial balance between the need for timely replenishment and the financial implications of preserving inventory, businesses can make informed decisions that align with their overarching goals. The element of expiry dates introduces a time-sensitive dimension to inventory management. In industries where products have a limited shelf life, managing stock effectively to avoid losses due to expiry is essential. By doing so, businesses can align their stocking strategies with product lifecycles, minimizing waste and maximizing profitability.

This literature review focuses on the replenishment period, where inventory experiences a continuous decrease due to the combined impact of demand and deterioration. The pioneering work has revolutionized inventory modelling by introducing the concept of no-shortage stock level [1]. An extension of this model, using a two-parametric Weibull distribution for variable deterioration rates, has provided insights into different distribution patterns [2] and, in inventory management, emphasized the significance of demand fluctuations [3]. And an inventory system with deteriorating items, challenging the assumption of a constant demand rate [4].

Mostly, models considers deterioration rate is constant over time. However, in reality, deterioration rate can fluctuating over time due to different factors like environmental factors, storage conditions, and other external factors. To address this problem, the EOQ model was developed with time-varying deterioration and inventory-dependent demand [5]. it can also be applied to declining products with shifting demand based on factors like selling price and advertising frequency. Moving forward, the exploration of time-dependent factors in inventory models will be considered by considering holding costs and demand functions that vary over time [6]. This temporal dimension adds a layer of sophistication, acknowledging the dynamic nature of both costs and demand, further enhancing the applicability of inventory models to real-world scenario [7]. A two-phase inventory model is proposed for non-instantaneous decay items with stock-dependent demand before and constant demand during decay and shortage periods [8].

A pricing strategy known as dynamic pricing was developed for products that have price sensitive demand [9]. A seasonal inventory model proposed with preservation technology reveals profit dynamics and determines optimal solutions for selling price, replenishment cycle, and preservation investment across business setups [10]. Also, the model focuses on optimized price and inventory control for items that deteriorate over time, experience partial backordering, and highlight the need for a comprehensive approach to decision-making [11]. The framework proposed by [12] presents strategies for aligning economic and environmental objectives, leading to cost reductions, lower carbon footprints, and improved operational stability. A comparison of deterministic and stochastic inventory modeling approaches reveals that deterministic models may oversimplify demand dynamics, whereas stochastic models, particularly those utilizing Particle Swarm Optimization (PSO), better address market uncertainties. This highlights the value of stochastic methods for enhancing adaptive decision-making, resource allocation, and risk management in complex supply chains. To maximize profits, [13] introduces a novel dynamic pricing approach that optimizes the number of price adjustments during sales seasons based on stochastic factors and real-time demand fluctuations. Further, A model for demand forecasting and dynamic pricing evaluates Holt-Winters Exponential Smoothing (HWES) and Autoregressive Integrated Moving Average (ARIMA) [14]. The study finds that ARIMA models outperform HWES in minimizing lost sales and mitigating stockouts under varying economic conditions.

Table 1: *Contribution table and highlighting the research gap*

Author(s)	Deterioration	Preservation	Freshness	Pr. Opt.	St. Opt.
[16]	Non-Inst	×	×	✓	×
[17]	Non-Inst	×	×	✓	×
[18]	Non-Inst	×	×	×	×
[19]	Inst	×	×	✓	×
[20]	Non-Inst	✓	×	×	✓
[21]	Non-Inst	✓	×	×	×
[22]	Non-Inst	×	×	✓	✓
[23]	Non-Inst	✓	×	✓	×
[24]	Inst.	✓	×	✓	✓
[25]	×	×	×	✓	✓
This Study	Inst	✓	✓	✓	✓

Non-Inst: Non-instantaneous deterioration; Pro. Opt.: Promotional Optimisation; St. Opt.: Stochastic Optimisation

Integrating promotional efforts, preservation technology, proportionate shortage time, and partial backlogging emphasizes the need for a holistic understanding of inventory dynamics in real-time scenarios. In their subsequent work, [15] extended their research to investigate profit optimization in conversion processes such as milk to butter and debenture to share. By employing Particle Swarm Optimization, the study addresses factors like conversion timing, quantity, cost, and duration, offering insights into strategic decision-making for inventory managers. This paper makes a significant contribution to the modelling of a retailer's business by incorporating promotional optimization and stochastic optimization. Our analysis has identified a research gap in this field, emphasizing the need for further exploration. Table 1 summarizes these contributions and gaps.

2. Problem Definition

In the traditional inventory models, which often rely on deterministic assumptions, fail to address these uncertainties effectively. Another issue like deterioration is particularly critical for businesses dealing with perishable goods. Conventional models often overlook the intricate dynamics of deterioration, leading to suboptimal decisions and financial losses. This research addresses this challenge by integrating strategies that account for the perishable nature of goods, providing insights for optimizing stock levels and minimizing financial impact. Preservation costs further complicate inventory management, requiring a balance between timely replenishment and the costs associated with preserving inventory. This study investigates optimal resource allocation strategies, considering preservation costs to enhance decision-making and align with broader business objectives. Promotional efforts also impact inventory dynamics and profitability, yet traditional models often neglect this aspect. The research examines the effect of promotional activities, including advertising frequency, on inventory management and profitability. By understanding the relationship between advertising strategies and inventory dynamics, businesses can refine their promotional efforts for optimal results.

This research aims to develop a comprehensive inventory management model that incorporates stochastic dynamics, addresses deterioration and preservation costs, integrates expiry dates, and optimizes promotional efforts. The objective is to provide decision-makers with actionable strategies to navigate modern market complexities and enhance overall inventory management performance.

3. NOTATIONS

Notations bearing usual tradition, utilized in the subsequent discussions, are laid down as follows:
Parameters:

τ : Maximum life span of the product,

γ : Advertising impact parameter associated with the advertising frequency,

a : Demand potential parameter,

α : Price sensitive parameter of the demand,

b : Time sensitive parameter of the demand,

ϵ : Continuous random variable associated with demand responsible for demand uncertainty,

λ : Preservation impact parameter,

η : Gain fraction of total demand in case of shortage,

C_h : Holding cost per unit per unit of time,

c_1 : Shortage Cost per unit per unit of time,

c_2 : Opportunity cost per unit of time,

C_0 : Set up cost per set up,

W : Backlogged demand,

Q : Total order quantity of without shortage case,

S : Initial stock level in case of shortage,

$f(u)$: Instantaneous freshness index,

Decision variables:

p : Selling price per unit,

T : Total inventory period of the model including with and without shortage cases,

t_1 : Shortage time period,

A : Advertisement frequency,

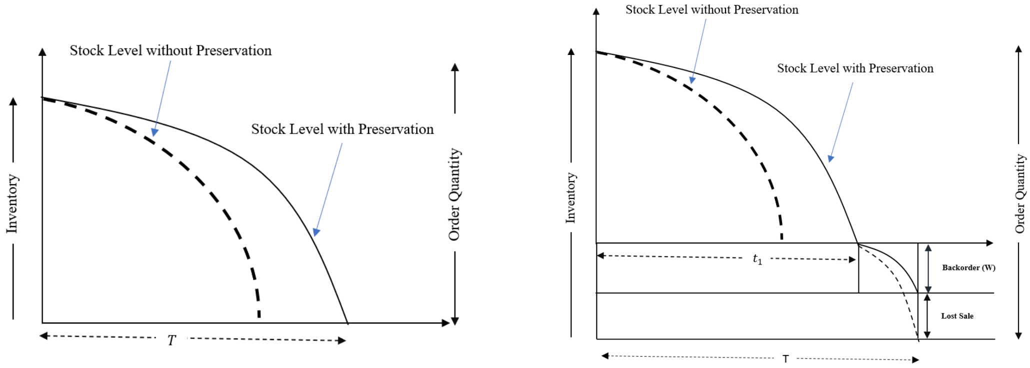
ξ : Per unit preservation cost,

$q(t)$: Instantaneous inventory level at variable time t,

Π_1, Π_2 : Total profit function for the model including without and with shortage respectively,

4. Assumptions

1. The proposed model is for single retailer model dealing with the perishable product having fixed self-life τ,
2. Demand is influenced by the freshness level of the product and the freshness index is determined by $f(t) = \frac{\tau-t}{\tau}, \quad t \in [0, \tau]$
3. The deterioration is considered instantaneous determined by $\theta(t) = \frac{1-m(\xi)}{1+\tau-t}$, where per unit preservation cost ξ is employed to reduce the rate of deterioration in $0 \le t \le T \le \tau$,
4. The preservation effort is a continuous and concave function of per unit preservation cost ξ, and determined as $m(\xi) = 1 - e^{-\alpha\xi}$. Thus, reduced rate of deterioration is $1 - m(\xi)$,
5. Price-sensitive and time dependent demand is employed under the influence of uncertain factor ϵ and determined as $D(t, p) = (a - \alpha p - bt + \epsilon)$, for $a > 0$, $b > 0$ and $p > 0$



(a) Stock level without shortage

(b) Stock level with shortage

Figure 1: *Comparison of stock levels with and without shortage*

6. The promotional effort is incorporated to enhance the demand with promotional efforts parameter A^γ. Where, the γ is the frequency of advertisement and gives a freedom to management that how many time it could advertise the product to get optimum profit,

7. The mathematical formulation section is categorised into two categories; 'with' and 'without'shortage. In shortage scenario, the demand is assumed to be partially backlogged with the gain fraction η and Lead time is to be considered zero.

5. Mathematical Formulation

This paper introduces a mathematical framework to help retailers to increase profits when restocking perishable items having a fix self-life τ. The proposed formulation focuses on optimizing profit of retailer who invests in an ordering cost (C_0) and restocking inventory regularly having purchasing cost (C_p) and selling price p. With negligible lead time, This model accounts for the critical relationship between freshness of stock level, demand level, as consumers strongly prefer fresh, perishable items and the advertisement frequency. It is simply to understand that the maximum lifespan of the product determines the maximum length of a business cycle (T) and as well as less deterioration. As a result, products that stay longer their maximum lifespan will be avid to sale for the customers and organisation's welfare. Additionally, consumers always struggle to purchase the best available product at a reasonable price, so the selling price is often the primary factor for customers. Therefore, the proposed formulation optimises the replenishment time T, optimal selling price p, advertising frequency A, and the preservation cost ξ.

The formulation is divided into two separate categories, with and without shortage. The further section focuses on the case of without shortage.

Case I: The model with no shortage

The model (refer to figure 1(a)) starts by examining a situation where a retailer gets the order quantity Q, of a perishable items from a supplier. After some time, the quality and quantity of the products reduces gradually due to both natural decay and consumer demand. Eventually, the quantity will reach zero at time $t = T$. In any instant t, the stock level $q(t)$ follows following differential equation:

$$\frac{dq(t)}{dt} + \frac{1 - m(\xi)}{1 + \tau - t} q(t) = -A^\gamma (a - \alpha p - bt + \epsilon) \frac{\tau - t}{\tau}, \quad 0 \le t \le T \tag{1}$$

subject to $q(T) = 0$, and $q(0) = Q$. For notational convenience consider $1 - m(\xi) = k$, and

$$\psi_1 = \frac{b}{k-3}; \quad \psi_2 = \frac{-a + p\alpha - \epsilon + b(\tau+1)}{k-1}; \quad \psi_3 = \frac{-a + p\alpha - \epsilon + b(\tau+2)}{k-2};$$

$$\phi_1 = (1+\tau)^3 C_p + \frac{C_h(1+\tau)^4}{4}; \quad \phi_2 = (1+\tau)C_p + \frac{C_h(1+\tau)^2}{2};$$

$$\phi_3 = (1+\tau)^2 C_p + \frac{C_h(1+\tau)^3}{3}; \quad \phi_4 = (1+\tau)^k C_p + \frac{C_h(1+\tau)^{(k+1)}}{k+1}$$

Equation (1), leads to

$$\begin{aligned} q(t) = \quad & \frac{A^\gamma}{\tau}\Big[-(1+\tau-t)\Big\{\psi_1(1+\tau-t)^2 + \psi_2 - \psi_3(1+\tau-t)\Big\} \\ & +(1+\tau-t)^k(1+\tau-T)^{(1-k)}\Big\{\psi_1(1+\tau-T)^2 + \psi_2 - \psi_3(1-T+\tau)\Big\}\Big] \end{aligned} \tag{2}$$

The retailer's lot size Q with $q(0) = Q$, will be

$$\begin{aligned} Q(T) = \quad & \frac{A^\gamma}{\tau}\Big[-\psi_1(1+\tau)^3 - \psi_2(1+\tau) + \psi_3(1+\tau)^2 \\ & +(1+\tau)^k\Big\{\psi_1(1+\tau-T)^{(3-k)} + \psi_2(1+\tau-T)^{(1-k)} - \psi_3(1+\tau-T)^{(2-k)}\Big\}\Big] \end{aligned} \tag{3}$$

From the above inventory level and retailer's lot size, there are several cost functions incorporated in the proposed profit function. The purchase cost (PC) of each cycle for the retailer's is as follows:

$$\begin{aligned} PC(T) = \quad & C_p Q = C_p \frac{A^\gamma}{\tau}\Big[-\psi_1(1+\tau)^3 - \psi_2(1+\tau) + \psi_3(1+\tau)^2 + \\ & (1+\tau)^k\Big\{\psi_1(1+\tau-T)^{(3-k)} + \psi_2(1+\tau-T)^{(1-k)} - \psi_3(1+\tau-T)^{(2-k)}\Big\}\Big] \end{aligned} \tag{4}$$

The cost associated with holding (CC) over the entire cycle

$$\begin{aligned} CC(T) = \quad & C_h \int_0^T q(t)dt \\ & = \frac{C_h A^\gamma}{\tau}\Big[\psi_1(1+\tau-t)^4\left(\frac{1}{4} - \frac{1}{k+1}\right) + \psi_2(1+\tau-T)^2\left(\frac{1}{2} - \frac{1}{k+1}\right) \\ & -\psi_3(1+\tau-t)^3\left(\frac{1}{3} - \frac{1}{k+1}\right) - \frac{(1+\tau)^{(k+1)}}{k+1}\Big[\psi_2(1+\tau-T)^{(1-k)} + \\ & \psi_1(1+\tau-T)^{(3-k)} - \psi_3(1+\tau-T)^{(2-k)}\Big] - \frac{\psi_1}{4}(1+\tau)^4 - \frac{\psi_2}{2}(1+\tau)^2 + \\ & \frac{\psi_3}{3}(1+\tau)^3\Big] \end{aligned} \tag{5}$$

The advertising cost (AC) with advertisement frequency A is

$$AC = G \times A \tag{6}$$

where G is the cost per advertisement. During the interval $[0, T]$, the total sales revenue (SR) is

$$\begin{aligned} SR(T) = \quad & p\int_0^T A^\gamma(a - \alpha p - bt + \epsilon)\frac{\tau - t}{\tau}dt \\ = \quad & \frac{pA^\gamma}{6\tau}\Big[2bT^3 + 6T(a - \alpha p + \epsilon)\tau - 3T^2(a - \alpha p + b\tau + \epsilon)\Big] \end{aligned} \tag{7}$$

Incorporating all the above cost, the profit earned by retailer in period $[0, T]$ is ,

$$\begin{aligned}\Pi_1(A,p,T) = \quad & SR - OC - AC - PC - CC \\ = \quad & \frac{pA^\gamma}{6\tau}\Big[2bT^3 + 6T(a-\alpha p+\epsilon)\tau - 3T^2(a-\alpha p+b\tau+\epsilon)\Big] - C_0 - GA \\ & + \frac{A^\gamma}{\tau}\Big[(\psi_1\phi_1 + \psi_2\phi_2 - \psi_3\phi_3) + \Big\{-\psi_1(1+\tau-T)^{(3-k)} - \psi_2(1+\tau-T)^{(1-k)} \\ & + \psi_3(1+\tau-T)^{(2-k)}\Big\}\phi_4 - C_h\Big\{\psi_1(1+\tau-T)^4\left(\frac{1}{4}-\frac{1}{k+1}\right) \\ & + \psi_2(1+\tau-T)^2\left(\frac{1}{2}-\frac{1}{k+1}\right) - \psi_3(1+\tau-T)^3\left(\frac{1}{3}-\frac{1}{k+1}\right)\Big\}\Big] \end{aligned} \tag{8}$$

Our primary objective is to determine the optimal advertisement frequency, per unit selling price, and replenishment time that will lead to the highest expected profit for the retailer. We determined the optimal advertisement frequency using an algorithmic approach, and obtained conditions in the theorem given below.

Theorem 5.1. *The proposed profit function $\Pi_1(A, p, T)$ is strictly concave for price p, for $T < 2\tau$.*
Proof: *Consider the profit function equation (8), we have*

$$\begin{aligned}\frac{\partial\Pi_1(A,p,T)}{\partial p} = \quad & \frac{A^\gamma}{6\tau}\Big[2bT^3 + 6T(a-\alpha p+\epsilon)\tau - 3T^2(a-\alpha p+b\tau+\epsilon)\Big] + \frac{pA^\gamma}{6\tau}\Big[-6T\alpha\tau \\ & + 3T^2\alpha\Big] + \frac{A^\gamma}{\tau}\left[\left[\frac{\alpha\phi_2}{k-1} - \frac{\alpha\phi_3}{k-2}\right] + \right. \\ & \left[-\frac{\alpha(1+\tau-t_1)^{1-k}}{k-1} + \frac{\alpha(1+\tau-T)^{2-k}}{k-2}\right]\phi_4 \\ & \left. -C_h\left[\frac{\alpha(1+\tau-t_1)^2(\frac{1}{2}-\frac{1}{k+1})}{k-1} - \frac{\alpha(1+\tau-t_1)^3(\frac{1}{3}-\frac{1}{k+1})}{k-2}\right]\right] \end{aligned} \tag{9}$$

Putting $\frac{\partial\Pi_1(A,p,T)}{\partial p} = 0$, we have optimal value of p denoted by p^*

$$\begin{aligned} p^* = \quad & \frac{1}{\alpha T(2-T)}\left[\frac{A^\gamma}{6\tau}\Big(2bT^3 + 6T\tau(a+\epsilon) - 3T^2(a+b\tau+\epsilon)\Big) + \frac{A^\gamma}{\tau}\left\{\left(\frac{\alpha\phi_2}{k-1} - \frac{\alpha\phi_3}{k-2}\right)\right.\right. \\ & \left. -C_h\left(\frac{\alpha(1+\tau-T)^2(\frac{1}{2}-\frac{1}{k+1})}{k-1} - \frac{\alpha(1+\tau-T)^3(\frac{1}{3}-\frac{1}{k+1})}{k-2}\right)\right\} \\ & \left. + \left(\frac{\alpha(1+\tau-T)^{2-k}}{k-2} - \frac{\alpha(1+\tau-T)^{1-k}}{k-1}\right)\phi_4\right] \end{aligned} \tag{10}$$

differentiating again the above equation, we have

$$\frac{\partial^2\Pi_1(A,p,T)}{\partial p^2} = 2A^\gamma\alpha T\left[-1 + \frac{T}{2\tau}\right] < 0, \quad for \quad T < 2\tau \tag{11}$$

Hence, for $T < 2\tau$, the profit function $\Pi_1(A, p, T)$ is concave with respect to selling price p, and attained maxima at p^* (refer to figure 2(a)).

After, finding the condition for the optimal solution of the price, we have shown the concave behaviour of the objective function with respect to the time T.

Theorem 5.2. *The objective function $\Pi_1(A, p, T)$ is concave in $T \in (0, \tau]$.*

Proof: *From the objective function $\Pi_1(A,p,T)$ (equation (8)), we have*

$$\begin{aligned}
\frac{\partial \Pi_1(A,p,T)}{\partial T} &= \Bigg[\frac{pA^\gamma}{6\tau}\left\{6bT^2+6(a-\alpha p+\epsilon)\tau-6T(a-\alpha p+b\tau+\epsilon)\right\}\\
&+\frac{A^\gamma}{\tau}\left\{(3-k)\psi_1(1+\tau-T)^{(2-k)}+(1-k)\psi_2(1+\tau-T)^{-k}\right.\\
&\left.-(2-k)\psi_3(1+\tau-T)^{(1-k)}\right\}\phi_4\\
&-C_h\frac{A^\gamma}{\tau}\left\{-4\psi_1(1+\tau-T)^3\left(\frac{1}{4}-\frac{1}{k+1}\right)\right.\\
&\left.-2\psi_2(1+\tau-T)\left(\frac{1}{2}-\frac{1}{k+1}\right)+3\psi_3(1+\tau-T)^2\left(\frac{1}{3}-\frac{1}{k+1}\right)\right\}\Bigg] \qquad (12)
\end{aligned}$$

The above equation leads to,

$$\begin{aligned}
\frac{\partial^2 \Pi_1(A,p,T)}{\partial T^2} = \quad & \frac{A^\gamma}{\tau}\Big[p\left[b(2T-1)+\alpha p-(a+\epsilon)\right]-(1+\tau-T)^{-k}\left[-\psi_1(k-2)\right.\\
&\left.(k-3)(1+\tau-T)+k(k-1)(1+\tau-T)^{-1}\psi_2+\psi_3(k-1)(k-2)\right]\phi_4\\
&-C_h\Big[12\psi_1(1+\tau-T)^2\left(\frac{1}{4}-\frac{1}{k+1}\right)\\
&+2\psi_2\left(\frac{1}{2}-\frac{1}{k+1}\right)-6\psi_3(1+\tau-T)\left(\frac{1}{3}-\frac{1}{k+1}\right)\Big]\Big] \qquad (13)
\end{aligned}$$

$$\frac{\partial^2 \Pi_1(A,p,T)}{\partial T^2} = \frac{A^\gamma}{\tau}(\delta_1-\delta_2) < 0$$

for $Max(\delta_1) < Min(\delta_2)$, where

$$\begin{aligned}
\delta_1 = \quad & p\left[b(2T-1)+\alpha p-(a+\epsilon)\right]-(1+\tau-T)^{-k}\left[-\psi_1(k-2)(k-3)(1+\tau-T)\right.\\
&\left.+k(k-1)(1+\tau-T)^{-1}\psi_2+\psi_3(k-1)(k-2)\right]\phi_4
\end{aligned}$$

and

$$\delta_2 = C_h\left[12\psi_1(1+\tau-T)^2\left(\frac{1}{4}-\frac{1}{k+1}\right)+2\psi_2\left(\frac{1}{2}-\frac{1}{k+1}\right)-6\psi_3(1+\tau-T)\left(\frac{1}{3}-\frac{1}{k+1}\right)\right]$$

Here

$$\frac{\partial \delta_1}{\partial T} > 0,\ \frac{\partial \delta_2}{\partial T} < 0$$

and thus, δ_1 is increasing in T and δ_2 is decreasing in T, Due to this, maximum and minimum value of $\delta_1$and δ_2 respectively is obtained on τ in the interval $[0,\tau]$.
From above,

$$\begin{aligned}
Max(\delta_1)-Min(\delta_2) = \quad & -\frac{1}{(3-k)(1+k)}\left[(3-k)(1+k)p(a-p\alpha+\epsilon-b(-1+2\tau))\right.\\
&+C_h\left[(a-\alpha p-\epsilon)(5-k)+b(2+5\tau-k\tau)\right]\\
&+(1+\tau)^k(C_h+C_p+C_pk+C_h\tau)(a(-3+5k-2k^2)\\
&\left.+(3-5k+2k^2)(p\alpha-\epsilon)+b(3\tau+2k^2(1+\tau)-k(4+5\tau)))\right] < 0
\end{aligned} \qquad (14)$$

Hence, the profit function $\Pi_1(A,p,T)$ (equation (8)) is concave with respect to T .

Using these obtained conditions for optimality of objective function, we have developed an algorithm to obtain the numerical solution of the problem.

Algorithm 1. *Algorithm to determine the optimal solution without shortage*
Input: *Demand, A_{max}, Parameters, Objective function*
Output: *Optimal value of p^*, T^*, A^*, π^**
Define Module 1 taking argument (μ, σ)

1. *Initialize $\epsilon = f(\mu, \sigma)$*
2. *Return ϵ*

Main Body:

1. *Initialize $T^* = 0, p^* = 0, A^* = 0, \xi^* = 0, \pi^* = -\infty$*
2. *for $A = 1, 2, ..., A_{max}$*
3. *Solve Eq. (8) and (11) for p and T.*
4. *Solve $\frac{\partial \pi_1}{\partial \xi} = 0$ for ξ.*
5. *If $(T < 2\tau)$ do*
6. *If $[\delta_1]_{T=\tau} < [\delta_2]_{T=\tau}$ do*
7. *Calculate $\pi_1(A, p, T, \xi)$*
8. *If $\pi_1(.) > \pi_1^*(.)$*
9. *Set $T^* = T, p^* = p, A^* = A, \xi^* = \xi, \pi^* = \pi_1$*
10. *end If*
11. *end If*
12. *end If*
13. *end for*
14. *Get $T^*, p^*, A^*, \xi^*, \pi^*$*

Now, we have another category for the formulation, in which the shortage is allowed. Further discussion have investigated the shortage condition and derived the condition for the optimal solution. where

Case II: The model with shortage

At the beginning of the sale period, the retailer purchases Q units of a perishable products from the supplier. These units are considered fresh and in their original state. Once the stock reaches to zero, the retailer experiences shortages, and the demand is accumulated with a fraction η in interval $[t_1, T]$ (Figure 1(b)). So, the inventory level at any instant is govern in equation 15 and 16.

$$\frac{dq_1(t)}{dt} + \frac{1 - m(\xi)}{1 + \tau - t} q(t) = -A^{\gamma}(a - \alpha p - bt + \epsilon)\frac{\tau - t}{\tau}, \quad 0 \le t \le t_1 \tag{15}$$

and

$$\frac{dq_2(t)}{dt} = -\eta A^{\gamma}(a - bt - \alpha p + \epsilon), \quad t_1 \le t \le T \tag{16}$$

with the help of $q_1(0) = S$,$q_1(t_1) = 0$,$q_2(t_1) = 0$, and $q_2(T) = -W$., the above equations, provide solution given as follows

$$\begin{aligned} q_1(t) = \quad & \frac{A^\gamma}{\tau}\Big[-\psi_1(1+\tau-t)^3 - \psi_2(1+\tau-t) + \psi_3(1+\tau-t)^2 + \\ & (1+\tau-t)^k(1+\tau-t_1)^{(1-k)}\Big\{\psi_1(1+\tau-t_1)^2 + \psi_2 - \psi_3(1-t_1+\tau)\Big\}\Big] \end{aligned} \tag{17}$$

and,

$$q_2(t) = \eta A^\gamma\left[(a-\alpha p+\epsilon)(t_1-t) - \frac{b(t_1^2-t^2)}{2}\right] \tag{18}$$

The given amounts for initial inventory and backlogged shortage respectively are as follows:

$$\begin{aligned} S = \quad & \frac{A^\gamma}{\tau}\Big[-\psi_1(1+\tau)^3 - \psi_2(1+\tau) + \psi_3(1+\tau)^2 + \\ & (1+\tau)^k\Big[\psi_1(1+\tau-t_1)^{(3-k)} + \psi_2(1+\tau-T)^{(1-k)} - \psi_3(1+\tau-T)^{(2-k)}\Big]\Big] \end{aligned} \tag{19}$$

and,

$$W = \eta A^\gamma\left[(a-\alpha p+\epsilon)(T-t_1) - \frac{b(T^2-t_1^2)}{2}\right] \tag{20}$$

Employing the above inventory level and order quantities, there are several costs associated with the inventory control. The cost incurred by the retailer for purchasing cost (PC) is given as,

$$\begin{aligned} PC = \quad & C_p(S+W) \\ = \quad & C_p\frac{A^\gamma}{\tau}\Big[-\psi_1(1+\tau)^3 - \psi_2(1+\tau) + \psi_3(1+\tau)^2 \\ & +(1+\tau)^k\Big[\psi_1(1+\tau-t_1)^{(3-k)} + \psi_2(1+\tau-t_1)^{(1-k)} - \psi_3(1+\tau-t_1)^{(2-k)}\Big] \\ & +C_p\eta A^\gamma\left[(a-\alpha p+\epsilon)(T-t_1) - \frac{b(T^2-t_1^2)}{2}\right]\Big] \end{aligned} \tag{21}$$

The carrying cost (CC) for each cycle $[0, t_1]$ is,

$$\begin{aligned} CC(t_1) = \quad & C_h\int_0^{t_1} q_1(t)dt \\ = \quad & \frac{C_hA^\gamma}{\tau}\Big[\psi_1(1+\tau-t_1)^4\left(\frac{1}{4}-\frac{1}{k+1}\right) + \psi_2(1+\tau-t_1)^2\left(\frac{1}{2}-\frac{1}{k+1}\right) \\ & -\psi_3(1+\tau-t_1)^3\left(\frac{1}{3}-\frac{1}{k+1}\right) - \frac{(1+\tau)^{(k+1)}}{k+1}\Big[\psi_2(1+\tau-t_1)^{(1-k)} \\ & +\psi_1(1+\tau-t_1)^{(3-k)} - \psi_3(1+\tau-t_1)^{(2-k)}\Big] - \frac{\psi_1}{4}(1+\tau)^4 \\ & -\frac{\psi_2}{2}(1+\tau)^2 + \frac{\psi_3}{3}(1+\tau)^3\Big] \end{aligned} \tag{22}$$

The cost of backorder (BC), and lost sales (LSC) per replenishment cycle respectively is given as,

$$\begin{aligned} BC(t_1, T) = \quad & -C_1\int_{t_1}^{T} q_2(t)dt \\ = \quad & -C_1\eta A^\gamma\left[(a-\alpha p+\epsilon)\left(t_1T - \frac{T^2}{2} - \frac{t_1^2}{2}\right) - \frac{b}{2}\left(t_1^2T - \frac{T^3}{3} - \frac{2t_1^3}{3}\right)\right] \end{aligned} \tag{23}$$

and

$$\begin{aligned} LSC(t_1, T) = & \quad C_2(1-\eta)\int_{t_1}^{T} A^{\gamma}(a-\alpha p - bt+\epsilon)dt \\ = & \quad C_2(1-\eta)A^{\gamma}\left[(a-\alpha p+\epsilon)(T-t_1) - b\frac{T^2-t_1^2}{2}\right] \end{aligned} \tag{24}$$

During inventory cycle, the total number of advertisements is A and the cost per advertisement is G. Therefore, the total advertisement cost is GA.
The total sales revenue (SR) during [0, T] is

$$\begin{aligned} SR(p, t_1) = & \quad p\int_{0}^{t_1} A^{\gamma}(a-\alpha p - bt+\epsilon)\frac{\tau - t}{\tau}dt + pW \\ = & \quad pA^{\gamma}\left[\frac{1}{6\tau}\left\{2bt_1^3 + 6t_1(a-\alpha p+\epsilon)\tau - 3t_1^2(a-\alpha p + b\tau+\epsilon)\right\} + \right. \\ & \quad \left. \eta\left\{(a-\alpha p+\epsilon)(T-t_1) - \frac{b(T^2-t_1^2)}{2}\right\}\right] \end{aligned} \tag{25}$$

Hence, accumulating all the above costs the profit function given is,

$$\begin{aligned} \Pi_2(A, p, t_1, T) = & \quad \frac{pA^{\gamma}}{6\tau}\left[2bt_1^3 + 6t_1(a-\alpha p+\epsilon)\tau - 3t_1^2(a-\alpha p+\tau+\epsilon)\right] - C_0 - GA \\ & + \frac{A^{\gamma}}{\tau}\left[(\psi_1\phi_1 + \psi_2\phi_2 - \psi_3\phi_3) + \left\{-\psi_1(1+\tau-t_1)^{(3-k)} - \psi_2(1+\tau-t_1)^{(1-k)}\right.\right. \\ & \left. + \psi_3(1+\tau-t_1)^{(2-k)}\right\}\phi_4 - C_h\left\{\psi_1(1+\tau-t_1)^4\left(\frac{1}{4} - \frac{1}{k+1}\right)\right. \\ & \left.\left. + \psi_2(1+\tau-t_1)^2\left(\frac{1}{2} - \frac{1}{k+1}\right) - \psi_3(1+\tau-t_1)^3\left(\frac{1}{3} - \frac{1}{k+1}\right)\right\}\right] \\ & + \eta A^{\gamma}\left[p\left\{(a-\alpha p+\epsilon)(T-t_1) - \frac{b(T^2-t_1^2)}{2}\right\}\right. \\ & \left. + C_1\left\{(a-\alpha p+\epsilon)\left(t_1T - \frac{T^2}{2} - \frac{t_1^2}{2}\right) - \frac{b}{2}\left(t_1^2T - \frac{T^3}{3} - 2\frac{t_1^3}{3}\right)\right\}\right] \\ & - C_2(1-\eta)A^{\gamma}\left[(a-\alpha p+\epsilon)(T-t_1) - \frac{b(T^2-t_1^2)}{2}\right] \end{aligned} \tag{26}$$

Our objective is to analyse the above profit function (equation (26)) and identify the optimal values for the frequency of advertisement, selling price, and replenishment time to achieve maximum profit for the retailer. In order to achieve this, we have formulated the following theorems .

Theorem 5.3. *For $t_1 < 2\tau$, the profit function $\Pi_2(A, p, t_1, T)$ is strictly concave for price p, where $t_1 \in (0, \tau)$.*

Proof: Consider the profit function equation (26), we obtained

$$\begin{aligned}\frac{\partial \Pi_2(A,p,t_1,T)}{\partial p} = \; & \frac{A^\gamma}{6\tau}\left[2bt_1^3+6t_1(a-\alpha p+\epsilon)\tau-3t_1^2(a-\alpha p+b\tau+\epsilon)\right]+\frac{pA^\gamma}{6\tau}\left[-6t_1\alpha\tau+3t_1^2\alpha\right] \\ & +\frac{A^\gamma}{\tau}\left\{\left(\frac{\alpha\phi_2}{k-1}-\frac{\alpha\phi_3}{k-2}\right)+\left(\frac{\alpha(1+\tau-t_1)^{2-k}-\frac{\alpha(1+\tau-t_1)^{1-k}}{k-1}}{k-2}\right)\phi_4\right. \\ & \left.-C_h\left(\frac{\alpha(1+\tau-t_1)^2(\frac{1}{2}-\frac{1}{k+1})}{k-1}-\frac{\alpha(1+\tau-t_1)^3(\frac{1}{3}-\frac{1}{k+1})}{k-2}\right)\right\}+ \\ & \eta A^\gamma\left[\left((a-\alpha p+\epsilon)(T-t_1)-b\frac{(T^2-t_1^2)}{2}-\alpha p(T-t_1)\right)\right. \\ & \left.-C_1\alpha\left(t_1T-\frac{T^2}{2}-\frac{t_1^2}{2}\right)\right]-C_2(1-\eta)A^\gamma\left[-\alpha(T-t_1)\right] \end{aligned} \tag{27}$$

Putting $\frac{\partial \Pi_2(A,p,T)}{\partial p}=0$, we have optimal value of p is,

$$\begin{aligned} p^* = \; & \frac{1}{\frac{A^\gamma}{\tau}\left[2t_1\alpha\tau-\alpha t_1^2\right]+\eta A^\gamma\left[2\alpha(T-t_1)\right]}\left[\frac{A^\gamma}{\tau}\left\{\frac{bt_1^3}{3}+t_1(a+\epsilon)\tau-\frac{t_1^2(a+b\tau+\epsilon)}{2}\right.\right. \\ & +\frac{\alpha\phi_2}{k-1}-\frac{\alpha\phi_3}{k-2}+\left(-\frac{\alpha(1+\tau-t_1)^{1-k}}{k-1}+\frac{\alpha(1+\tau-t_1)^{2-k}}{k-2}\right)\phi_4 \\ & \left.-C_h\left(\frac{\alpha(1+\tau-t_1)^2(\frac{1}{2}-\frac{1}{k+1})}{k-1}-\frac{\alpha(1+\tau-t_1)^3(\frac{1}{3}-\frac{1}{k+1}))}{k-2}\right)\right\} \\ & +\eta A^\gamma\left[\left((a+\epsilon(T-t_1)-b\frac{(T^2-t_1^2)}{2}-\alpha p(T-t_1)\right)\right. \\ & \left.\left.-C_1\alpha\left(t_1T-\frac{T^2}{2}-\frac{t_1^2}{2}\right)\right]+C_2(1-\eta)A^\gamma\alpha(T-t_1)\right] \end{aligned} \tag{28}$$

Differentiating again, we have,

$$\frac{\partial^2 \Pi_2(A,p,t_1,T)}{\partial p^2}=2A^\gamma\alpha\left[t_1\left(-1+\frac{t_1}{2\tau}\right)-\eta(T-t_1)\right]<0 \tag{29}$$

Thus, for $T<t_1<2\tau$, the profit function $\Pi_2(A,p,T)$ is concave and attain the maxima on the optimal price p^*.

Preposition 1. *For $a>0$, $b>0$, $p>0$, $Max(\delta_1+\delta_3)<Min(\delta_2)$,*

$$\begin{aligned}\delta_1 = \; & \frac{A^\gamma}{\tau}\left[p\left[b(2t_1-1)+\alpha p-(a+\epsilon)\right]-(1+\tau-t_1)^{-k}\left[-\psi_1(k-2)(k-3)(1+\tau-t_1)+\right.\right. \\ & \left.\left.k(k-1)(1+\tau-t_1)^{-1}\psi_2+\psi_3(k-1)(k-2)\right]\phi_4\right]\end{aligned}$$

$$\delta_2 = C_h\left[12\psi_1(1+\tau-t_1)^2\left(\frac{1}{4}-\frac{1}{k+1}\right)+2\psi_2\left(\frac{1}{2}-\frac{1}{k+1}\right)-6\psi_3(1+\tau-t_1)\left(\frac{1}{3}-\frac{1}{k+1}\right)\right]$$

$$\delta_3 = -A^\gamma\eta\left[b-C_1((a-\alpha p+\epsilon)-b(T-2t_1))\right]-C_2A^\gamma b(1-\eta)$$

Proof: From theorem (5.2), we have, $Max(\delta_1)<Min(\delta_2)$. So, we have added negative quantity ($\delta_3<0$) in the RHS, resultant maximum value of $\delta_1+\delta_3$ is still down then we have,

$$Max(\delta_1+\delta_3)<Min(\delta_2)$$

Now, $\frac{dTP(p)}{dp} = 0$ implies a unique value of p using Mathematica software, which is complex in nature. Clearly, $\frac{d^2TP(p)}{dp^2}$ is negative (see figure 2(b))

Theorem 5.4. *Objective function $\Pi_2(A, p, t_1, T)$ concave for t_1 in the interval $[0, \tau]$.*

Proof: From the equation (26), we may write,

$$\begin{aligned}
\frac{\partial \Pi_2}{\partial t_1} = \quad & \frac{pA^\gamma}{6\tau}\left[6bt_1^2 + 6(a - \alpha p + \epsilon)\tau - 6t_1(a - \alpha p + b\tau + \epsilon)\right] \\
& + \frac{A^\gamma}{\tau}\left[(3-k)\psi_1(1+\tau-t_1)^{(2-k)} + (1-k)\psi_2(1+\tau-t_1)^{-k}\right. \\
& \left. -(2-k)\psi_3(1+\tau-t_1)^{(1-k)}\right]\phi_4 \\
& -C_h\frac{A^\gamma}{\tau}\left[-4\psi_1(1+\tau-t_1)^3\left(\frac{1}{4}-\frac{1}{k+1}\right) - 2\psi_2(1+\tau-t_1)\left(\frac{1}{2}-\frac{1}{k+1}\right)\right. \\
& \left. +3\psi_3(1+\tau-t_1)^2\left(\frac{1}{3}-\frac{1}{k+1}\right)\right] \\
& +A^\gamma\eta\left[p(-(a-\alpha p+\epsilon)+bt_1) + C_1((a-\alpha p+\epsilon)(T-t_1) - b(t_1T - t_1^2))\right] \\
& -C_2(1-\eta)A^\gamma\left[bt_1 - (a-\alpha p+\epsilon)\right] \qquad (30)
\end{aligned}$$

Above equation leads to,

$$\frac{\partial \Pi_2(A, p, t_1, T)}{\partial t_1} = \delta_1 - \delta_2 + \delta_3$$

where, $\delta_1, \delta_2, \delta_3$ is given in above preposition. From the preposition 1, $\frac{\partial \Pi_2(A,p,t_1,T)}{\partial T} < 0$.

Theorem 5.5. *For $c_2 < \eta\left[b(p + c_2) + c_1 D(p, t)\right]$, the objective function is concave for T in the interval $[0, \tau]$.*
Proof: *From the equation (26), we have*

$$\begin{aligned}
\frac{\partial \Pi_2(A, p, t_1, T)}{\partial T} = \quad & \eta A^\gamma\left[p\left[(a-\alpha p+\epsilon) - bT\right] + C_1\left\{(a-\alpha p+\epsilon)(t_1 - T) - \frac{b}{2}\left(t_1^2 - T^2\right)\right\}\right] \\
& -C_2(1-\eta)A^\gamma\left[(a-\alpha p+\epsilon) - bT\right] \qquad (31)
\end{aligned}$$

or,

$$\frac{\partial \Pi_2^2(A, p, t_1, T)}{\partial T^2} = A^\gamma\eta\left[-bP - C_1(a - \alpha p + \epsilon + bT)\right] + C_2(1-\eta)A^\gamma b \qquad (32)$$

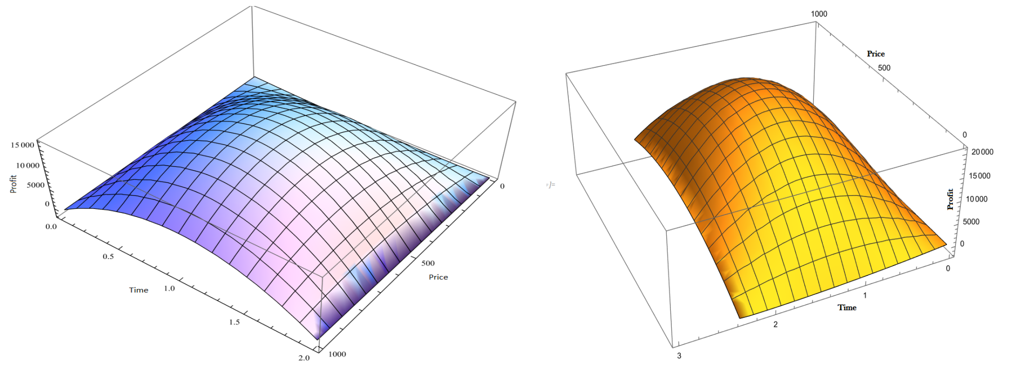
For the given positive values of parameters and $c_2 < \eta\left[b(p + c_2) + c_1 D(p, t)\right]$, the objective function is concave in T.

After obtaining the conditions for optimality of the objective function, we developed an algorithm to determine the algorithm to get numerical solution.

Algorithm 2. *Algorithm to determine the optimal solution without shortage*
Input: *Demand, A_{max}, Parameters, Objective function*
Output: *Optimal value of p^*, T^*, A^*, π^**
Define Module 1 taking argument (μ, σ)

1. *Initialize $\epsilon = f(\mu, \sigma)$*
2. *Return ϵ*

Main Body:



(a) Concavity of the profit function with respect to time and price without shortages

(b) Concavity of the profit function with respect to time and price with shortages

Figure 2: *Three dimensional concavity of profit function*

Initialize $t_1 = 0, T^* = 0, p^* = 0, A^* = 0, \xi^* = 0, \pi^* = -\infty$

for $A = 1, 2, ..., A_{max}$

    Solve Eq. (27), (30) and (31) for p, t_1 *and* T.

    Solve $\frac{\partial \pi_1}{\partial \xi} = 0$ *for* ξ.

    If $(T < 2\tau)$ *and* $c_2 < \eta\, [b(p + c_2) + c_1 D(p,t)]$ *do*

        If $[\delta_1 + \delta_3]_{T=\tau} < [\delta_2]_{T=\tau}$ *do*

            Calculate $\pi_1(A, p, T, \xi)$

            If $\pi_1(.) > \pi_1^*(.)$

                Set $t_1^* = t_1, T^* = T, p^* = p, A^* = A, \xi^* = A, \pi^* = \pi_1$

            end If

        end If

    end If

end for

Get $t_1^*, T^*, p^*, A^*, \xi^*, \pi^*$

After obtaining the algorithms, we have derived numerical results for this formulations given in the further sections.

6. Empirical Evaluation

A numerical example illustrates the model's adaptability and efficacy, providing insights into replenishment time, optimal pricing, preservation costs, and overall profitability. This research contributes to evolving inventory management strategies, particularly in industries with volatile demand and deteriorating items.

Example 1. ***Example of model without shortage:*** *In the retail store, research investigates an inventory management model that integrates stochastic demand and advertising techniques. The example incorporates essential parametric values such as the price-sensitive factor* (α) *is 0.1, a setup cost* (C_0) *is \$520, a purchasing cost* (C_p) *is \$5, a potential demand* (a) *is 100 units, a holding cost per unit* (C_h) *is \$1, a time-sensitive parameter* (b) *is 0.1, a maximum product lifetime* (τ) *is 2 unit of time, an advertising frequency* (A) *is 1 per advertisement, an advertising impact parameter* (γ) *is 0.1, an advertisement cost* (G) *is \$50, a stochastic demand factor* (ϵ) *is 20, and a preservation cost parameter* (λ) *is 0.0025.*

The model adapts pricing and preservation strategies, resulting in a replenishment time (T) *is 0.986282 units of time, an optimal price* (p) *is \$607.44, an optimal preservation cost is \$277.259, and an optimal profit is \$16,963.9.*

This example illustrates how the model can adapt in the real world, where dynamic strategies are aligned with market demands and deterioration. It offers valuable insights into inventory management, demonstrating the model's effectiveness in addressing challenges and adapting to market dynamics.

Example 2. ***Example for the model with shortage:*** *The example is formulated precisely to examine a system that includes shortages. Key parameters are as follows: Price sensitivity* (α) *is 0.1, setup cost* (C_0) *is \$520, purchasing cost* (C_p) *is \$5, potential demand* (a) *is 100 units, time-sensitivity parameter* (b) *is 0.1, holding cost* (C_h) *is \$1 per unit per unit of time, maximum product lifetime* (τ) *is 2 unit of time, advertising frequency* (A) *is 1 per advertisement, advertisement impact* (γ) *is 0.1, advertisement cost* (G) *at \$50 per advertisement, stochastic demand factor* (ϵ) *is 20, preservation cost impact parameter* (λ) *is 0.0025, backlogging fraction* (η) *is 0.6, shortage cost* (c_1) *is \$1, and opportunity cost* (c_2) *is \$1.*

The numerical outcomes reveal a replenishment time (T) *is 1.28626 units of time, an optimal shortage time* (t_1) *is 1.09332 units, an optimal price* (p) *is \$606.71, an optimal preservation cost is \$277.26, and an optimal profit \$20,916.3.*

The numerical results show a relationship among these elements, leading to optimal shortage time, price, preservation cost, and profit. This emphasizes the model's versatility in addressing broader challenges, such as unexpected demand fluctuations.

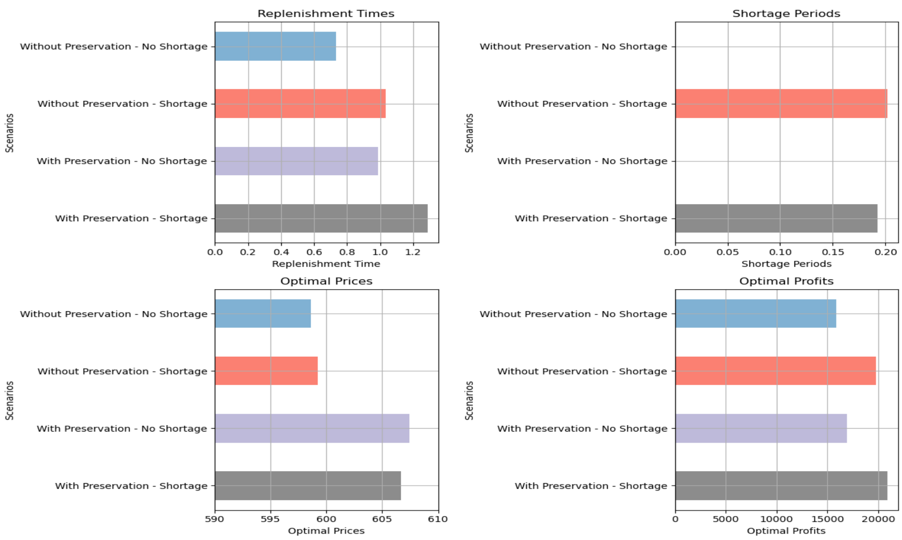
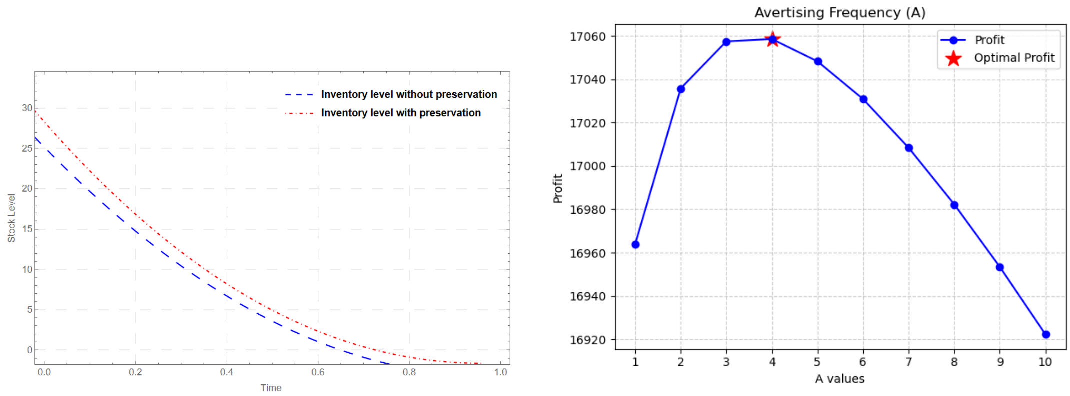


Figure 3: *Market dynamics in different scenarios*

In both scenarios, the model proves adaptable to various operational conditions, integrating multiple parameters for optimal decision-making in inventory management. Beyond numerical outcomes, the discussion provides managerial insights and practical implications.

From a managerial perspective, the model that includes without shortages emphasizes the importance of adopting forward-looking strategies (refer to Figure 3). It is observed that dynamic nature of pricing and strategic advertising effectively enhance overall profitability and competitiveness.It reveals that managers can optimize resource allocation, refine pricing strategies, and



(a) Inventory level and replenishment time

(b) Optimal value of advertisement frequency (A)

Figure 4: *Graphical representation of inventory level and optimal advertisement level*

boost sales efficiently. On the other hand, the example that deals with shortages focuses on mitigating risks and uncertainties. The model helps managers make informed decisions about backlogging, shortage, and opportunity costs. This is particularly relevant in real-world scenarios where unanticipated disruptions can impact inventory levels.

Further investigation considers models with and without shortages and the influence of preservation costs. The analysis reveals that the optimal replenishment time varies significantly across different scenarios. When considering preservation costs, the optimal replenishment time for the model without shortages is approximately 0.986282. This shows that businesses attempt to balance the need for timely replenishment and the financial implications of preserving inventory. However, the counterpart with shortages experiences a longer optimal replenishment time of 1.28626. This extension may be due to the complex interaction between shortage mitigation strategies and the time required to replenish stock, considering stochastic demand and expiration date-related deterioration.

Moreover, analyzing optimal prices provides valuable insights into the pricing strategies that optimize profit. Without shortages with preservation, the optimal price is \$598.624. This reveals that businesses can maximize profit under these conditions by setting a slightly lower price. However, when shortages are introduced into the model, the optimal price increases to \$606.711. This increase in price highlights the difficulty of maintaining a balance between price and shortage. Businesses may use higher prices to make up for potential profit losses caused by shortages. The complexity of determining the best pricing strategy becomes more evident when considering the costs of preserving deteriorating stock levels.

In the scenario with shortages, the model experiences a shortage period, representing a period during which demand exceeds the available stock. The duration of this shortage period is unaffected by preservation costs. The model without preservation costs exhibits a less shortage period of 0.19294 compared to 0.2021 in the model with preservation costs. This indicates the complex relationship between preservation expenses, reordering time, and periods of shortage. Therefore, decision-makers need to balance these factors carefully to optimize overall profit.

The findings also emphasize the strategic significance of pricing decisions in managing perishable inventory. The observed variations in optimal prices highlight the need for businesses to retailer their pricing strategies based on the specific dynamics of their operational environment. Higher prices may mitigate profit losses associated with shortages, but the impact on overall profitability should be carefully assessed. This sensitive understanding of optimal pricing gives managers actionable insights to navigate the delicate balance between pricing and inventory management. It also shows that the different advertising frequencies $(A = 1, 2, 3, ..., 10)$ impact on optimal profit differently (refer to Figure 4(a) and Figure 4(b)). Notably, the highest profit occurs at $A = 4$, emphasizing the crucial role of strategic decision-making in advertising frequency for businesses aiming to maximize profitability within the given inventory management model. The rise at $A = 4$ suggests an optimal balance in allocating resources to advertising efforts. Increasing advertising frequency may lead to higher costs but declining revenue, while lowering frequency may result in suboptimal market reach. The finding at $A = 4$ indicates an optimal spot where the benefits from increased advertising frequency align with the costs, resulting in the highest profit. From a managerial perspective, this insight provides actionable recommendations for businesses to optimize their advertising strategies. Managers can strategically allocate resources, focusing on the frequency that maximizes profit, aligning with the trend in marketing where data-driven decisions play a vital role in resource allocation and campaign effectiveness.

Additionally, the result underscores the dynamic nature of advertising strategies. Managers should continuously evaluate and adjust advertising frequencies based on real-time data and market conditions. The findings contribute valuable insights to the literature on inventory management and advertising strategies, guiding practitioners in making informed decisions impacting the bottom line. The research proposes that businesses can achieve optimal profits in the given inventory model context by strategically selecting an advertising frequency, with the peak observed at A = 4. This knowledge empowers managers to fine-tune their advertising strategies, balance cost and impact, and ultimately enhance overall business performance.

7. ANALYSIS OF UNCERTAINTY

In the proposed model, the uncertainty in demand (represented by ϵ) is intricately linked to various factors, each of which can be categorized into specific probability distributions. Based on Consumer behaviour and preferences, it is observe that there is in subject to unpredictable shifts and societal influences, making a uniform distribution suitable for capturing this diversity. Due to market dynamics there is fluctuation on economic conditions and currency instability, align with the characteristics of normal distribution by the central limit theorem, providing a logical fit for capturing the aggregate impact of these macroeconomic factors. The rapid pace of technological advances and disruptive innovations reflects the uncertainty inherent in exponential processes, making the exponential distribution a suitable choice for modelling this source of variability. Retail and revenue disruptions, regulatory changes, global events, and competitive actions characterized by various potential outcomes resonate with the triangular distribution, allowing for a flexible representation of uncertainties that span different magnitudes.

Assumes that $F(\epsilon)$ is the distribution function of random variable ϵ, and $f(\epsilon)$ is the density function of ϵ. The presented graph examines the influence of various distributions on a mathematical model representing time-dependent demand (Figure 6). Four distributions—Uniform, Normal, Exponential, and Triangular are analyzed for their impact on the demand function. Each subplot illustrates the demand profile with different levels of randomness, providing insights into the dynamic relationship between time and demand. The graph is a versatile illustration applicable to scenarios ranging from inventory management to supply chain dynamics. Including mean and standard deviation, annotations offer a comprehensive understanding of the slight variations introduced by different distributions, contributing to a deeper understanding of time-dependent demand fluctuations (refer to figure 5).

This study delves into profit maximization in dynamic business models, focusing on the relationship between external factors and the resulting profit function (refer to figure 4(a)). The study reveals a profit function sensitive to changes in external factors represented by epsilon. The objective is to understand how different epsilon distributions, modelled by normal and uniform distributions, impact the profit function. Numerical simulations reveal distinct profit patterns for different epsilon distributions. The deterministic case shows a continuous increase in profit, lacking adaptability to varying external conditions. Introducing uncertainty through normal distributions with different means and standard deviations reveals varying profit landscapes. Higher mean values correlate with increased profitability. Uniform distributions introduce a more comprehensive range of possibilities, with broader and flatter curves suggesting potential volatility in business performance. The results highlight the sensitivity of the profit function to

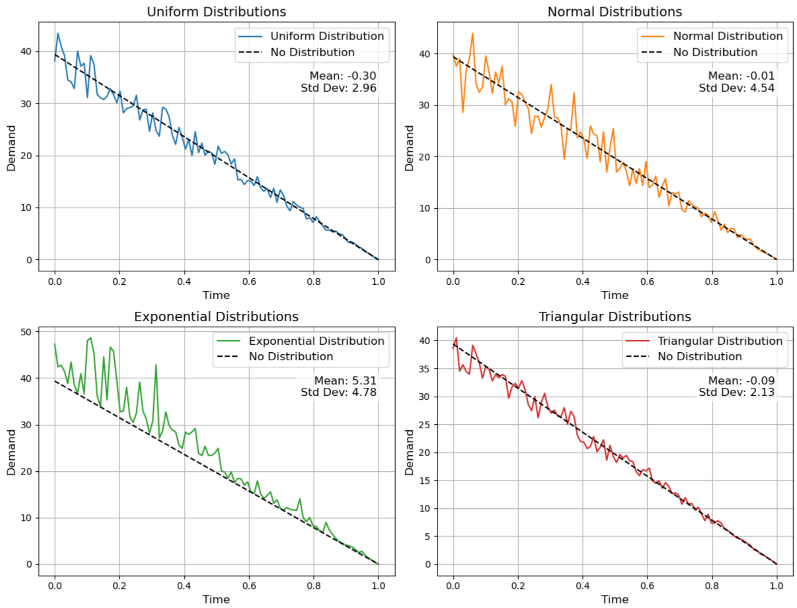


Figure 5: *Demand under the influence of various uncertainty factors*

changes in external factors, emphasizing the need for adaptive strategies to navigate uncertainties. This study provides a nuanced perspective on the impact of epsilon distributions on profit maximization in business models, emphasizing the importance of adaptive strategies to address external influences. It is graphically illustrate in figure 6.

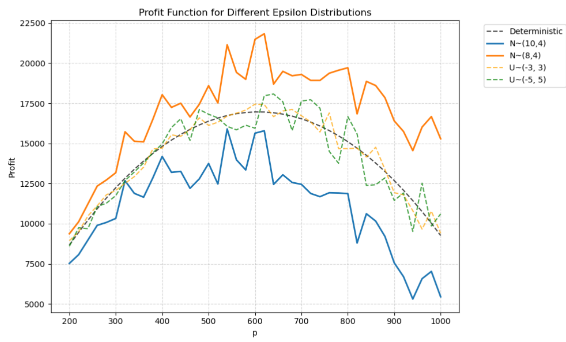


Figure 6: *Optimal profit under the influence of stochastic demand*

8. Conclusion and Further Work

This framework provides decision-makers with a robust approach to navigating the challenges of volatile markets. Through exploring managerial insights and results, this study offers practical guidance, emphasizing the importance of adaptability in decision-making:

1. The strategy for inventory management emphasizes the need for flexible and data-driven approaches to decision-making in pricing, advertising, and for managing retail market scenario. Decision-makers should recognize and adapt to the uncertainties inherent in the business environment.

2. The stochastic demand factor (ϵ) introduces variability, influencing optimal replenishment time and pricing. Different epsilon distributions, modelled by normal, exponential, triangular and uniform distributions, show distinct profit patterns. Higher mean values in

normal distributions correlate with increased profitability, offering valuable insights for decision-makers.

3. The optimal replenishment time (T) exhibits significant variation between scenarios, shows the model's adaptability to different operational conditions.

4. Optimal prices vary based on shortages, emphasizing the delicate balance required to optimize pricing strategies considering both price and shortage considerations. This managerial insight guides decision-makers in shaping pricing policies for overall profitability.

5. The highest profit occurs at an optimal advertising frequency of $A = 4$, indicating an optimal balance in resource allocation to advertising efforts. This result highlights the strategic importance of aligning advertising frequency with cost-effectiveness, offering actionable recommendations for businesses aiming to maximize profitability through advertising strategies. This insight guides decision-makers in making informed choices in deteriorating items, where preservation costs must be carefully weighed against potential benefits.

6. Continuous evaluation and adjustment of advertising frequencies based on real-time data emerge as crucial for optimizing advertising strategies. This insight aligns with contemporary marketing trends, emphasizing the importance of data-driven decision-making in resource allocation and campaign effectiveness. By adopting a dynamic approach, managers should ensure that advertising strategies remain aligned with market conditions.

The future scope outlined in the study suggests promising avenues for further research, ensuring the continued relevance and applicability of the model in addressing evolving business dynamics.

- Explore incorporating machine learning algorithms to enhance the model's adaptability. Machine learning can predict market trends and optimize pricing and advertising strategies, providing valuable insights for decision-makers in dynamic environments.

- Investigate the sustainability aspects of inventory management. Evaluate the environmental impact of different inventory strategies, ensuring alignment with corporate sustainability goals for responsible business practices.

- Examine the influence of real-time data analytics on advertising strategies. Understand how real-time data shapes decision-making in advertising, particularly in dynamic market conditions, offering insights for businesses aiming to stay competitive.

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