RT&A, No 4(80)
Volume 19, December, 2024

A NOVEL APPROACH TO DISTRIBUTION GENERATION WITH APPLICATIONS IN ELECTRICAL ENGINEERING

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Abstract

Many fields use standard distributions to model lifetime data. However, datasets from areas such as engineering and medical sciences frequently deviate from these standard distributions. This highlights the necessity for developing new distribution models that can accommodate significant variations in data patterns to better align with real-world observations. In this manuscript, we introduce a novel technique called the PNJ Transformation technique (named using the initials of its authors) for generating probability distributions. Using this technique, we developed a new and improved version of the Power function (PF) distribution, named the PNJ Power function (PNJ-PF) distribution. The PNJ-PF distribution offers superior flexibility compared to PF Distribution in terms of probability density function (pdf) and hazard rate function. We investigated the statistical properties of the PNJ-PF distribution and describe the maximum likelihood estimation (MLE) procedure for its parameters. To demonstrate the effectiveness and adaptability of the PNJ-PF distribution, we apply it to a simulated and two real-life datasets and compared proposed model fit with the traditional Power function model and other competitive models based on the various goodness-of-fit measures, such as the Akaike Information Criterion (AIC), Bayesian Information Criterion(BIC), Corrected AIC, Hannan−Quinn Information Criterion (HQIC) and these results are also justified graphically, further demonstrating the superiority and flexibility of the PNJ-PF distribution.

Power function distribution, hazard rate function, survival function, mean residual life, Maximum likelihood estimation

1. Introduction

Choosing the right statistical distribution is crucial for accurate data analysis across various fields such as medicine and engineering. While traditional distributions serve as foundational tools, their limitations in fitting complex real-world data necessities the need for enhanced models. The world of probability distributions encompasses a variety of extensions, from continuous to discrete, symmetric to asymmetric, designed to effectively capture the complexities and variability found in real world datasets. This evolution is paralleled by advancements in data science, which have transformed our capacity to derive insights from vast datasets. Understanding and utilizing data science techniques is crucial for meaningful analysis and applications in the context of probability distributions.

Lee [7] classified transformation techniques as composite methods because they seek to develop new distributions through the combination of existing ones or by integrating additional parameters into existing distributions. By adding extra parameters to base distributions, we aim to boost flexibility and enhance model accuracy, ensuring a better aligning with the characteristics of real-world data. Many innovative transformation techniques, extensively documented in statistical literature, have been devised to introduce novel distributions. Significant contributions include Exponentiated technique for analyzing bathtub failure rate data using Weibull distribution by [12] , Alpha power transformation by [9], Tangent exponential transformation by [6], innovative transformation applied to Weibull distribution by [8], innovative SMP transformation studied on Lomax distribution by [14], Quadratic Rank Transmutation method (QRTM) proposed by [15]. Despite numerous methods to analyze real-world data, there's a persistent demand to find new ways to create different types of distributions. This shows a continued interest in exploring fresh approaches that can handle the complexities found in real data. This study explores innovative transformation technique for generating probability distributions, with a focus on improving the adaptability and practical utility of the Power function distribution. Power function distribution also called the inverse of Pareto distribution [5] is a simple life time model. And was often employed in the assessment of reliability of semiconductor devices and electrical components. Meniconi [11] were the first who proposed the probability density function (pdf) and cumulative distribution function (cdf) of two parameter PF distribution with scale parameter λ and shape parameter β and are given respectively as,

$$f(x,\lambda,\eta) = \frac{\eta x^{\eta-1}}{\lambda^{\eta}} \ ; \quad 0 < x < \lambda, \lambda > 0, \eta > 0$$

$$F(x,\lambda,\eta) = \left(\frac{x}{\lambda}\right)^{\eta} \ ; \quad 0 < x < \lambda, \lambda > 0, \eta > 0$$

Various extended models of PF distribution has been proposed in literature, some of them include Weibull-PF [16], Transmuted-PF [18], Transmuted Weibull-PF [17], Exponentiated-PF [3] ,among others.

2. PNJ Transformation and its Properties

Let X be a continuous random variable and $F(x)$ be its cumulative distribution function (cdf), then the cdf of PNJ transformation technique for $x \in \mathbb{R}$, is given as follows

$$F_{PNJ}(x) = \begin{cases} \frac{e^{log(\zeta)F(x)} - S(x)}{\zeta} \ ; & \zeta > 1 \\ F(x) \ ; & \zeta = 1 \end{cases} \tag{1}$$

$F_{PNJ}(x)$ is an absolute continuous distribution function, If $F(x)$ is an absolute continuous distribution function. Clearly, $F_{PNJ}(x)$ is a valid cdf, as it satisfied all the properties of valid cdf function , such as,

i $F_{PNJ}(-\infty) = 0$ and $F_{PNJ}(\infty) = 1$

ii $F_{PNJ}(x)$ is a monotonically increasing function of x.

iii $F_{PNJ}(x)$ is right continuous.

iv $0 \leq F_{PNJ}(x) \leq 1$.

The corresponding probability density function (pdf) of $F_{PNJ}(x)$ for $x \in \mathbb{R}$ is given as follows

$$f_{PNJ}(x) = \begin{cases} \frac{f(x}{\zeta}\{log(\zeta)e^{log(\zeta)F(x)} + 1\} \ ; & \zeta > 1 \\ f(x) \ ; & \zeta = 1 \end{cases} \tag{2}$$

The survival function $S_{PNJ}(x)$ for PNJ transformation is given by

$$S_{PNJ}(x) = \frac{(\zeta + S(x)) - e^{log(\zeta)F(x)}}{\zeta} \ ; \qquad \zeta > 1 \tag{3}$$

The hazard rate function $h_{PNJ}(x)$ for PNJ transformation is given by

$$h_{PNJ}(x) = \frac{f(x)(log(\zeta)e^{log(\zeta)F(x)} + 1)}{\zeta + S(x) - e^{log(\zeta)F(x)}} \; ; \qquad \zeta > 1 \tag{4}$$

The $h_{PNJ}(x)$ of PNJ transformation in terms of survival $S(x)$ and hazard rate function $h(x)$ of f can be written as

$$h_{PNJ}(x) = h(x)\bar{F(x)}\frac{(log(\zeta)e^{log(\zeta)F(x)} + 1)}{\zeta + S(x) - e^{log(\zeta)F(x)}} \; ; \qquad \zeta > 1 \tag{5}$$

The PNJ transformation technique, represented by the cumulative distribution function: (1) , maintains consistency with base distributions when $\zeta = 1$, ensuring no added complexity.
One of its primary advantages of PNJ transformation technique is the flexibility and adaptability introduced by the parameter ζ, which allows the transformation to smoothly adjust to different dataset characteristics. This adaptability is crucial in dealing with varied and complex data in applied sciences. Additionally, by manipulating ζ, new distributions can be generated to match specific real-world data characteristics. The innovative parameterization of ζ influences the shape and characteristics of the resulting distribution, making it customizable for specific modeling needs. Theoretical foundation, the technique preserves the essential properties of a cdf, ensuring reliability in statistical analysis.

3. PNJ Power function distribution and its properties

The cdf of PNJ-PF distribution can be obtained from (1) by taking $F(x) = F(x, \lambda, \eta)$, the cdf of the PF distribution, and is given by

$$F_{PNJ}(x, \Theta) = \frac{1}{\zeta}\{e^{log(\zeta)\left(\frac{x}{\lambda}\right)^{\eta}} + \left(\frac{x}{\lambda}\right)^{\eta} - 1\} \; ; \qquad \zeta > 1 \tag{6}$$

and the corresponding pdf of PNJ-PF distribution is given by

$$f_{PNJ}(x, \Theta) = \frac{\eta x^{\eta-1}}{\zeta\lambda^{\eta}}\{log(\zeta)e^{log(\zeta)\left(\frac{x}{\lambda}\right)^{\eta}} + 1\} \; ; \qquad \zeta > 1 \tag{7}$$

where, $0 < x < \lambda$, $\Theta = (\zeta, \lambda, \eta)$,$\lambda > 0$ is a scale parameter and $\zeta > 1, \eta > 0$ are shape parameter.

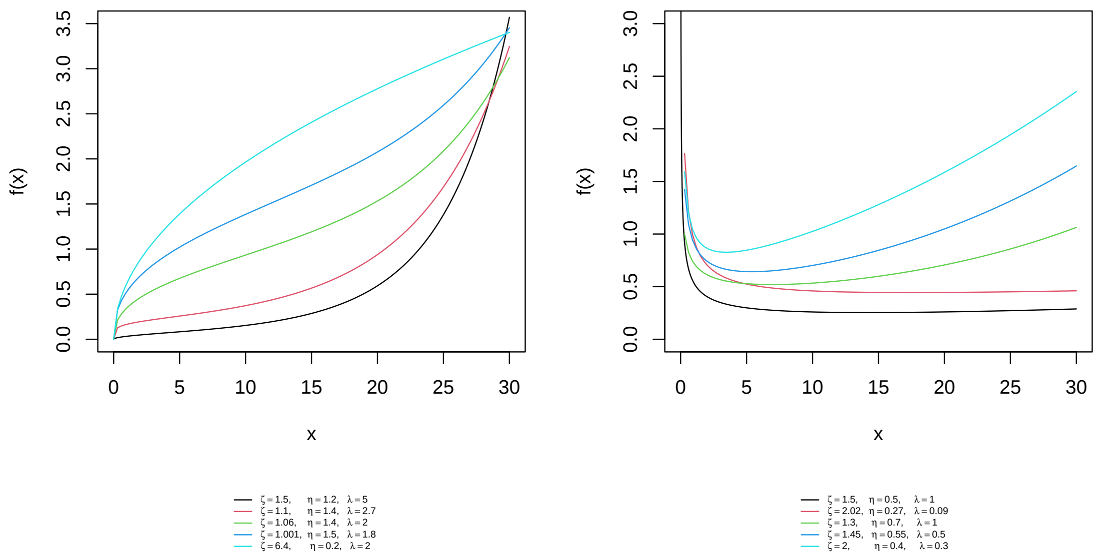

Figure 1: *Plots of the PNJ-PF density for various combinations of ζ and η and λ.*

Figure 1 depicts some different curves of the Pdf for different combination of PNJ-PF parameters ζ, η and λ. It is noted from figure (1) that the density curves for PNJ-PF distribution can be decreasing, decreasing-increasing, and increasing.
The survival function $S_{PNJ}(x, \Theta)$ and the hazard rate function $h_{PNJ}(x, \Theta)$ for $0 < x < \lambda$ are, respectively, given by

$$S_{PNJ}(x, \Theta) = \frac{1}{\zeta}\{1 + \zeta - \left(\frac{x}{\lambda}\right)^{\eta} - e^{log(\zeta)\left(\frac{x}{\lambda}\right)^{\eta}}\} \; ; \qquad \zeta > 1 \tag{8}$$

$$h_{PNJ}(x, \Theta) = \frac{\eta x^{\eta-1}\{log(\zeta)e^{log(\zeta)\left(\frac{x}{\lambda}\right)^{\eta}} + 1\}}{\lambda^{\eta}\{1 + \zeta - \left(\frac{x}{\lambda}\right)^{\eta} - e^{log(\zeta)\left(\frac{x}{\lambda}\right)^{\eta}}\}} \; ; \zeta > 1 \tag{9}$$

The Plots of the hazard rate function of the PNJ-PF distribution for selected parameter values are displayed in Figure (2). It is noted that the PNJ-PF distribution possesses increasing, J-shaped, decreasing, constant ,and bathtub shape hazard rate function.

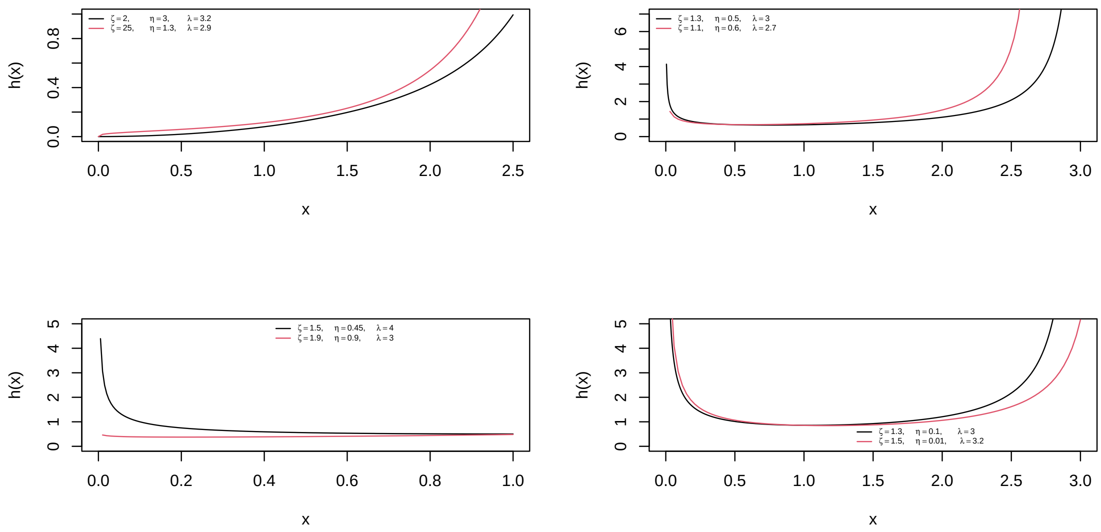

Figure 2: *Plots of the PNJ-PF hazard rate function for various combinations of ζ and η and λ.*

Colorary I: (Stochasting Ordering) The ratio of the densities of transformed variable X_{PNJ} and original random variable (i.e. PF X) is given by

$$R(x) = \frac{1}{\zeta}\{log(\zeta)e^{log(\zeta)\left(\frac{x}{\lambda}\right)^{\eta}} + 1\} \tag{10}$$

and the first order derivative of R(x) is given by

$$R^{'}(x) = \frac{\eta x^{\eta-1}}{\zeta\lambda^{\eta}}\{log(\zeta)e^{(log(\zeta))^{2}\left(\frac{x}{\lambda}\right)^{\eta}}\} \tag{11}$$

This expression $R^{'}(x)$ is always positive for $x > 0, \zeta > 1, \eta > 0$ and $\lambda > 0$, which implies X_{PNJ} exhibits higher likelihood ratio than the original variable X i.e, $X_{PNJ} \geq_{lr} X$. Based on the chain of implications within various stochastic orders, it can concluded that $X_{PNJ} \geq_{hr} X$ and hence $X_{PNJ} \geq_{mlr} X$.

4. Statistical properties of PNJ-PF distribution

In this section, the essential probabilistic and statistical characteristics of the proposed model are presented.

4.1. Moments and associated measures

Theorem I: Let X follow the PNJ-PF distribution with parameters $\zeta > 1$,$\eta > 0$ and $\lambda > 0$; then, the r-th ordinary moment $E(x^r)$ of X has the form

$$E(X^r) = \frac{\eta\lambda^r}{\zeta}\{\sum_{j=0}^{\infty}\frac{(log(\zeta))^{j+1}}{j!(r+\eta(j+1))} + \frac{1}{\eta+r}\} \tag{12}$$

Proof:

$$E(X^r) = \int_0^{\lambda} x^r f_{PNJ}(x)dx$$

$$E(X^r) = \frac{\eta}{\zeta\lambda^n}\int_0^{\lambda} x^{r+\eta-1}\{log(\zeta)e^{log(\zeta)\left(\frac{x}{\lambda}\right)^{\eta}} + 1\}dx$$

After some algebra, the r-th ordinary moment of X reduces to

$$E(X^r) = \frac{\eta\lambda^r}{\zeta}\{\sum_{j=0}^{\infty}\frac{(log(\zeta))^{j+1}}{j!(r+\eta(j+1))} + \frac{1}{\eta+r}\}$$

Corollary II: The first and second ordinary moments can be obtained by substituting $r = 1, 2,$ in (12), respectively. The expressions for the mean $E(X)$ and variance of PNJ-PF distribution are given, respectively, by

$$E(X) = \frac{\eta\lambda}{\zeta}\{\sum_{j=0}^{\infty}\frac{(log(\zeta))^{j+1}}{j!(1+\eta(j+1))} + \frac{1}{\eta+1}\}$$

and,

$$V(X) = \frac{\eta\lambda^2}{\zeta}\left[\{\sum_{j=0}^{\infty}\frac{(log(\zeta))^{j+1}}{j!(2+\eta(j+1))} + \frac{1}{\eta+2}\} - \frac{\eta}{\zeta}\{\sum_{j=0}^{\infty}\frac{(log(\zeta))^{j+1}}{j!(1+\eta(j+1))} + \frac{1}{\eta+1}\}^2\right]$$

Theorem II: Let $X \sim PNJ - PF(\zeta, \eta, \lambda)$, then r-th incomplete moment $I_r(x)$ of X are

$$I_r(x) = \frac{\eta t^{\eta+r}}{\zeta\lambda^{\eta}}\{(\frac{t}{\lambda})^{\eta j}\sum_{j=0}^{\infty}\frac{(log(\zeta))^{j+1}}{j!(r+\eta(j+1))} + \frac{1}{\eta+r}\} \tag{13}$$

Proof: The r-th incomplete moments $I_r(x)$ are defined as

$$I_r(x) = \int_0^{t} x^r f_{PNJ}(x)dx$$

$$I_r(x) = \int_0^{t} x^r\frac{\eta x^{\eta-1}}{\zeta\lambda^{\eta}}\{log(\zeta)e^{log(\zeta)\left(\frac{x}{\lambda}\right)^{\eta}} + 1\}dx$$

through algebraic simplification, the r-th incomplete moment $I_r(x)$ of X can be expressed as

$$I_r(x) = \frac{\eta t^{\eta+r}}{\zeta\lambda^{\eta}}\{(\frac{t}{\lambda})^{\eta j}\sum_{j=0}^{\infty}\frac{(log(\zeta))^{j+1}}{j!(r+\eta(j+1))} + \frac{1}{\eta+r}\}$$

Corollary III: The first incomplete moments $I(x)$ can be obtained by substituting $r = 1$, in (18), as

$$I(x) = \frac{\eta t^{\eta+1}}{\zeta\lambda^{\eta}}\{(\frac{t}{\lambda})^{\eta j}\sum_{j=0}^{\infty}\frac{(log(\zeta))^{j+1}}{j!(1+\eta(j+1))}+\frac{1}{\eta+1}\}$$

Theorem III: If $X \sim PNJ - PF(\zeta, \eta, \lambda)$, then the MGF $(M_X(t))$ of X is given by

$$M_X(t) = \frac{\eta}{\zeta\lambda^{\eta}}\sum_{k=0}^{\infty}\frac{\lambda^{\eta+k}t^k}{k!}\{\sum_{j=0}^{\infty}\frac{(log(\zeta))^{j+1}}{j!(k+\eta(j+1))}+\frac{1}{\eta+k}\} \tag{14}$$

Proof: The MGF $(M_X(t))$ is defined as

$$M_X(t) = \int_0^{\lambda} e^{tx} f_{PNJ}(x)dx$$

using 7, we have

$$M_X(t) = \int_0^{\lambda} e^{tx}\frac{\eta x^{\eta-1}}{\zeta\lambda^{\eta}}\{log(\zeta)e^{log(\zeta)(\frac{x}{\lambda})^{\eta}}+1\}dx$$

after simplifying by using the series expansion e^{ax} and some algebraic calculations, we get the final expression for MGF as

$$M_X(t) = \frac{\eta}{\zeta\lambda^{\eta}}\sum_{k=0}^{\infty}\frac{\lambda^{\eta+k}t^k}{k!}\{\sum_{j=0}^{\infty}\frac{(log(\zeta))^{j+1}}{j!(k+\eta(j+1))}+\frac{1}{\eta+k}\}$$

4.2. Mean residual life and mean waiting time

Theorem IV: Let $X \sim PNJ - PF(\zeta, \eta, \lambda)$, then The mean residual life function, say $\mu(t)$ of x , is given by

$$\mu(t) = \frac{\eta}{1+\zeta-(\frac{x}{\lambda})^{\eta}-e^{log(\zeta)(\frac{x}{\lambda})^{\eta}}}\{\sum_{j=0}^{\infty}\frac{(log(\zeta))^{j+1}}{j!(1+\eta(j+1))}(\lambda-\frac{t^{\eta(j+1)+1}}{\lambda^{\eta(j+1)}})-\frac{1}{\eta+1}(\lambda-\frac{t^{\eta+1}}{\lambda^{\eta}})\}-t \tag{15}$$

Proof: The mean RLF is defined as

$$\mu(t) = \frac{1}{S_{PNJ}(t)}\left(E(t)-\int_0^t xf_{PNJ}(x)dx\right)-t \tag{16}$$

where

$$E(t) = \frac{\eta\lambda}{\zeta}\{\sum_{j=0}^{\infty}\frac{(log(\zeta))^{j+1}}{j!(1+\eta(j+1))}+\frac{1}{\eta+1}\} \tag{17}$$

and

$$\int_0^t xf_{PNJ}(x)dx = \frac{\eta t^{\eta+1}}{\zeta\lambda^{\eta}}\{(\frac{t}{\lambda})^{\eta j}\sum_{j=0}^{\infty}\frac{(log(\zeta))^{j+1}}{j!(1+\eta(j+1))}+\frac{1}{\eta+1}\} \tag{18}$$

Substituting (8), (17) and (18) in (16), we get final expression for $\mu(t)$ as

$$\mu(t) = \frac{\eta}{1+\zeta-(\frac{t}{\lambda})^{\eta}-e^{log(\zeta)(\frac{t}{\lambda})^{\eta}}}\{\sum_{j=0}^{\infty}\frac{(log(\zeta))^{j+1}}{j!(1+\eta(j+1))}(\lambda-\frac{t^{\eta(j+1)+1}}{\lambda^{\eta(j+1)}})-\frac{1}{\eta+1}(\lambda-\frac{t^{\eta+1}}{\lambda^{\eta}})\}-t$$

Theorem V: Let $X \sim PNJ - PF(\zeta, \eta, \lambda)$, then the mean waiting time of X, say $\bar{\mu}(t)$, is

$$\bar{\mu}(t) = t - \frac{\eta t^{\eta+1}}{\lambda^{\eta}(e^{log(\zeta)(\frac{t}{\lambda})^{\eta}} + (\frac{t}{\lambda})^{\eta} - 1)}\{(\frac{t}{\lambda})^{\eta j}\sum_{j=0}^{\infty}\frac{(log(\zeta))^{j+1}}{j!(1+\eta(j+1))} + \frac{1}{\eta+1}\} \tag{19}$$

Proof: the $\bar{\mu}(t)$, is defined as

$$\bar{\mu}(t) = t - \frac{1}{F_{PNJ}(t)}\int_0^t x f_{PNJ}(x)dx. \tag{20}$$

Substituting (6) and (18) in (20), we get

$$\bar{\mu}(t) = t - \frac{\eta t^{\eta+1}}{\lambda^{\eta}(e^{log(\zeta)(\frac{t}{\lambda})^{\eta}} + (\frac{t}{\lambda})^{\eta} - 1)}\{(\frac{t}{\lambda})^{\eta j}\sum_{j=0}^{\infty}\frac{(log(\zeta))^{j+1}}{j!(1+\eta(j+1))} + \frac{1}{\eta+1}\}$$

4.3. Entropy

Theorem VI: Let $X \sim PNJ - PF(\zeta, \lambda, \eta)$, then the Renyi entropy of X is

$$H_R(x) = \frac{1}{1-\nu}log\left[\left(\frac{\eta}{\zeta\lambda^{\eta}}\right)^{\nu}\frac{\lambda^{\nu(\eta-1)}}{\nu(\eta-1)+1}\sum_{k=0}^{\infty}({}^{\nu}C_k)log^k(\zeta)\sum_{j=0}^{\infty}\frac{(klog(\zeta))^j}{j!}\right] \quad ; \qquad \nu > 0, \nu \neq 1 \tag{21}$$

Proof: The Renyi entropy of X is defined as

$$H_R(x) = \frac{1}{1-\nu}log\int_0^{\lambda} f_{PNJ}^{\nu}(x)dx$$

using (7) ,we have

$$H_R(x) = \frac{1}{1-\nu}log\int_0^{\lambda}(\frac{\eta x^{\eta-1}}{\zeta\lambda^{\eta}})^{\nu}\{log(\zeta)e^{log(\zeta)(\frac{x}{\lambda})^{\eta}} + 1\}^{\nu}dx$$

after integrating, the Renyi entropy reduces to

$$H_R(x) = \frac{1}{1-\nu}log\left[\left(\frac{\eta}{\zeta\lambda^{\eta}}\right)^{\nu}\frac{\lambda^{\nu(\eta-1)}}{\nu(\eta-1)+1}\sum_{k=0}^{\infty}({}^{\nu}C_k)log^k(\zeta)\sum_{j=0}^{\infty}\frac{(klog(\zeta))^j}{j!}\right]$$

Collorary IV: The renyi entropy (21) is useful for computing the entropy measures of Havrda and Charvat $H_H(x)$, as well as Arimoto entropy $H_A(x)$. And the final expressions for these entropy measures are respectively given by

$$H_H(x) = \frac{1}{1-\nu}\left[\left(\frac{\eta}{\zeta\lambda^{\eta}}\right)^{\nu}\frac{\lambda^{\nu(\eta-1)}}{\nu(\eta-1)+1}\sum_{k=0}^{\infty}({}^{\nu}C_k)log^k(\zeta)\sum_{j=0}^{\infty}\frac{(klog(\zeta))^j}{j!} - 1\right]$$

and

$$H_A(x) = \frac{1}{2^{\nu-1}-1}\left[\left(\frac{\eta}{\zeta\lambda^{\eta}}\right)^{\frac{1}{\nu}}\frac{\nu\lambda^{\frac{(\eta-1)}{\nu}}}{(\eta-1)+\nu}\sum_{k=0}^{\infty}({}^{\frac{1}{\nu}}C_k)log^k(\zeta)\sum_{j=0}^{\infty}\frac{(klog(\zeta))^j}{j!} - 1\right]$$

4.4. Order Statistics

Let $X_1, X_2, ..., X_m$ be a random sample of size m from NPJ-PF, and let $X_{r:m}$ denote the rth order statistic, then, the pdf of $X_{r:m}$, say $f_{r:m}(x)$ is given by

$$f_{r:m}(x) = \frac{m!}{(r-1)!(m-r)!} F_{PNJ}(x)^{r-1} f_{PNJ}(x)(1-F_{PNJ}(x))^{m-r}. \tag{22}$$

Substituting (6) and (7) in (22), we get

$$f_{r:m}(x) = \frac{m!\eta x^{\eta-1\{log(\zeta)e^{log(\zeta)\left(\frac{x}{\lambda}\right)^{\eta}}+1\}}}{(r-1)!(m-r)!\lambda^{\eta}\zeta^{m}} \left[\{e^{log(\zeta)\left(\frac{x}{\lambda}\right)^{\eta}} + \left(\frac{x}{\lambda}\right)^{\eta} - 1\}\right]^{r-1} \left[\{1+\zeta - \left(\frac{x}{\lambda}\right)^{\eta} - e^{log(\zeta)\left(\frac{x}{\lambda}\right)^{\eta}}\}\right]^{m-r} \tag{23}$$

The minimum and maximum OS densities are obtained, respectively, by substituting $r = 1$ and $r = m$ in (23), and the expressions are respectively given by

$$f_{1:m}(x) = \frac{m\eta x^{\eta-1\{log(\zeta)e^{log(\zeta)\left(\frac{x}{\lambda}\right)^{\eta}}+1\}}}{\lambda^{\eta}\zeta^{m}} \left[\{1+\zeta - \left(\frac{x}{\lambda}\right)^{\eta} - e^{log(\zeta)\left(\frac{x}{\lambda}\right)^{\eta}}\}\right]^{m-1}$$

and

$$f_{m:m}(x) = \frac{m\eta x^{\eta-1\{log(\zeta)e^{log(\zeta)\left(\frac{x}{\lambda}\right)^{\eta}}+1\}}}{\lambda^{\eta}\zeta^{m}} \left[\{e^{log(\zeta)\left(\frac{x}{\lambda}\right)^{\eta}} + \left(\frac{x}{\lambda}\right)^{\eta} - 1\}\right]^{m-1}$$

4.5. Stress Strength Reliability

Theorem VII: let X_1 and X_2 be independent strength and stress random variables respectively, where $X_1 \sim PNJ - PF(\zeta_1, \eta_1, , \lambda)$ and $X_2 \sim PNJ - PF(\zeta_2, \eta_2, , \lambda)$, then the stress strength reliability defined as the probability that the strength X_1 exceeds the stress X_2 $\mathbb{P}(X_1 > X_2)$, say SSR, is

$$\begin{aligned} SSR = \frac{\eta_1}{\zeta_1\zeta_2} & \left[\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(log(\zeta_1))^{j+1}(log(\zeta_2))^{k}}{j!k!(\eta_1(j+1)+\eta_2 k)} - \sum_{l=0}^{\infty} \frac{(log(\zeta_1))^{l+1}}{l!(\eta_1(l+1))}\right. \\ & \left. + \sum_{m=0}^{\infty} \frac{(log(\zeta_1))^{m+1}}{m!(\eta_1(m+1)+\eta_2)} + \sum_{n=0}^{\infty} \frac{(log(\zeta_2))^{n}}{n!(\eta_1+\eta_2 n)} + \frac{1}{\eta_1+\eta_2}\right] \end{aligned} \tag{24}$$

Proof: The stress strength reliability $\mathbb{P}(X_1 > X_2)$, say SSR, is defined as

$$SSR = \int_0^{\lambda} f_1(x)F_2(x)dx$$

using (6) , (7), we have

$$SSR = \frac{\eta_1}{\zeta_1\zeta_2(\lambda)^{\eta_1}} \int_0^{\lambda} \left[x^{\eta_1-1}\{log(\zeta_1)e^{log(\zeta_1)\left(\frac{x}{\lambda}\right)^{\eta_1}} + 1\}\right] \left[e^{log(\zeta_2)\left(\frac{x}{\lambda}\right)^{\eta_2}} + \left(\frac{x}{\lambda}\right)^{\eta_2} - 1\right] dx$$

after simplifying and using series expansion e^{ax} we get the final expression for the stress strength reliability SSR as

$$\begin{aligned} SSR = \frac{\eta_1}{\zeta_1\zeta_2(\lambda)^{\eta_1}} & \left[\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(log(\zeta_1))^{j+1}(log(\zeta_2))^{k}}{j!k!(\lambda)^{\eta_1 j}(\lambda)^{\eta_2 k}} \frac{(\lambda)^{\eta_1(j+1)+\eta_2 k}}{\eta_1(j+1)+\eta_2 k} - \sum_{l=0}^{\infty} \frac{(log(\zeta_1))^{l+1}}{l!(\lambda)^{\eta_1 l}} \frac{(\lambda)^{\eta_1(l+1)}}{\eta_1(l+1)}\right. \\ & \left. + \sum_{m=0}^{\infty} \frac{(log(\zeta_1))^{m+1}}{m!(\lambda)^{\eta_1 m}(\lambda)^{\eta_2}} \frac{(\lambda)^{\eta_1(m+1)+\eta_2}}{\eta_1(m+1)+\eta_2} + \sum_{n=0}^{\infty} \frac{(log(\zeta_2))^{n}}{n!(\lambda)^{\eta_2 n}} \frac{(\lambda)^{\eta_1+\eta_2 n}}{\eta_1+\eta_2 n} + \frac{1}{(\lambda)^{\eta_2}} \frac{\lambda^{\eta_1+\eta_2}}{\eta_1+\eta_2}\right] \end{aligned}$$

$$SSR = \frac{\eta_1}{\zeta_1\zeta_2}\left[\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(log(\zeta_1))^{j+1}(log(\zeta_2))^k}{j!k!(\eta_1(j+1)+\eta_2 k)} - \sum_{l=0}^{\infty}\frac{(log(\zeta_1))^{l+1}}{l!(\eta_1(l+1))} + \sum_{m=0}^{\infty}\frac{(log(\zeta_1))^{m+1}}{m!(\eta_1(m+1)+\eta_2)}\right.$$
$$\left. + \sum_{n=0}^{\infty}\frac{(log(\zeta_2))^n}{n!(\eta_1+\eta_2 n)} + \frac{1}{\eta_1+\eta_2}\right]$$

5. Statistical Inference

5.1. Parameter Estimation

Let $X_1, X_2, ..., X_m$ be a random sample from PNJ-PF distribution, then the logarithm of the likelihood function is

$$l = mlog\eta - mlog(\zeta) - m\eta log(\lambda) + (\eta-1)\sum_{i=1}^{m} log(x_i) + mlog\left[log(\zeta)e^{log(\zeta)(\frac{x}{\lambda})^{\eta}} + 1\right] \tag{25}$$

The MLEs of ζ, λ and η are obtained by partially differentiating (25) with respect to the corresponding parameters and equating to zero, we have

$$\frac{\partial l}{\partial \zeta} = \frac{-m}{\zeta} + \frac{me^{log(\zeta)(\frac{x}{\lambda})^{\eta}}(log(\zeta)(\frac{x}{\lambda})^{\eta}+1)}{\zeta\left[(log(\zeta)e^{log(\zeta)(\frac{x}{\lambda})^{\eta}}+1\right]} \tag{26}$$

$$\frac{\partial l}{\partial \eta} = \frac{m}{\eta} - mlog\lambda + \sum_{i=1}^{m} log(x_i) + \frac{mx^{\eta}(log(\zeta))^2 e^{log(\zeta)(\frac{x}{\lambda})^{\eta}} log(x)}{\lambda^{\eta}\{(log(\zeta)e^{log(\zeta)(\frac{x}{\lambda})^{\eta}}+1\}} \tag{27}$$

$$\frac{\partial l}{\partial \lambda} = -\frac{m\eta}{\lambda} - \frac{m\eta(log(\zeta))^2 e^{log(\zeta)(\frac{x}{\lambda})^{\eta}} x^{\eta}}{\lambda^{\eta+1}\left[(log(\zeta)e^{log(\zeta)(\frac{x}{\lambda})^{\eta}}+1\right]} \tag{28}$$

The above three equations (26),(27) and (28) are not in closed form. Thus, it is difficult to calculate the values of the parameters ζ, η and λ. However, R software can be used to get the MLE.

5.2. Simulation study

A simulation study was conducted using R Software to examine the behavior of the Maximum Likelihood Estimates (MLEs) with varying sample sizes. Three sets of samples (n=25, n=100, and n=500), each replicated 100 times, were generated from the PNJ-PF distribution with different parameter values $\zeta = (1.5, 2)$ $\lambda = (1.5, 2)$, and $\eta = (1.5, 2)$, to effectively check the impact of small ($n = 25$), medium ($n = 100$), and large ($n = 500$) sample sizes on the accuracy and precision of the MLEs, demonstrating improved performance with larger sample sizes. For each configuration, the average MLEs and their corresponding mean squared errors (MSEs) were computed. The results are summarized in Tables 1.

Table 1: *Average values of MLEs and the corresponding MSEs.*

	Parameter			MLE			MSE		
n	ζ	η	λ	$\hat{\zeta}$	$\hat{\eta}$	$\hat{\lambda}$	$\hat{\zeta}$	$\hat{\eta}$	$\hat{\lambda}$
25	1.5	1	1.5	1.4413	0.8666	1.3183	0.9136	0.1296	0.2748
100				1.4621	0.9889	1.4822	0.0992	0.0111	0.0233
500				1.4778	0.9907	1.4845	0.0378	0.0103	0.0228
25			2	1.3489	0.7429	1.5336	1.1309	0.2406	0.9311
100				1.4444	0.933	1.8812	0.2646	0.0625	0.2434
500				1.4963	0.9864	1.9793	0.0515	0.0105	0.0407
25		1.5	1.5	1.4128	1.3403	1.3610	0.4165	0.2194	0.2053
100				1.4768	1.4287	1.4394	0.1824	0.0963	0.0911
500				1.4961	1.483	1.4856	0.0497	0.0245	0.0228
25			2	1.7193	1.2146	1.727	1.9832	0.3490	0.5283
100				1.4868	1.4192	1.9175	0.2886	0.1039	0.1624
500				1.5197	1.4926	1.9991	0.0374	0.0023	0.0001
25	2	1	1.5	1.8251	0.9219	1.3773	0.7054	0.0888	0.1833
100				1.9543	0.9816	1.4667	0.1491	0.0214	0.0457
500				1.9940	0.9887	1.4848	0.0561	0.0104	0.0228
25			2	2.1983	0.9447	2.1166	0.2833	0.0088	0.0378
100				2.0079	0.9652	1.9369	0.2614	0.0325	0.1221
500				2.0035	0.9884	1.9802	0.0686	0.0106	0.0406
25		1.5	1.5	1.8863	1.3938	1.3909	0.7179	0.1897	0.1598
100				1.9681	1.4936	1.4829	0.1917	0.0342	0.0229
500				1.9916	1.5017	1.5001	0.0227	0.0020	0.0000
25			2	2.0949	1.4191	1.9156	1.7986	0.1553	0.1645
100				2.0333	1.4422	1.9402	0.4624	0.0874	0.1218
500				2.0258	1.4913	1.9983	0.0347	0.0034	0.0001

The results of 1 shows that the MLEs are stable and closely approximate the true parameter values. Additionally, as the sample size increases, the MSE consistently decreases across all scenarios, indicating enhanced estimate reliability with larger sample sizes.
Additionally, a random sample of 51 observations has been generated from the PNJ-PF distribution with parameters $\zeta = 15$, $\eta = 1$, and $\lambda = 20$ using R programming language.This sample representing the quantiles of our proposed model, serves to demonstrate theoretical concepts and to compare the fit of the proposed model with baseline and several competitive models. The results are displayed in Table 2 and Table 3 and the resulting simulated dataset, along with its corresponding R code, is provided below.

```
> Data<-function(n,m,zeta,eta,lambda)
+ {set.seed(0)
+ library(zipfR)
+ cdf<-function(x,zeta,eta,lambda)
+ {fn<-((1/(zeta))*(exp(log(zeta)*((x/(lambda))^(eta)))-{1-(x/(lambda))^(eta)}))}
+ data=c() + U=runif(n,0,1)
+ for(i in 1:length(U)){
+ fn<-function(x){cdf(x,zeta,eta,lambda)-U[i]}
+ uni<-uniroot(fn,c(0,100000))
+ data=c(data,uni$root)}
+ return(data)}
> Simulateddata<-Data(51,1,15,1,20)
```

> Simulateddata

[1] 19.216206	10.997038	13.139771	16.053109	19.307662	9.370801	19.229723
[8] 19.590336	17.049291	16.705115	4.028329	9.491038	8.629201	17.322909
[15] 13.347840	18.127826	15.085951	17.630277	19.941461	13.277794	18.197640
[22] 19.514095	9.660749	16.951754	6.882131	11.036435	13.382222	1.028522
[29] 13.318377	18.997202	12.559603	14.868757	16.369590	15.028740	8.922347
[36] 18.640806	17.130323	18.349609	6.183355	17.690021	13.800205	18.585186
[43] 16.901957	18.247629	15.809617	15.512915	18.305730	1.727778	14.800077
[50] 17.773593	17.381194					

Table 2: *Maximum Likelihood Estimates (with standard errors in parentheses) for simulated data set.*

		Estimates	
Model	$\hat{\zeta}$	$\hat{\eta}$	$\hat{\lambda}$
PNJ-PF	29.01068 (0.17446)	0.99048 (0.17443)	19.94143 (0.00003)
ZTP-PF	0.01308 (0.23971)	2.38893 (0.14642)	19.94143 (0.09328)
EP	3.40437 (0.12304)	0.69908 (0.12318)	19.94143 (0.00011)
EPF	5.18796 (0.23553)	1.15104 (0.23563)	19.94171 (0.00010)
PF	-	2.37968 (NaN)	19.94143 (0.00001)

Table 3: *Comparison of PNJ-PF Distribution with other competitive models for simulated data set.*

Model	$-2ll$	AIC	BIC	AICC	HQIC
PNJ-PF	270.8637	276.8637	282.6592	277.3744	279.0783
ZTP-PF	276.0330	282.0330	287.8285	282.5436	284.2476
EP	275.9709	281.9709	287.7664	282.4816	284.1855
EPF	271.3136	277.3136	283.1090	277.8242	279.5282
PF	275.9709	279.9709	283.8346	280.2209	281.4473

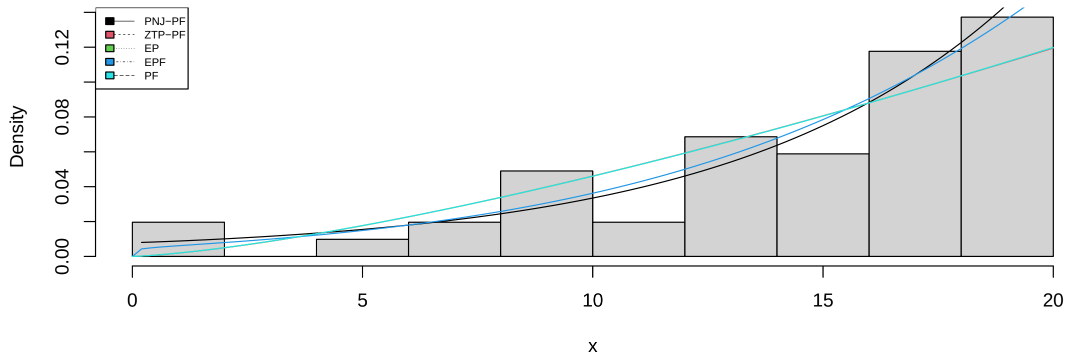

Figure 3: *The relative histogram and the fitted density functions of PNJ-PF and other competing distributions for simulated data set.*

The results of simulated dataset presented in Table 2 and Table 3,clearly demonstrate that the PNJ-PF distribution exhibits the lowest $-2ll$, AIC, BIC, AICC, and HQIC values among all the other competitive models and base line model.These findings are further supported by figure 3. Consequently, our proposed model offers a superior fit and outperforms base model of PF distribution as well as other mentioned competing models.

6. Applications

In this section, we explore two datasets related to electrical engineering to highlight the relevance and versatility of the PNJ-PF distribution. This analysis will demonstrate how the PNJ-PF distribution can be effectively utilized to model and interpret various types of data within the engineering field, illustrating its broad applicability and effectiveness. The data set first represents the times of 30 electronic components exposed to power-line voltage spikes during electric storms published first by [10] and is given as follows: 275,13, 147, 23, 181, 30, 65, 10, 300, 173, 106, 300, 300, 212, 300, 300, 300, 2,307 ,261, 293, 88, 247, 28, 143, 300, 23, 300, 80, 245, 266.
The second data set represents the failure times of first 50 electronic devices. which was originally published by [1], The data is given as follows: 0.1, 0.2, 1.0, 1.0, 1.0, 1.0, 1.0, 2.0, 3.0, 6.0, 7.0, 11.0, 12.0, 18.0, 18.0, 18.0, 18.0, 18.0, 21.0, 32.0, 36.0, 40.0, 45.0, 45.0, 47.0, 50.0, 55.0, 60.0, 63.0, 63.0, 67.0, 67.0, 67.0, 67.0, 72.0, 75.0, 79.0, 82.0, 82.0, 83.0, 84.0, 84.0, 84.0, 85.0, 85.0, 85.0, 85.0, 85.0, 86.0, 86.0.
We compare the fit of the proposed PNJ-PF distribution with its base-model (two parameter Power function (PF) distribution) and with several more related competitive models, namely Exponentiated Power function(EPF) [2], Zero Truncated Poisson Power function(ZTP-PF)[13] and Exponentiated Power(EP) [4] Distribution, their corresponding density functions for $0 < x < \lambda$ are as follows

$$\text{ZTP-PF} \qquad f(x) = \frac{\zeta \eta x^{\eta-1} \exp\left(-\zeta \left(\frac{x}{\lambda}\right)^{\eta}\right)}{\lambda^{\eta}\left(\exp\left(-\zeta \left(\frac{x}{\lambda}\right)^{\eta}\right)\right) - \exp(-\zeta)}$$

$$\text{EPF} \qquad f(x) = \frac{\zeta\eta}{\lambda^{\eta}} \frac{1}{x^{1-\eta}} \left(1 + \left(\frac{x}{\lambda}\right)^{\eta}\right)^{-(1-\zeta)}$$

$$\text{EP} \qquad f(x) = \frac{\zeta \eta x^{\zeta\eta-1}}{\lambda^{\zeta\eta}}$$

For comparison, we use criterion of -2log-likelihood(-2ll), along with various information criteria (AIC, BIC,AICc, HQIC). And the results are displayed in Table 4, Table 5, Table 6 and Table 7.

Table 4: *Maximum Likelihood Estimates (with standard errors in parentheses) for first data set.*

Model	Estimates $\hat{\zeta}$	$\hat{\eta}$	$\hat{\lambda}$
PNJ-PF	9.88183	0.67367	307.01898
	(0.10531)	(0.15274)	(0.04681)
ZTP-PF	0.00000	1.01032	307.00001
	(0.00047)	(0.18195)	(0.18121)
EP	1.09892	0.80050	307.00001
	(0.39168)	(0.28722)	(0.10448)
EPF	3.79113	0.66570	307.00001
	(0.15531)	(0.16302)	0.00719
PF	-	1.00657	307.00001
		(NaN)	(0.00173)

Table 5: *Comparison of PNJ-PF Distribution with other competitive models for first data set.*

Model	$-2ll$	AIC	BIC	AICC	HQIC
PNJ-PF	349.4228	355.4228	359.7247	356.3116	356.8251
ZTP-PF	355.0636	361.0636	365.3656	361.9525	362.4660
EP	355.6021	361.6021	365.9040	362.4909	363.0044
EPF	351.7101	357.7101	362.0120	358.5989	359.1124
PF	355.0632	359.0632	361.9312	359.4918	359.9981

Table 6: *Maximum Likelihood Estimates (with standard errors in parentheses) for second dataset.*

Model	Estimates $\hat{\zeta}$	$\hat{\eta}$	$\hat{\lambda}$
PNJ-PF	9.884284	0.478196	86.000000
	(0.078434)	(0.084724)	(0.006096)
ZTP-PF	0.000001	0.739749	86.000003
	0.021726	0.102805	0.081023
EP	0.998401	0.728003	86.000003
	(0.379407)	(0.379788)	0.000197
EPF	3.820468	0.472193	86.000001
	(0.168872)	(0.093803)	(0.075191)
PF	-	0.726714	86.000002
		(NaN)	(0.001117)

Table 7: *Comparison of PNJ-PF Distribution with other competitive models for second data set.*

Model	$-2ll$	AIC	BIC	AICC	HQIC
PNJ-PF	431.2207	437.2207	442.9568	437.7424	439.4050
ZTP-PF	439.7738	445.7738	451.5098	446.2955	447.9581
EP	439.7582	445.7582	451.4943	446.2799	447.9425
EPF	434.5294	440.5294	446.2654	441.0511	442.7137
PF	439.7582	443.7582	447.5822	444.0135	445.2144

From Table 4, Table 5, Table 6 and Table 7, it is clearly evident that PNJ-PF distribution has lowest $-2ll$, AIC, BIC, AICC and HQIC values among all the other competitive models and base line model. Therefore provide superior fit and outperforms base model of PF distribution as well as other mentioned competing models.

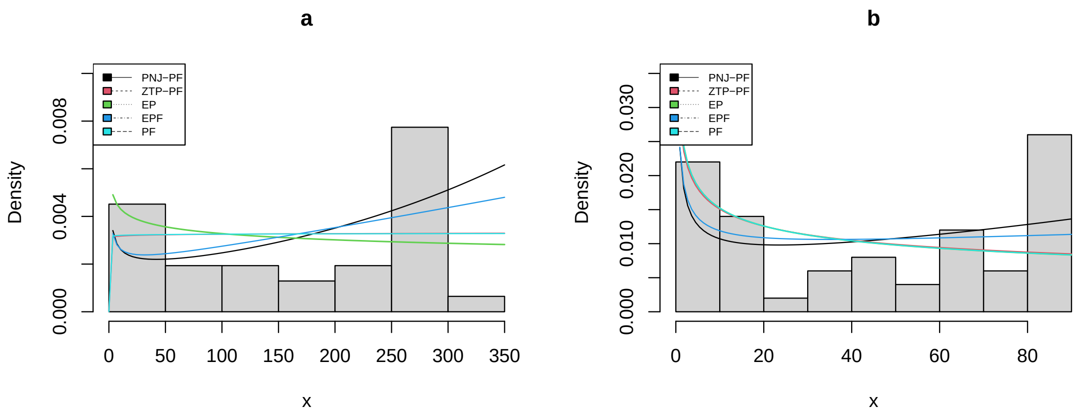

Figure 4: *(a) The relative histogram and the fitted density functions of PNJ-PF and other competing distributions for data set first. (b) The relative histogram and the fitted density functions of PNJ-PF and other competing distributions for data set second.*

The relative histogram and the fitted density functions of PNJ-PF and other competing distributions of the data set first and second are shown in Figures 4. This graphical representation clearly validate the results in Tables 4, Table 5, Table 6 and Table 7.

7. Conclusion

This manuscript introduces an innovative and versatile method known as the PNJ method for generating probability distributions. The PNJ method is specifically tailored to the two-parameter Power function (PF) distribution, resulting in the development of a new three-parameter PNJ-PF distribution. The paper thoroughly examines the various statistical and reliability characteristics of the PNJ-PF model, emphasizing its adaptable and flexible shapes for both density and hazard functions. To illustrate the PNJ-PF model's effectiveness, the study applies it to a simulated and two real-world datasets and conducts a comprehensive comparison with base model and other competing models using goodness-of-fit analysis. The findings clearly demonstrate that the PNJ-PF model outperforms base model and all other competing models in these datasets, showcasing its superior performance and effectiveness. Also by offering a novel approach that significantly enhances the accuracy and reliability of hazard rate modeling, this manuscript positions the PNJ-PF model as an essential tool for researchers and practitioners in engineering field. The innovative contributions of this study have the potential to bring about substantial advancements in the field, enabling more precise and effective decision-making based on strong statistical foundations.

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