INVENTORY MODEL FOR PROBABILISTIC DETERIORATION WITH RELIABILITY-DEPENDENT DEMAND AND TIME USING CLOUDY-FUZZY ENVIRONMENT

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Abstract

Inventory control is vital in supply chain management, especially for perishable goods. The paper depicts a probabilistic inventory model for robust products where deterioration and demand change over time and depend on reliability. This paper also talks about the conventional back order reliability inventory model in a fuzzy, cloudy environment. This is because products deteriorate and demand fluctuates all the time. This study shows a novel approach to modeling inventory that deals with these problems. It does this by including uniform distribution deterioration, demand that depends on both time and product reliability, and cloudy-fuzzy numbers to show uncertainty. Although we start with the crisp model and fuzzifying it to obtain a decision under the cloudy fuzzy demand rate (which is an extension of dense fuzzy) demand rate, before putting it to use in practice. For ranking the fuzzy numbers, a new defuzzification method was used. Subsequently, extensive analysis is done to compare the crisp, general fuzzy solutions to the cloudy fuzzy solutions. The numerical examples and graphical are examined to demonstrate that the novel approach is useful in the model itself. The suggested model aims to maintain high service reliability while minimizing the total cost of inventory. Numerical analyses indicate that the model is effective, exhibiting that it can lower costs and improve reliability compared to older models using MATLAB software. This study builds a strong framework for managing inventory in supply lines for perishable goods, which opens up opportunities for more progress in this area.

Keywords: Probabilistic inventory model, Uniform Distribution Deterioration, Time and Reliability-Dependent Demand, Cloudy-Fuzzy Environment, Defuzzification.

1. Introduction and literature survey

Inventory management is an important part of the supply chain, especially for perishable products is an essential field to investigate because of the complexity associated with demand variability, deterioration and uncertainty in system parameters. This literature review carefully investigates at the most significant developments in inventory models, time and reliability-dependent demand, focusing on uniform probabilistic distribution deterioration, and the application of cloudy-fuzzy numbers for handling uncertainties.This study provides an advanced accounting model that takes these complicated issues into account by consisting of three key features: demand that is dependent on both time and product reliability, uniform distribution deterioration over time and uncertainty represented by cloudy fuzzy numbers. Uniform distribution of deterioration shows how gradually and unpredictable nature the quality of perishable goods declines. Time and reliability dependent demand recognizes that customers choices about purchase are affected by the age the products are and how reliable they are perceived to be. Cloudy-fuzzy numbers provide a robust framework for handling the inherent uncertainties in demand and deterioration rates, combining the concepts of fuzzy logic and probability.

The purpose of this study is to develop up with an mathematically analytical optimization model that minimizes the total cost of inventory, which includes holding costs, deterioration cost, ordering cost, and reliability improvement costs. By making use of the cloudy-fuzzy number approach, we want to make an inventory management tool that is more accurate and flexible. so it can handle uncertainty as well as variations that happen in real-world supply chains.

1.1. Literature survey on Inventory Models with Uniform Probabilistic Distribution Deterioration

Different approaches have been used to study deterioration in inventory models in the conventionally began. In this way, Ghare and Schrader's [1] seminal work presented a model with constant deterioration rates. Real-life situations, on the other hand, frequently include uncertain deterioration, which is why probabilistic models have been developed. Uniform distribution a widely used probability distribution in inventory modelling is particularly relevant due to its simplicity and applicability when little is known about the distribution's parameters. Chang and Dye [2] examined into probabilistic deterioration in inventory systems, using a uniform distribution to model the uncertain the rate of deterioration. This approach was extended by Lo et al. [3] who applied a uniform probabilistic distribution to model deteriorating inventory with partial backlogging. Given that the uniform distribution can be used to model uncertainty in a number of ways, it is often used in inventory research. This is especially true for perishable products that whose rates of deterioration can vary.

Recent studies, such as those by Mahata, G and Mahata, P., [4] have these models were made even better by including uniform probability distributions to demonstrate different rates of degradation. These enhancements have made inventory models more precise and helpful, especially in environments in which there is much of uncertainty. Kumar, P and Dutta, D., [5] a deteriorating Inventory Model with Completely Backlogged Shortages and Uniformly Distributed Random Demand.

In the current market situation, it's hard to get an appropriate choice of the items customers want, so a probabilistic demand technique is more effective to deal with uncertainty. Pioneering the development of such models, Shah, N. H., [6] represent a probabilistic inventory model with allowed payment delays. Contributing to this, Shah, N. H. and Shah, Y. K., [7] included products that are getting worse and trade credit policy over a discrete time interval to the model. Several other researchers, including Shah, N. H., [8] created inventory models into probabilistic uncertain demand, trade credit financing, and shortages. De, L. N. and Goswami, A., [9] dedicated attention to joint inventory costs for customers and retailers, creating an EOQ model for deteriorating goods financed by trade credit. Khedlekar, et al. [10] Studied optimal replenishment decisions considering price, inventory under probabilistic demand and promotion strategies. Aldurgam et al. [11] demonstrated inventory control models for uncertain demand based on probability distributions, with a range of factors and assumptions. Probability distributions are often used to demonstrate how probabilistic demand.

1.2. Literature Survey on Time and Reliability-Dependent Demand in Inventory Models

Inventory management heavily relies on the relationship between demand and product reliability, particularly when it comes to perishable items. Models that take this dynamic into account are necessary since consumer demand frequently decreases as product quality deteriorates over time.

Their research presents the crucial it is to integrating reliability into demand forecasts in order to maximize inventory levels and minimize waste. The work that suggested A model of inventory where demand is a function of both reliability and time represents further advancements in this field. This method offers a more accurate depiction of customer behavior, especially in markets where demand is heavily influenced by product quality. Reliability has been progressively incorporated into demand modelling, with multi-echelon supply chain models containing reliability-dependent demand.

In the current competitive market, producing a high-reliable, dependable product is necessary to maintain a positive reputation in the marketplace. As a result, product quality, which is usually seen as a subset of reliability, is intimately tied to reliability. A product of superior quality is likely to be very reliable in the marketplace for an extended period of time, and vice versa. Reliability is a measure of the probability of failure-free operation over a specified period of time and is concerned with minimizing failure during that period. In actuality, it would be more sensible to view the demand as contingent on both time and reliability. As a result, demand for highly reliable products rises with time; that is, time typically influences a significant portion of customers to purchase high-reliability goods. Abdulla et al. [12] created an inventory model that is optimised while taking reliability into account. Due to the imperfect production that Krishnamoorthi and Panayappan [13] worked on, there are shortages and rework. Sarkar et al. [14] studied optimal reliability in the case of defective items.Pal et al. [15] Customers have many options available to them, but they tend to purchase highly dependable products to save money on future inconveniences and recurrent maintenance costs.

An economic manufacturing model with reliability is investigated by Bag et al. [16]. Sarkar et al. [17] have looked at a study on goods of defective quality and the impact of reliability. Tripathy, P. K., and Pattnaik, M. [18] built a model for inventories that takes reliability into account. Paul et al. [19] have created a production inventory model that takes reliability and unpredictability into account. Bhunia et al.[20] have investigated a production-inventory model that takes flexible reliability into account and a variable demand rate. The interval valued inventory cost criteria have been taken into consideration. An inventory model is examined by Khara et al. [21] who established a reliability model that takes into account the cost factors while accounting for the noninstantaneous deterioration rate of the products. focus of an imperfect production system is the demand rate, which is dependent on reliability. Abdel-Aleem et al. [22] has created an inventory model for a system of production with reliability. Rathore, H. [23] established a reliability model that takes into account the cost factors while accounting for the noninstantaneous deterioration rate of the products. Yadav et al. [24] perceived as a generalization of other research on deterioration and investments in green technologies.

1.3. Literature Survey on Cloudy-Fuzzy Numbers in Inventory Management

For a long time, inventory management has used fuzzy logic to deal with uncertainty in variables like lead times, deterioration rates, and demand. Zadeh [25] created fuzzy sets, which are now widely utilized in inventory models. Nevertheless, conventional fuzzy models frequently fail to fully express the degree of uncertainty present in real-world situations. It must be evident to all inventors that uncertainty gradually leaves the system over time. The traditional understanding of fuzzy set theory, however, makes the assumption that fuzziness is continuous across time. Therefore, De and Mahata [26] developed an inventory model with backorders under cloudy fuzzy and came up with the concept of cloudy fuzzy numbers. De and Mahata, [27] for mediocre quality items, a cloudy fuzzy EOQ model with reasonable proportionate discounts was created. Shah and Patel [28] discussed the best ordering practices for retailers using supplier credits in cases of fuzzy and cloudy fuzzy demand. Padiyar et al. [29] offered a supply chain model for a production process using cloudy fuzzy computing in a few real-world situations.

1.4. Literature Survey on Integration of Uniform Probabilistic Distribution Deterioration, Time and Reliability-Dependent Demand, and Cloudy-Fuzzy Numbers

An important development in inventory modelling is the combination of cloudy-fuzzy numbers, time and reliability-dependent demand, and uniform probabilistic distribution deterioration. Especially for perishable commodities, this method makes inventory systems more realistic and thorough to depict. Many academics have looked at classical inventory models in the study of the inventory system, which have a constant demand rate or that increase or decrease linearly. Numerous researchers have hypothesized that he function of time, stock, and price the demand rate is decreasing. However, the market has seen that, for a while, the demand pattern does not accurately reflect the growth in specific commodities, such as freshly introduced products, hardware devices, cosmetics, stylish clothing, mobile phones, and electronic items. Cash on hand is the most significant and practical factor in the inventory system, and it is a crucial factor in demand patterns. A salaried individual receives their pay at the beginning of each month or fortnight, thus their cash on hand is at its maximum during that period. As a result, they will make an effort to purchase more necessities at the beginning of the cycle. Therefore, the demand will automatically increase. They will purchase only essential and limited products starting in the middle of the time as their financial reserves diminish. As a result of limited purchasing options, demand will naturally decline. Thus, cash on hand is essential to the field of inventory management. Numerous scholars developed inventory models that took into account various demand categories and time-varying demand (Chang, H. J. and Dye, C. Y. [30], Khanra, et al. [31] and Sett, et al. [32]) price-sensitive demand model (Chanda, U. and Kumar, A. [33], Pal, S. and Mahapatra, G. S. [34]). Several researchers (Goli et al. [35] and [36], Shabani et al. [37]) examined a fuzzy deterioration rate and fuzzy demand rate inventory model. Barzegar et al. [38] examined vendor-managed inventory systems with constrained storage space and partial backordering under stochastic demand. . Modak and Kelle [39] demonstrated stochastic demand that is sensitive on pricing and delivery time. Shah and Vaghela [40] examined a time-and effort-dependent demand under-inflation and reliability model using an imperfect production inventory model. Barman et al. [41] offered a cloudy-fuzzy model of backordered inventories with inflation.

To a limited extent, this is accurate; yet, other academics have created various inventory models based on the premise that the holding cost remains constant throughout the inventory cycle. In the current day, buyers are drawn to products with a high degree of reliability; As a result, holding times are shortened and demand is automatically raised. This implies that for a product with high reliability, the holding cost will naturally decreases, resulting in a decreases in the overall cost. A lot of researchers talked about nonlinear holding costs in inventory models (San-Jose, et al. [42] and [43]) and time-varying holding cost (Alfares, H. K. and Ghaithan, A. M. [44], Pervin, et. al. [45]). Pervin et al. [46] talked about a two-echelon inventory model that included variable holding costs for degrading items and stock-dependent demand. Reliability that depends on holding costs in inventory system modelling is quite uncommon, nevertheless. This inventory system research takes reliability into account while assessing holding charges. By creating an analytical optimization model that takes into account cloudy-fuzzy numbers, time and reliability-dependent demand, and uniform probabilistic distribution deterioration, the current research seeks to close this gap. An enhanced framework for handling perishable inventory in uncertain situations is anticipated from this concept. Yadav et al. [47] regarded as interval valued the inventory parameters.This represents uncertainty in another manner.We don't employ any uncertainty elimination techniques in this case. Yadav et al. [48] used price sensitive demand or time sensitive demand. Dutta, Anurag, et al. [49] use of the data by National Aeronautics and Space Administration to train our model.

2. Preliminary Concept

2.1. Normalized General Triangular Fuzzy Number (NGTFN)

Let X be a NGTFN which is the form $\tilde{X} = (X_1, X_1, X_1)$,then its membership function which is defined by

$$\mu(\tilde{X}) = \begin{cases} 0, & \text{if } X < X_1 \text{ and } X > X_2 \\ \frac{X - X_1}{X_2 - X_1}, & \text{if } X_1 \leq X \leq X_2 \\ \frac{X_3 - X}{X_3 - X_2}, & \text{if } X_2 \leq X \leq X_3 \end{cases} \tag{1}$$

Now, the right and left α - cuts of $\mu(\tilde{X})$ are

$$R(\alpha) = X_3 + (X_3 - X_2)\alpha \ and \ L(\alpha) = X_1 + (X_2 - X_1)\alpha \tag{2}$$

It should be noted that the following formula can be used to determine the fuzziness measure:

2.2. Ranking Index

if $R(\alpha)$ and $L(\alpha)$ are the right and left $\alpha-$ cuts of a fuzzy number $\tilde{X}$ then Yager's [50] Ranking Index's defuzzification rule is provided by.

$$I(\tilde{X}) = \frac{1}{2}\int_0^1 [R(\alpha) + L(\alpha)]d\alpha \tag{3}$$

Note that the measure of fuzziness (degree of fuzziness d_f) can be found using the equation $d_f = \frac{U_b - L_b}{2g}$, where L_b and U_b are the lower bounds and upper bounds of the fuzzy numbers respectively and g being their respective mode.

2.3. Cloudy Normalized Triangular Fuzzy Number (CNTFN)

A fuzzy number which is the form $\tilde{A} = (a_1, a_2, a_3)$ is called cloudy triangular fuzzy number. An The set itself converges to a crisp singleton in infinite time. i.e., both $a_1, a_3 \to a_2$. as time $t \to \infty$ Considering the fuzzy number(extension of De and Beg [51]).

$$\tilde{A} = \left\langle a_2\left(1 - \frac{\rho}{1+t}\right), a_2, a_2\left(1 + \frac{\sigma}{1+t}\right)\right\rangle \tag{4}$$

Note that $\lim_{t\to\infty} a_2\left(1 - \frac{\rho}{1+t}\right) = a_2$ and $\lim_{t\to\infty} a_2\left(1 + \frac{\sigma}{1+t}\right) = a_2$ so $\tilde{A} \to a_2$. The following are the membership functions for $0 \leq t$:

$$\mu(x,t) = \begin{cases} 0, & \text{if } X < a_2\left(1 - \frac{\rho}{1+t}\right) \text{ and } X > a_2\left(1 + \frac{\sigma}{1+t}\right) \\ \frac{X - a_2\left(1 - \frac{\rho}{1+t}\right)}{\frac{\rho a_2}{1+t}}, & \text{if } a_2\left(1 - \frac{\rho}{1+t}\right) \leq x \leq a_2 \\ \frac{a_2\left(1 + \frac{\sigma}{1+t}\right) - x}{\frac{\sigma a_2}{1+t}}, & \text{if } a_2 \leq x \leq a_2\left(1 + \frac{\sigma}{1+t}\right) \end{cases} \tag{5}$$

The graphical representation of CNTFN (Figure 1) is obtained in the manner described below:

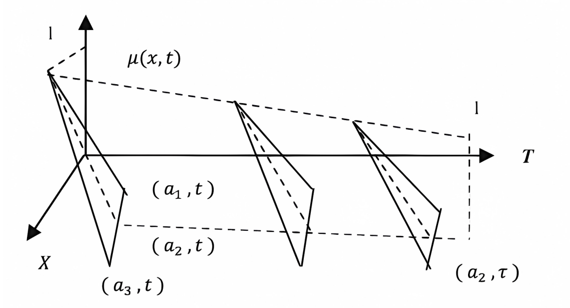

Figure 1: *Membership function of CNTFN*

2.4. Ranking Index on CNTFN

Let consider right and left α-cuts of $\mu(x,t)$ from equation (5) noted as $R(\alpha,t)$ and $L(\alpha,t)$ respectively. After that, under the temporal extension of Yager's ranking index(Extended De and Beg's [51]), the defuzzification formula is provided by

$$I(\tilde{A}) = \frac{1}{2T} \int \int_{\alpha=0,t=0}^{\alpha=1,t=T} [R^{-1}(\alpha,t) + L^{-1}(\alpha,t)] d\alpha dt \tag{6}$$

Here, α and t are independent variables. Let $\tilde{A}$ be a CNTFN stated in equation(4). Then the its membership function equation(5). Now taking the right and left α-cuts of $\mu(x,t)$ from equation(5) we get

$$R^{-1}(\alpha,t) = a_2\left(1 + \frac{\sigma}{1+t} - \frac{\alpha\sigma}{1+t}\right) \ and \ L^{-1}(\alpha,t) = a_2\left(1 - \frac{\rho}{1+t} + \frac{\alpha\rho}{1+t}\right) \tag{7}$$

Thus using equation(6), we have

$$I(\tilde{A}) = \frac{a_2}{2T}\left[2T + \frac{\sigma-\rho}{2} log(1+T)\right] \tag{8}$$

Again equation(8) can be rewritten as

$$I(\tilde{A}) = a_2\left[1 + \frac{\sigma-\rho}{4T} log(1+T)\right] \tag{9}$$

Obviously, $\lim\limits_{T\to\infty} \frac{log(1+T)}{T} = 0$ and therefore $I(\tilde{A}) \to a_2$ as $T \to \infty$ Note that the factor may be referred to as $\frac{log(1+T)}{T}$ cloud index (CI) and in actuality, the time T is measured in days. Figure 2 depicts the cloud index's nature.

3. Assumptions and notations

3.1. Assumptions

1. Demand rate is depends on both time and product reliability, which is $D(r,t) = mtr^n$, where $m, n > 0$ are constants.

2. Shortages are permitted and the backlog is fully..

3. Deterioration rate parameter is depending on uniform distribution. Which is $\alpha = f(\xi) = \frac{1}{\gamma_2-\gamma_1}(\gamma_1 < \xi < \gamma_2)$ similar to Shah, Nita H., et al. [8] as in demand rate.

4. Here $\alpha \sim U(5,10); \alpha = 0.2$ is uniform distribution deterioration parameter.

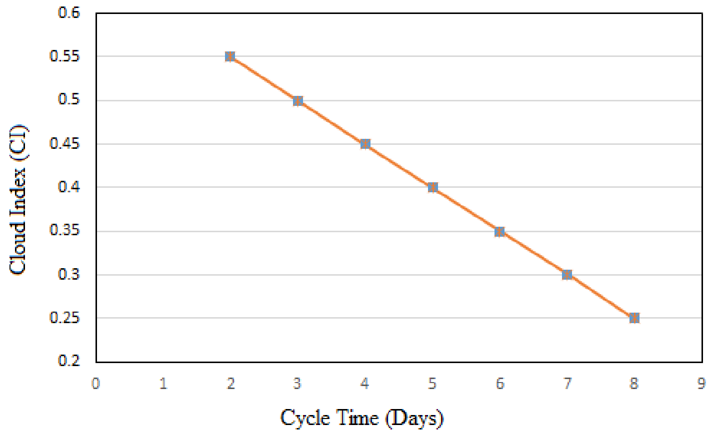

Figure 2: *Character of fuzziness across time*

3.2. Notations

Table 1 a description of the notations used for the created mathematical model is given.

Table 1: *Notations*

Notation	Units	Description
p	\$/unit	Deterioration Cost
h	\$/unit	Holding Cost
q	\$/unit	Shortage cost.
A	\$/unit	ordering cost.
n	Reliability parameter	Which is lies always between $0 < n < 1$
m	–	shape parameter $m > 0$
$I_1(t)$	units	Inventory Level at time t over the ordering cycle $(0, t_1)$
$I_2(t)$	units	Inventory Level at time t over the ordering cycle (t_1, T)
Q	units	Maximum inventory level at time t in $[0, T]$ for ordering cycle i.e.$I_1(0) = Q$.
R	Decision Variable	Shortage Level.
T	Units	Length of total replenishment cycle(days).
$TIC(r, T)$	\$/Units	Total Average inventory cost unit time.
r	–	Reliability of item of the inventory system.
T	Years	Length of the cycle.
$\widetilde{TIC}(r, T)$	\$/Units	Fuzzy Total Average inventory cost unit time.

4. Mathematical Model Formulation

The stock level at $t = 0$ to $t = T$ is characterised as follows in the differential equations:

$$\frac{dI_1(t)}{dt} + \alpha I_1(t) = -(mtr^n); \quad 0 \le t \le t_1 \tag{10}$$

with boundary conditions (B.C.) $I_1(t_1) = 0$ and $I_1(0) = Q$.

$$\frac{dI_2(t)}{dt} = -(mtr^n); \quad t_1 \le t \le T \tag{11}$$

with the boundary conditions (B.C.) $I_2(t_1) = 0$ and $I_2(T) = -R$.
The solutions to equations (10) and equation (11) are equation (12) and equation(13), respectively.

$$I_1(t) = \frac{mr^n}{\alpha^2}\left[1 - e^{\alpha(t_1 - t)}(1 - \alpha t_1)\right]; \quad 0 \le t \le t_1 \tag{12}$$

$$I_2(t) = \frac{mr^n}{2}(T^2 - t_1^2); \quad t_1 \le t \le T \tag{13}$$

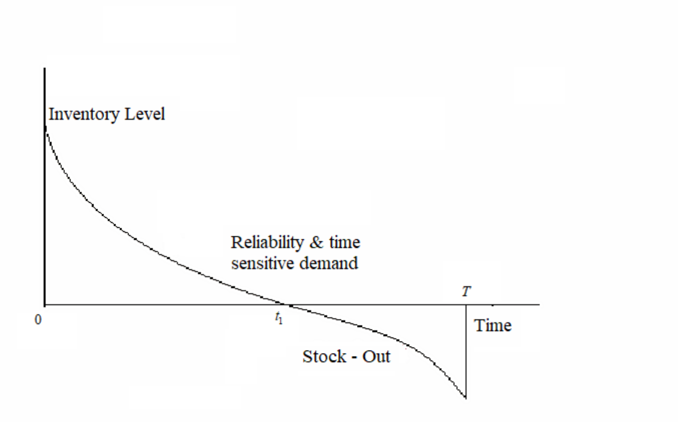

Figure 3: *Mathematical Model between inventory and time*

The following elements make up the inventory system's total cost per unit of time.

(a). **Opportunity Cost :**

$$OC = A \tag{14}$$

(b). **Deterioration cost :**

$$DC = p \cdot \alpha \int_0^{t_1} I_1(t)dt$$

$$DC = -\frac{pmr^n}{2\alpha^2}(2e^{\alpha t_1} + \alpha^2 t_1^2 - 2\alpha t_1 e^{\alpha t_1} - 2) \tag{15}$$

(c). **Holding Cost :**

$$HC = h \int_0^{t_1} I_1(t)dt$$

$$HC = -\frac{hmr^n}{2\alpha^3}(2e^{\alpha t_1} + \alpha^2 t_1^2 - 2\alpha t_1 e^{\alpha t_1} - 2) \tag{16}$$

(d). **Shortage Cost :**

$$SC = q \int_{t_1}^{T} [-I_2(t)]dt$$

$$SC = \frac{qmr^n}{6}(T - t_1)^2(T + 2t_1) \tag{17}$$

Hence, the total cost function per unit time is:

(e). **Total Inventory Cost (TIC):**

$$TIC = \frac{1}{T}[OC + DC + HC + SC]$$

$$TIC = \frac{1}{T}[A - \frac{pmr^n}{2\alpha^2}(2e^{\alpha t_1} + \alpha^2 t_1^2 - 2\alpha t_1 e^{\alpha t_1} - 2) - \frac{hmr^n}{2\alpha^3}(2e^{\alpha t_1} + \alpha^2 t_1^2 - 2\alpha t_1 e^{\alpha t_1} - 2) + \frac{qmr^n}{6}(T - t_1)^2(T + 2t_1)] \tag{18}$$

Thus, our problem is given by

$$\begin{cases} Minimize\ TIC = \frac{1}{T}[A - \frac{pmr^n}{2\alpha^2}(2e^{\alpha t_1} + \alpha^2 t_1^2 - 2\alpha t_1 e^{\alpha t_1} - 2) - \frac{hmr^n}{2\alpha^3}(2e^{\alpha t_1} + \alpha^2 t_1^2 \\ -2\alpha t_1 e^{\alpha t_1} - 2) + \frac{qmr^n}{6}(T - t_1)^2(T + 2t_1)] \\ Subject\ to\ Q = \frac{mr^n}{\alpha^2}\left[1 - e^{\alpha t_1}(1 - \alpha t_1)\right], R = \frac{mr^n}{2}(T^2 - t_1^2) \end{cases} \tag{19}$$

5. Developing a Fuzzy Mathematical Model

Over the course of the inventory run period, let the demand rate follow a general fuzzy and cloudy fuzzy. Then taking $\tilde{X}$ as follows

$$\tilde{X} = \begin{cases} \langle X_1, X_2, X_3 \rangle \ for\ NGTFN \\ \left\langle X\left(1 - \frac{\rho}{1+t}\right), X, X\left(1 + \frac{\sigma}{1+t}\right) \right\rangle \ for\ CNTFN,\ where\ 0 < \rho, \sigma < 1\ and\ T > 0 \end{cases} \tag{20}$$

And the corresponding fuzzy problem is given by

$$\begin{cases} Minimize\ \widetilde{TIC} = \frac{1}{T}[A - \frac{\tilde{p}\tilde{m}r^{\tilde{n}}}{2\tilde{\alpha}^2}(2e^{\tilde{\alpha} t_1} + \tilde{\alpha}^2 t_1^2 - 2\tilde{\alpha} t_1 e^{\tilde{\alpha} t_1} - 2) - \frac{\tilde{h}\tilde{m}r^{\tilde{n}}}{2\tilde{\alpha}^3}(2e^{\tilde{\alpha} t_1} + \tilde{\alpha}^2 t_1^2 \\ -2\tilde{\alpha} t_1 e^{\tilde{\alpha} t_1} - 2) + \frac{\tilde{q}\tilde{m}r^{\tilde{n}}}{6}(T - t_1)^2(T + 2t_1)] \\ Subject\ to\ Q = \frac{\tilde{m}r^{\tilde{n}}}{\tilde{\alpha}^2}\left[1 - e^{\tilde{\alpha} t_1}(1 - \tilde{\alpha} t_1)\right], R = \frac{\tilde{m}r^{\tilde{n}}}{2}(T^2 - t_1^2) \end{cases} \tag{21}$$

Now, Using equation (3), the order amount and shortage quantity under NGTFN are given by the membership function for the fuzzy objective

$$\mu_1(T\tilde{I}C) = \begin{cases} 0, & \text{if } TIC < TIC_1 \text{ and } TIC > TIC_2 \\ \frac{TIC - TIC_1}{TIC_2 - TIC_1}, & \text{if } TIC_1 \leq TIC \leq TIC_2 \\ \frac{TIC_3 - TIC}{TIC_3 - TIC_2}, & \text{if } TIC_2 \leq TIC \leq TIC_3 \end{cases} \tag{22}$$

where,

$$TIC_1 = \frac{1}{T}\Bigg[A - \frac{p_1 m_1 r^{n_1}}{2\alpha_1^2}(2e^{\alpha_1 t_1} + \alpha_1^2 t_1^2 - 2\alpha_1 t_1 e^{\alpha_1 t_1} - 2) \\ - \frac{h_1 m_1 r^{n_1}}{2\alpha_1^3}(2e^{\alpha_1 t_1} + \alpha_1^2 t_1^2 - 2\alpha_1 t_1 e^{\alpha_1 t_1} - 2) + \frac{q_1 m_1 r^{n_1}}{6}(T - t_1)^2(T + 2t_1)\Bigg]$$

$$TIC_2 = \frac{1}{T}\Bigg[A - \frac{p_2 m_2 r^{n_2}}{2\alpha_2^2}(2e^{\alpha_2 t_1} + \alpha_2^2 t_1^2 - 2\alpha_2 t_1 e^{\alpha_2 t_1} - 2) \\ - \frac{h_2 m_2 r^{n_2}}{2\alpha_2^3}(2e^{\alpha_2 t_1} + \alpha_2^2 t_1^2 - 2\alpha_2 t_1 e^{\alpha_2 t_1} - 2) + \frac{q_2 m_2 r^{n_2}}{6}(T - t_1)^2(T + 2t_1)\Bigg]$$

$$TIC_3 = \frac{1}{T}\Bigg[A - \frac{p_3 m_3 r^{n_3}}{2\alpha_3^2}(2e^{\alpha_3 t_1} + \alpha_3^2 t_1^2 - 2\alpha_3 t_1 e^{\alpha_3 t_1} - 2) \\ - \frac{h_3 m_3 r^{n_3}}{2\alpha_3^3}(2e^{\alpha_3 t_1} + \alpha_3^2 t_1^2 - 2\alpha_3 t_1 e^{\alpha_3 t_1} - 2) + \frac{q_3 m_3 r^{n_3}}{6}(T - t_1)^2(T + 2t_1)\Bigg]$$

$$\mu_2(\tilde{Q}) = \begin{cases} 0, & \text{if } Q < Q_1 \text{ and } Q > Q_2 \\ \frac{Q-Q_1}{Q_2-Q_1}, & \text{if } Q_1 \leq Q \leq Q_2 \\ \frac{Q_3-Q}{Q_3-Q_2}, & \text{if } Q_2 \leq Q \leq Q_3 \end{cases} \tag{23}$$

where

$$Q_i = \frac{m_i r^{n_i}}{\alpha_i^2}\left[1 - e^{\alpha_i t_1}(1-\alpha_i t_1)\right]; \quad i = 1,2,3$$

$$\mu_3(\tilde{R}) = \begin{cases} 0, & \text{if } R < R_1 \text{ and } R > R_2 \\ \frac{R-R_1}{R_2-R_1}, & \text{if } R_1 \leq R \leq R_2 \\ \frac{R_3-R}{R_3-R_2}, & \text{if } R_2 \leq R \leq R_3 \end{cases} \tag{24}$$

where

$$R_i = \frac{m_i r^{n_i}}{2}(T^2 - t_1^2); \quad i = 1,2,3$$

The fuzzy goal, fuzzy order quantity, and fuzzy shortage quantity index values are obtained, respectively, using equation (2) and equation (3).

$$\begin{aligned} I(\widetilde{TIC}) &= \frac{1}{4}(TIC_1 + 2TIC_2 + TIC_3) \\ &= \frac{(m_1 r^{n_1} + m_2 r^{n_2} + m_3 r^{n_3})}{4}\Bigg[\Bigg(\frac{q_1}{6T}(T-t_1)^2(T+2t_1) + \frac{2q_2}{6T}(T-t_1)^2(T+2t_1) \\ &\quad + \frac{q_3}{6T}(T-t_1)^2(T+2t_1)\Bigg) + \Bigg(-\frac{p_1}{2T\alpha_1^2}(2e^{\alpha_1 t_1} + \alpha_1^2 t_1^2 - 2\alpha_1 t_1 e^{\alpha_1 t_1} - 2) \\ &\quad - \frac{p_2}{2T\alpha_2^2}(2e^{\alpha_2 t_1} + \alpha_2^2 t_2^2 - 2\alpha_1 t_1 e^{\alpha_2 t_1} - 2) - \frac{p_3}{2T\alpha_3^2}(2e^{\alpha_3 t_1} + \alpha_3^2 t_1^2 - 2\alpha_3 t_1 e^{\alpha_3 t_1} - 2)\Bigg) \\ &\quad + \Bigg(-\frac{h_1}{2T\alpha_1^3}(2e^{\alpha_1 t_1} + \alpha_1^2 t_1^2 - 2\alpha_1 t_1 e^{\alpha_1 t_1} - 2) - \frac{h_2}{2T\alpha_2^3}(2e^{\alpha_2 t_1} + \alpha_2^2 t_1^2 - 2\alpha_2 t_1 e^{\alpha_2 t_1} - 2) \\ &\quad - \frac{h_3}{2T\alpha_3^3}(2e^{\alpha_3 t_1} + \alpha_3^2 t_1^2 - 2\alpha_3 t_1 e^{\alpha_3 t_1} - 2)\Bigg)\Bigg] + \frac{4A}{T} \end{aligned} \tag{25}$$

$$\begin{aligned} I(\widetilde{Q}) = \frac{1}{4}(Q_1 + 2Q_2 + Q_3) = \frac{(m_1 r^{n_1} + m_2 r^{n_2} + m_3 r^{n_3})}{4}\Bigg[\Bigg(\frac{1}{\alpha_1^2}\{1 - e^{\alpha_1 t_1}(1-\alpha_1 t_1)\}\Bigg) \\ + \Bigg(\frac{1}{\alpha_2^2}\{1 - e^{\alpha_2 t_1}(1-\alpha_2 t_1)\}\Bigg) + \Bigg(\frac{1}{\alpha_3^2}\{1 - e^{\alpha_3 t_1}(1-\alpha_3 t_1)\}\Bigg)\Bigg] \end{aligned}$$

$$I(\widetilde{R}) = \frac{1}{4}(R_1 + 2R_2 + R_3) = \frac{(m_1 r^{n_1} + m_2 r^{n_2} + m_3 r^{n_3})(T^2 - t_1^2)}{2}$$

Particular Case:

(i) if $m_1 r^{n_1} \to m_2 r^{n_2}$ and $m_3 r^{n_3} \to m_2 r^{n_2} \to m r^n$
Then $I(\widetilde{TIC}) \to \frac{(m_1 r^{n_1} + m_2 r^{n_2} + m_3 r^{n_3})}{4}\Bigg[\Bigg(\frac{q_1}{6T}(T-t_1)^2(T+2t_1) + \frac{2q_2}{6T}(T-t_1)^2(T+2t_1)$
$+ \frac{q_3}{6T}(T-t_1)^2(T+2t_1)\Bigg) + \Bigg(-\frac{p_1}{2T\alpha_1^2}(2e^{\alpha_1 t_1} + \alpha_1^2 t_1^2 - 2\alpha_1 t_1 e^{\alpha_1 t_1} - 2) - \frac{p_2}{2T\alpha_2^2}(2e^{\alpha_2 t_1} + \alpha_2^2 t_2^2$

$$-2\alpha_1 t_1 e^{\alpha_2 t_1} - 2) - \frac{p_3}{2T\alpha_3^2}(2e^{\alpha_3 t_1} + \alpha_3^2 t_1^2 - 2\alpha_3 t_1 e^{\alpha_3 t_1} - 2)\Bigg) + \Bigg(- \frac{h_1}{2T\alpha_1^3}(2e^{\alpha_1 t_1} + \alpha_1^2 t_1^2 - 2\alpha_1 t_1 e^{\alpha_1 t_1}$$

$$-2) - \frac{h_2}{2T\alpha_2^3}(2e^{\alpha_2 t_1} + \alpha_3^2 t_1^2 - 2\alpha_3 t_1 e^{\alpha_3 t_1} - 2)\Bigg)\Bigg],$$

$$I(\widetilde{Q}) \to \frac{mr^n}{\alpha^2}\Big[1 - e^{\alpha t_1}(1 - \alpha t_1)\Big] \text{ and } I(\widetilde{R}) \to \frac{mr^n}{2}(T^2 - t_1^2)$$

(ii) if $t_1 \to T, I(\widetilde{TIC}) \to -\frac{pmr^n}{2T\alpha^2}\Bigg(2e^{\alpha T} + \alpha^2 T^2 - 2\alpha T e^{\alpha T} - 2\Bigg)$

$$-\frac{hmr^n}{2T\alpha^3}\Bigg(2e^{\alpha T} + \alpha^2 T^2 - 2\alpha T e^{\alpha T} - 2\Bigg) + \frac{A}{T},$$

On the other hand, applying equation (5), the fuzzy objective, fuzzy order quantity and fuzzy shortage quantity membership functions under the cloudy fuzzy model are provided by

$$\omega_1(TIC, T) = \begin{cases} 0, & \text{if } TIC < TIC_{11} \text{ and } TIC > TIC_{21} \\ \frac{TIC - TIC_{11}}{TIC_{21} - TIC_{21}}, & \text{if } TIC_{21} \le TIC \le TIC_{21} \\ \frac{TIC_{31} - TIC}{TIC_{31} - TIC_{21}}, & \text{if } TIC_{21} \le TIC \le TIC_{31} \end{cases} \tag{26}$$

where,

$$TIC_{11} = \frac{1}{T}\Bigg[\frac{q_1}{6}(T - t_1)^2(T + 2t_1) - \frac{p}{2\alpha^2}(2e^{\alpha t_1} + \alpha^2 t_1^2 - 2\alpha t_1 e^{\alpha t_1} - 2)$$

$$-\frac{h}{2\alpha^2}(2e^{\alpha t_1} + \alpha^3 t_1^2 - 2\alpha t_1 e^{\alpha t_1} - 2)\Bigg] mr^n \Bigg(1 - \frac{\rho}{1+T}\Bigg) + \frac{A}{T}$$

$$TIC_{21} = \frac{1}{T}\Bigg[\frac{q_1}{6}(T - t_1)^2(T + 2t_1) - \frac{p}{2\alpha^2}(2e^{\alpha t_1} + \alpha^2 t_1^2 - 2\alpha t_1 e^{\alpha t_1} - 2)$$

$$-\frac{h}{2\alpha^2}(2e^{\alpha t_1} + \alpha^3 t_1^2 - 2\alpha t_1 e^{\alpha t_1} - 2)\Bigg] mr^n + \frac{A}{T}$$

$$TIC_{31} = \frac{1}{T}\Bigg[\frac{q_1}{6}(T - t_1)^2(T + 2t_1) - \frac{p}{2\alpha^2}(2e^{\alpha t_1} + \alpha^2 t_1^2 - 2\alpha t_1 e^{\alpha t_1} - 2)$$

$$-\frac{h}{2\alpha^2}(2e^{\alpha t_1} + \alpha^3 t_1^2 - 2\alpha t_1 e^{\alpha t_1} - 2)\Bigg] mr^n \Bigg(1 + \frac{\sigma}{1+T}\Bigg) + \frac{A}{T}$$

$$\omega_2(Q, T) = \begin{cases} 0, & \text{if } Q < Q_{11} \text{ and } Q > Q_{21} \\ \frac{Q - Q_{11}}{Q_{21} - Q_{11}}, & \text{if } Q_{11} \le Q \le Q_{21} \\ \frac{Q_{31} - Q}{Q_{31} - Q_{21}}, & \text{if } Q_{21} \le Q \le Q_{31} \end{cases} \tag{27}$$

where

$$Q_{11} = \Bigg(1 - \frac{\rho}{1+T}\Bigg)\frac{mr^n}{\alpha^2}\Bigg[1 - e^{\alpha t_1}(1 - \alpha t_1)\Bigg], Q_{21} = \frac{mr^n}{\alpha^2}\Bigg[1 - e^{\alpha t_1}(1 - \alpha t_1)\Bigg]$$

$$Q_{31} = \Bigg(1 + \frac{\sigma}{1+T}\Bigg)\frac{mr^n}{\alpha^2}\Bigg[1 - e^{\alpha t_1}(1 - \alpha t_1)\Bigg]$$

$$\omega_3(R, T) = \begin{cases} 0, & \text{if } R < R_{11} \text{ and } R > R_{21} \\ \frac{R - R_{11}}{R_{21} - R_{11}}, & \text{if } R_{11} \le R \le R_{21} \\ \frac{R_{31} - R}{R_{31} - R_{21}}, & \text{if } R_{21} \le R \le R_{31} \end{cases} \tag{28}$$

where

$$R_{11} = \Bigg(1 - \frac{\rho}{1+T}\Bigg)\frac{mr^n}{2}(T^2 - t_1^2), R_{21} = \frac{mr^n}{2}(T^2 - t_1^2), R_{31} = \Bigg(1 + \frac{\sigma}{1+T}\Bigg)\frac{mr^n}{2}(T^2 - t_1^2)$$

The cloudy fuzzy objective, cloudy fuzzy order quantity and cloudy fuzzy shortage quantity index values are obtained by using equation (6).
$J(\widetilde{TIC}) = \frac{1}{\tau}\int_{T=0}^{\tau}\frac{1}{4}[TIC_{11}+2TIC_{21}+TIC_{31}]dT = \frac{1}{4\tau}\int_{T=0}^{\tau}[TIC_{11}+2TIC_{21}+TIC_{31}]dT$

$$J(\widetilde{TIC}) = \frac{1}{4\tau}\int_{T=0}^{\tau}\Bigg[\frac{4A}{T}+\Bigg\{\left(4+\frac{\sigma-\rho}{1+T}\right)\left(\frac{q_1}{6T}(T-t_1)^2(T+2t_1)-\frac{p}{2T\alpha^2}(2e^{\alpha t_1} + \alpha^2t_1^2-2\alpha t_1e^{\alpha t_1}-2)-\frac{h}{2T\alpha^2}(2e^{\alpha t_1}+\alpha^3t_1^2-2\alpha t_1e^{\alpha t_1}-2)\right)\Bigg\}mr^n\Bigg]dT$$

$$J(\widetilde{TIC}) = -\frac{pmr^n}{2\alpha^2}(2e^{\alpha t_1}+\alpha^2t_1^2-2\alpha t_1e^{\alpha t_1}-2)+\frac{mr^n log\left(\frac{\tau}{\epsilon}\right)}{2\tau}\Bigg\{\frac{2A}{mr^n}+\left(1+\frac{\sigma-\rho}{4}\right)(c_1+c_2)t_1^2\Bigg\}+\frac{c_2mr^n}{4}(\tau-4t_1)+\frac{(\sigma-\rho)mr^n}{4}\Bigg[c_2+\Bigg\{-\frac{p}{\alpha^2}(2e^{\alpha t_1}+\alpha^2t_1^2-2\alpha t_1e^{\alpha t_1}-2)-c_1t_1^2-c_2(1+t_1)^2\Bigg\}\frac{log|1+\tau|}{\tau}\Bigg] \quad (29)$$

$J(\widetilde{Q}) = \frac{1}{\tau}\int_{T=0}^{\tau}\frac{1}{4}[Q_{11}+2Q_{21}+Q_{31}]dT = \frac{1}{4\tau}\int_{T=0}^{\tau}[Q_{11}+2Q_{21}+Q_{31}]dT$

$J(\widetilde{Q}) = \frac{mr^n}{4\tau}\Big\{2\tau^2+(\sigma-\rho)(\tau-log|1+\tau|)\Big\}$

$J(\widetilde{R}) = \frac{1}{\tau}\int_{T=0}^{\tau}\frac{1}{4}[R_{11}+2R_{21}+R_{31}]dT = \frac{1}{4\tau}\int_{T=0}^{\tau}[R_{11}+2R_{21}+R_{31}]dT$

$J(\widetilde{R}) = \frac{1}{4\tau}\int_{T=0}^{\tau}\Big\{4(T-t_1)+\frac{(\sigma-\rho)}{1+T}(T-t_1)\Big\}mr^n dT$

$$J(\widetilde{R}) = \frac{mr^n}{4\tau}\Big[2(\tau^2-2t_1\tau)+(\sigma-\rho)\Big\{\tau-(1+t_1)log|1+\tau|\Big\}\Big] \quad (30)$$

Analysis of stability and specific cases:

(i) if $(\sigma-\rho)\to 0$ then $J(\widetilde{TIC}) \to \frac{pmr^n}{2\alpha^2}(2e^{\alpha t_1}+\alpha^2t_1^2-2\alpha t_1e^{\alpha t_1}-2)+\frac{mr^n log\left(\frac{\tau}{\epsilon}\right)}{2\tau}\Big\{\frac{2A}{mr^n}+(c_1+c_2)t_1^2\Big\}+\frac{c_2mr^n}{4}(\tau-4t_1), J(\widetilde{Q}) \to \frac{mr^n\tau}{2} and J(\widetilde{R}) \to \frac{mr^n}{2\tau}\Big[(\tau^2-2t_1\tau)\Big]$

(ii) If $\sigma\to 0, \rho\to 0$ then the model reduces (i). But the case of the classical back-order EOQ model. So we choose ϵ so that the aforementioned becomes the classical model.Then we take

$\frac{pmr^n}{2\alpha^2}(2e^{\alpha t_1}+\alpha^2t_1^2-2\alpha t_1e^{\alpha t_1}-2)+\frac{mr^n log\left(\frac{\tau}{\epsilon}\right)}{2\tau}\Big\{\frac{2A}{mr^n}+(c_1+c_2)t_1^2\Big\}+\frac{c_2mr^n}{4}(\tau-4t_1)$
$= \Big[\frac{p}{2\alpha^2}(2e^{\alpha t_1}+\alpha^2t_1^2-2\alpha t_1e^{\alpha t_1}-2)+\frac{log\left(\frac{\tau}{\epsilon}\right)}{2\tau}(c_1+c_2)t_1^2+\frac{c_2}{4}(\tau-4t_1)\Big]mr^n+\frac{log\left(\frac{A\tau}{\epsilon}\right)}{\tau}$

Comparing we get, $log\left(\frac{\tau}{\epsilon}\right) = \frac{1}{T}$ and $\frac{log\left(\frac{\tau}{\epsilon}\right)}{\tau}t_1^2+\frac{(\tau-4t_1)}{4} = (T-t_1)^2$ giving $\tau = 2$, and $T = 1 \Rightarrow \epsilon \to 2e^{-2} \ll 1$.
Hence, model is stable, and the case, we write, $\tau = 2T$, and hence, $J(\widetilde{Q}) \to mr^nT$ and $J(\widetilde{R}) \to mr^nT(T-2t_1)$, which corresponds to the results of the classical back-order model.

6. Numerical Illustration

Crisp Model: Let us consider $A = \$30$ per cycle, $h = \$5$ per unit per year, $p = \$1.6$ per unit per year, $q = \$3.6$ per unit per year, $n = 0.8$, $\alpha = 0.2$, $m = 4.8$ i.e. $D = mtr^n = 550$.
Fuzzy Model: Let us demand rate $\langle D_1, D_2, D_3 \rangle = \langle m_1 tr_1^n, m_2 tr_2^n, m_3 tr_3^n \rangle = \langle 500, 550, 650 \rangle$ units keeping the other parameter same as crisp.
Cloudy Fuzzy Model: Let us take $\sigma = 0.16$, $\rho = 0.14$, $\epsilon = 0.001$. For the model's numerical illustration.

Take in mind that we determine the model of The following formula can be used to determine the fuzzy demand rate, which is the computation of fuzziness; mean of $\langle 500, 550, 650 \rangle = 566.67$, median =550, so mode(m) = 3 × median - 2× mean = 3 × 550- 2 ×566.67 = 516.66. Since fuzzy demand components L_b and U_b are the lower bounds and upper bounds, respectively, Then apply degree of fuzziness $d_f = \frac{U_b - L_b}{2g}$ and cloud index $CI = \frac{(1+T)}{T}$.

6.1. Numerical Discussion

Table 2 symbolizes, in a crisp environment, for the 1 month or 30 days (approximate) cycle time with 3.6 days During a shortage, the average optimal inventory cost is assumed to be the value \$18078.52 per cycle, but The fuzzy solution costs more, preserving the total average value of \$19225.62 per cycle. Moreover, it is astonishing that Every time we envision a cloudy, fuzzy environment, the average inventory cost decreases to \$14122.59 per cycle time simply by increasing the of the cycle time to 38.70 days alone. Additionally, we note that by this point, there is less fuzziness from 0.536 to 0.213.

Table 2 Optimal Solution of Model

Model	Cycle time T* (days)	Inventory period t_1^*	Shortage quantity R* units	Ordering quantity A* units	Minimum cost TIC*($)	$d_f = \frac{U_b - L_b}{2g}$	$CI = \frac{(1+T)}{T}$
Crisp	14.28	10.92	207.38	708.39	18078.52	—	—
Fuzzy	14.16	10.92	213.26	786.77	19225.62	0.536	—
Clody Fuzzy	36.70	31.68	786.52	1032.56	14122.59	—	0.213

Tables 3 and 4 show that, during the cycle time duration of 33 to 41 days, the average inventory cost is given by both the crisp and general fuzzy models, and both show a rise with cycle time duration; however, the cloudy fuzzy model's objective function assumes a U-shaped curve ™ at 20 days cycle time keeping the minimum possible average inventory cost globally. Furthermore, our observation shows that the hazy fuzzy model's solution is not viable at cycle times shorter than 33 days. In comparison to the results produced in both a crisp and a fuzzy environment, the order quantity and shortage quantity increase in the cloudy fuzzy model. Nonetheless, we can observe that they are naturally increasing in all situations if we look at the trend values of the order quantity and shortage quantity that they take.

Table 3 Objective values under various cycle times

Cycle time	Crisp model				Fuzzy model			
T (days)	t_1^*	R*	q*	TIC**	t_1^*	R*	q*	TIC**
33	27.23	412.56	2186.52	18318.73	27.23	424.29	2245.38	19536.66
34	28.65	445.13	2331.10	18386.04	28.65	456.64	456.64	19605.42
35	30.07	477.70	2475.68	18453.35	30.07	488.99	2532.86	19674.18
36	31.49	510.27	2620.26	18520.66	31.49	521.34	2676.60	19742.94
37	32.91	542.84	2764.84	18587.97	32.91	553.69	2820.34	19811.70
38	34.33	575.41	2909.42	18655.28	34.33	586.04	2964.08	19880.46
39	35.75	607.98	3054.00	18722.59	35.75	618.39	3107.82	19949.22
40	37.17	640.55	3198.58	18789.90	37.17	650.74	3251.56	20017.98

Table 4 Cloudy fuzzy values under several cycle times

Cycle time	Crisp model			
T (days)	t_1^*	R*	q*	TIC**
33	31.28	55.61	1123.54	14307.62
34	31.37	193.39	1194.96	14251.64
35	31.46	331.17	1266.38	14216.28
36	31.55	468.95	1337.80	14208.96
37	**31.64**	**606.73**	**1409.22**	**14207.46**
38	31.73	744.51	1480.64	14218.66
39	31.82	882.29	1552.06	14243.50
40	31.91	1020.07	1623.48	14277.98

7. Sensitivity analysis

In this section, sensitivity analysis is carried out to examine the impact of parameter value changes on the optimal values.To conduct these experiments, one parameter was modified by ±10% and ±20% at a time, while the other parameters were left at their initial values.The results of the sensitivity analysis are shown in Table 5 below.

Table 5 Cloudy Fuzzy Model Sensitivity Analysis

Parameters	Change %	Cycle Time (days) T*	Inventory Period (days) t_1^*	Shortage quantity R*	Order quantity A*	Average Total Cost TIC*	$\frac{(TIC^*-TIC_*)}{TIC_*}100\%$
p	+20	18.71	14.37	335.74	618.81	10536.57	-18.36
	+10	18.65	14.76	383.28	635.76	10416.36	-20.01
	-10	18.39	15.74	406.71	652.37	10308.71	-21.79
	-20	18.33	15.86	448.35	666.71	10216.73	-22.86
h	+20	19.23	15.16	383.57	693.58	10432.61	-19.53
	+10	19.19	15.44	403.82	694.39	10316.79	-20.20
	-10	19.47	16.21	453.36	698.81	10198.71	-22.46
	-20	19.43	16.56	476.89	701.16	10035.16	-23.31
q	+20	16.12	12.71	326.92	597.47	9736.00	-24.76
	+10	17.25	13.69	360.10	647.38	9854.39	-23.53
	-10	23.33	18.57	501.67	855.43	10516.47	-17.77
	-20	27.53	21.51	559.76	995.36	11009.58	-14.83
α	+20	...	...	No	Feasible	...	...
	+10	19.50	18.16	95.23	698.39	13266.71	+0.91
	-10	19.64	12.79	213.36	701.72	7735.46	-41.36
	-20	19.77	10.58	55.11	706.36	5866.73	-54.07
$D = mr^n$	+20	17.56	15.38	221.77	943.46	**15000.61**	+14.65
	+10	18.22	15.38	241.43	851.59	13124.67	+0.28
	-10	21.35	19.76	247.56	533.01	7456.43	-43.13
	-20	24.13	16.65	133.55	418.05	**5539.05**	-57.79
A	+20	22.35	15.86	335.93	718.28	10618.71	-18.76
	+10	21.31	15.79	368.21	687.29	10463.72	-19.54
	-10	18.01	15.64	471.78	593.91	10016.75	-23.37
	-20	16.79	15.57	77.74	543.17	9135.36	-24.81

Parameters	Change %	Cycle Time (days) T*	Inventory Period (days) t_1^*	Shortage quantity R*	Order quantity A*	Average Total Cost TIC*	$\frac{(TIC^* - TIC_*)}{TIC_*}100\%$
ϵ	+20	18.93	15.58	446.76	671.36	10118.91	-22.05
	+10	19.21	15.58	438.87	681.72	10148.63	-21.67
	-10	20.12	15.54	401.77	712.35	10280.25	-20.65
	-20	20.53	15.54	386.13	730.14	10348.72	-20.43
ρ	+20	...	...	No	Feasible	...	...
	+10	...	...	No	Feasible	...	...
	-10	20.76	15.79	381.72	746.56	10450.73	-19.63
	-20	21.53	15.79	358.26	776.18	10605.32	-18.57
σ	+20	21.86	18.63	358.41	788.56	10756.43	-18.36
	+10	20.97	15.84	385.65	755.93	10554.37	-19.84
	-10	...	...	No	Feasible	...	...
	-20	...	...	No	Feasible	...	...

Here are a few insights from the sensitivity analysis findings.

1. If the deterioration cost (p) increases, the total average inventory cost (TIC) rises increases due to the shortage quantity and order quantity r decrease at the constant rate. At the same time, both the cycle length and the product reliability increasing.

2. If the holding cost (h) increase, then total average inventory cost (TIC) increase at constant rate due to the shortage quantity, and order quantity decrease at constant rate. At the sametime, the cycle length decreases.

3. If the parameters q are increased, the total average inventory cost (TIC) decrease at constant rate due to the shortage quantity and order quantity rapidly decrease at constant rate. while the product's reliability also decrease.

4. If the deterioration rate (α) increases, the total average inventory cost (TIC) rises highly rapidly increases due to the shortage quantity Oscillate and order quantity minor decrease. At the same time, the cycle length decreased and the product reliability increasing.

5. If the parameters m and n are increased in demand rate, the total average inventory cost (TIC) rises quickly due to the shortage quantity oscillation and order quantity increase. At the same time, the cycle length decreased and the product reliability increasing.

6. If the ordering cost A are increased in demand rate, the total average inventory cost (TIC) smoothly increasing at constant rate due to the shortage quantity Oscillate and order quantity increase. At the same time, both the cycle length and the product reliability increasing.

8. Graphical Representations of the Model

We will create the cloudy fuzzy model's graphs in order to more clearly justify the recently proposed approach. Figure 4 illustrates that there is a significant disparity in the average inventory expenses of the crisp and fuzzy models in comparison to the cloudy fuzzy model.Furthermore, we've noticed that, everywhere, the fuzzy model yields the highest value of the objective function, while the cloudy fuzzy model yields the lowest value.Therefore, the inventory practitioner, especially the decision maker (DM) should choose the solution in a fuzzy, unclear the environment. The cloudy fuzzy objective function's "U-tern," which appears at 37 days of cycle time and is hence convex, is depicted in Figure 5. Figures 6, 7, and 8 illustrate that the back-order quantity curves for all three model scenarios intersect at around 38 days of cycle time, resulting in a shortage quantity of approximately 600 units of the model. For crisp and fuzzy models, the back-order curve is probably made up of overlapped lines with lesser gradients, while for hazy fuzzy models, it is a straight line that constantly gets greater gradient values.

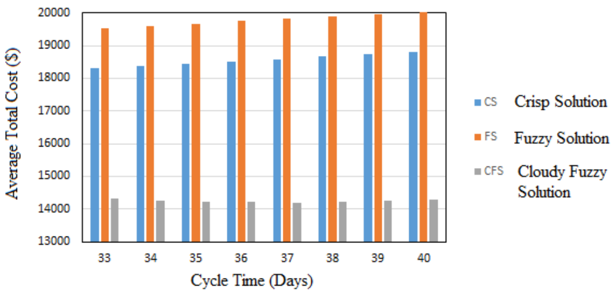

Figure 4: *Total Average cost and Cycle time for all three models*

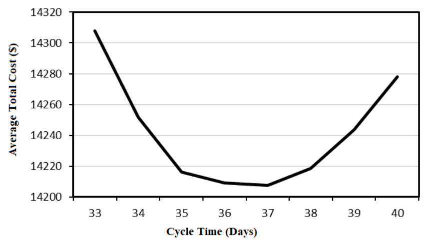

Figure 5: *Total cost variation over cloudy fuzzy model*

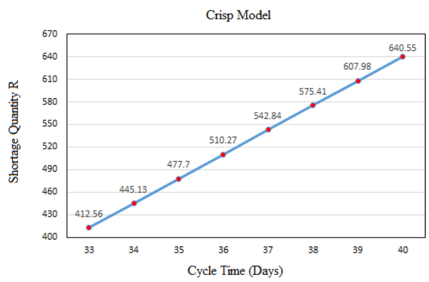

Figure 6: *Shortage Level of Crisp model*

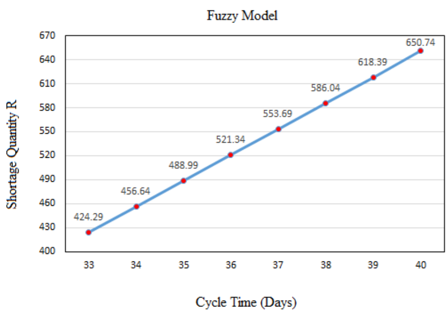

Figure 7: *Shortage Level of Fuzzy model*

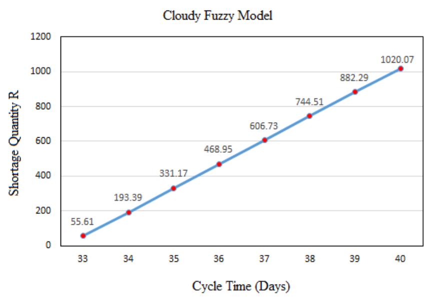

Figure 8: *Shortage Level of Cloudy Fuzzy model*

9. Conclusion

We have covered a back order EOQ model in a cloudy, fuzzy environment in this article. All inventory models are examined using crisp models, fuzzy models, intuitionistic fuzzy models, or fuzzy stochastic environments in the literature. However, the idea of "cloudy fuzzy" in decision-making scenarios is relatively recent. Nonetheless, we observe that the degree of fuzziness and the cloud index decrease as the cycle duration assumes greater values. Lower inventory costs do not equate to less fuzziness. Because, at its optimal, the fuzzy parameters may eventually begin to converge toward a crisp number with decreasing fuzziness. We propose to investigate the model in a cloudy fuzzy environment by introducing parametric flexibility because the crisp minimization issue yields higher values and is impractical in practice. Therefore, the average inventory cost is likely to converge with the cost derived just from the crisp model if we increase the cycle duration beyond its global optimum. Thus, we are searching the model minimum, which is the primary focus of the model, amid a significant degree of fuzziness.As a result, it is quite simple for any DM to comprehend and decide appropriately.

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