

Unbiased Exponential Type Estimators of Population Mean Using Auxiliary Variable as an Attribute In Double Sampling

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Abstract

In this paper, unbiased ratio-cum-product exponential type estimators for estimating the population mean have been introduced, specifically within the framework of a double sampling plan. The large sample properties of these estimators are investigated by deriving their bias and mean square error (MSE) expressions. The findings indicate that, under optimal conditions, the proposed estimators are not only unbiased but also more efficient than traditional methods, including the sample mean and the double sampling ratio and product type estimators developed by Naik and Gupta [11] and Singh et al. [17]. To further substantiate the theoretical results, we conducted a numerical study, which demonstrates the practical effectiveness of the proposed estimators in improving estimation accuracy.

Keywords: Exponential Estimator, Auxiliary Attribute, Proposed Estimator, Population, Optimum Value.

1. INTRODUCTION

In sampling theory, the precision of an estimate can be enhanced by incorporating information from an auxiliary variable, particularly when this variable is highly correlated with the study variable. The method of utilizing this auxiliary information depends on its form. For quantitative auxiliary information, estimators like Cochran's ratio estimator [4], Robson's product type estimator [12], Bahl and Tuteja's exponential type estimators [2], and other estimators developed by AL-Omari and Bouza [1], Shalabh and Tsai [15], Singh and Vishwakarma [16], Mehta et al. [10], and Hussain et al. [5, 6, 7] are commonly employed.

However, in many practical scenarios, the auxiliary information is qualitative in nature-meaning the auxiliary variable is an attribute correlated with the study variable. For example, Jhaji et al. [8] and Shabir & Gupta [17] illustrated cases such as (a) the height of an individual being dependent on their sex, and (b) crop yield depending on the variety of the crop. In such cases, traditional estimators that rely on quantitative auxiliary information are not suitable due to the point biserial correlation between the study and auxiliary variables. To address this, researchers like Naik and Gupta [11], Jhaji et al. [8], Singh et al. [17], Shabir & Gupta [17, 14], and Abd-Elfattah et al. [14] have developed ratio and product type estimators for the population mean that leverage prior knowledge of the parameters of the auxiliary attribute, thereby improving estimation accuracy in situations where the auxiliary information is qualitative. The proposed exponential estimators typically assume that the population proportion P is known beforehand. However, in real-world scenarios, researchers often encounter situations where this population proportion is not known in advance. To address this, the method of double sampling is employed.

In response to this challenge, Naik and Gupta [11] and Singh et al. [17] introduced double sampling ratio and product type estimators. However, these estimators are biased, meaning they can potentially under or overestimate the true population mean, which introduces inaccuracies in the estimation process.

Taking into account the above discussion, we propose unbiased ratio-cum-product exponential type estimators of the population mean, denoted as T_{de1} and T_{de2} , utilizing auxiliary information in the form of an attribute. Theoretical expressions for the Bias and Mean Square Error (MSE) up to the first order approximation have been derived for these proposed estimators. These results are then compared, both theoretically and empirically, with some existing double sampling ratio and product type estimators.

2. METHODOLOGY AND EXISTING ESTIMATORS

Use the Simple Random Sampling Without Replacement (SRSWOR) procedure to select a sample of size n from a population consisting of N units. In this context, each unit in the population is associated with two variables: Y_i , which represents the value of the study variable for the i^{th} unit, and ϕ_i , which represents the auxiliary attribute for that same unit ($i = 1, 2, \dots, N$). The auxiliary attribute ϕ is assumed to be dichotomous, meaning that it takes on only two possible values: 0 or 1. A value of 1 indicates the presence of the attribute, while a value of 0 indicates its absence. This complete dichotomy within the population allows for a clear classification of units into two distinct categories based on whether the attribute ϕ is present or absent. This classification is crucial for the estimation process, as it allows researchers to leverage the auxiliary information to improve the precision of estimates. For instance, if the attribute ϕ is highly correlated with the study variable Y , this information can be used to develop more efficient estimators of the population mean or other parameters of interest. The SRSWOR method ensures that every unit in the population has an equal chance of being selected, and no unit can be selected more than once, maintaining the randomness and representativeness of the sample. This procedure is fundamental in sampling theory, particularly when auxiliary information is available and can be used to refine estimates and reduce potential bias.

Consider $A = \sum_{i=1}^N \phi_i$ and $a = \sum_{i=1}^n \phi_i$, the number of units possessing the attribute ϕ in the population and sample respectively. Therefore $P = \frac{A}{N}$ and $p = \frac{a}{n}$ is the proportion of units possessing the given attribute ϕ in the population and sample respectively. When the value of P is unknown, the method of double sampling can be applied. This involves allocating a portion of the budget to gather information on an auxiliary variable. First, a large preliminary sample of size η is collected, with p_1 representing the proportion of units possessing the attribute ϕ in this sample. Then, a smaller, second-phase sample of size n is drawn, which is nested within the first-phase sample ($n < \eta$). In this second sample, p represents the proportion of units with the attribute ϕ , and \bar{y} is the mean of the study variable Y . Some formulas that have been used to compute various measures in this case are presented below as

Population Estimates	Sample Estimates
$\bar{Y} = \frac{1}{N} \sum_{i=1}^N X_i$, is the mean of study variable.	$\bar{y} = \frac{1}{n} \sum_{i=1}^n x_i$, is the mean of study variable.
$S_y^2 = \frac{1}{N-1} \sum_{i=1}^N (Y_i - \bar{Y})^2$, is the mean square of study variable.	$s_y^2 = \frac{1}{n-1} \sum_{i=1}^n (y_i - \bar{y})^2$, is the mean square of study variable.
$S_\phi^2 = \frac{1}{N-1} \sum_{i=1}^N (\phi_i - P)^2$, is the mean square of auxiliary variable.	$s_\phi^2 = \frac{1}{n-1} \sum_{i=1}^n (\phi_i - p)^2$, is the mean square of auxiliary variable.
$S_{y\phi} = \frac{1}{N-1} \sum_{i=1}^N (Y_i \phi_i - NP\bar{Y})^2$, is the covariance between study variable and auxiliary attribute.	$s_{y\phi} = \frac{1}{n-1} \sum_{i=1}^n (y_i \phi_i - np\bar{y})^2$, is the covariance between study variable and auxiliary attribute.
Further, $C_y = \frac{S_y}{\bar{Y}}$ and $C_p = \frac{S_\phi}{P}$, is the coefficient of variation of Y and ϕ respectively.	
$\rho_{pb} = \frac{S_{y\phi}}{S_y S_\phi}$, is the correlation between Y and ϕ .	

$\hat{\beta} = \frac{s_{y\phi}}{s_{\phi}^2}$, is the sample regression coefficient.

$\gamma = \frac{1-f}{n}, \gamma_1 = \left(\frac{1}{n} - \frac{1}{N}\right), \gamma_2 = \left(\frac{1}{n} - \frac{1}{n}\right), \gamma_3 = \gamma + \gamma_1$, where $f = \frac{n}{N}$ is the sampling fraction.

To obtain the Bias and MSE expressions when the auxiliary variable is an attribute in single sampling plan, consider

$$e_0 = \frac{\bar{y} - \bar{Y}}{\bar{Y}} \quad \text{and} \quad e_{\phi} = \frac{p - P}{P}.$$

The expected values of quantities $e_0, e_{\phi}, e_0^2, e_{\phi}^2$, and e_0e_{ϕ} are obtained as

$$E(e_0) = E(e_{\phi}) = 0, \quad E(e_0^2) = \gamma C_y^2, \quad E(e_{\phi}^2) = \gamma C_p^2, \quad E(e_0e_{\phi}) = \gamma C_{yp}.$$

To obtain the Bias and MSE expressions when the auxiliary variable is an attribute in double sampling plan, consider

$$e_0 = \bar{Y}^{-1}(\bar{y} - \bar{Y}), \quad e_{\phi} = P^{-1}(p - P), \quad e'_{\phi} = P^{-1}(p_1 - P).$$

Therefore, the following expected values are obtained as

$$E(e_0) = E(e_{\phi}) = E(e'_{\phi}) = 0, \quad E(e_0^2) = \gamma C_y^2, \quad E(e_{\phi}^2) = \gamma C_p^2.$$

$$E(e_0^2) = \gamma_1 C_p^2, \quad E(e_0e_{\phi}) = \gamma C_{yp}, \quad E(e_0e'_{\phi}) = \gamma_1 C_{yp}, \quad E(e_{\phi}e'_{\phi}) = \gamma_1 C_p^2.$$

When auxiliary information is unavailable, the sample mean serves as a reliable estimator for the population mean. The sample mean, denoted by \bar{y} , is computed by averaging the observations in a sample and provides a practical approach to estimating the population mean

$$t_1 = \bar{y} = \frac{1}{n} \sum_{i=1}^n y_i.$$

The estimator t_1 is unbiased and its variance is as

$$V(t_1) = \gamma \bar{Y}^2 C_y^2. \tag{1}$$

Naik and Gupta [11] were the pioneers in proposing double sampling ratio and product type estimators for the population mean, specifically tailored for scenarios where the auxiliary variable is presented as an attribute. Their groundbreaking work introduced a method to effectively utilize categorical or binary auxiliary data to improve the accuracy of population mean estimates

$$T_{ngr} = \bar{y} \left(\frac{p_1}{p} \right).$$

$$T_{ngp} = \bar{y} \left(\frac{p}{p_1} \right).$$

The Bias and MSE of the estimator T_{ngr} and T_{ngp} is as

$$\text{Bias}(T_{ngr}) = \bar{Y}(\gamma - \gamma_1)(C_p^2 - C_{yp}).$$

$$\text{MSE}(T_{ngr}) = \bar{Y}^2(\gamma C_y^2 + \gamma_2 C_p^2 - 2\gamma_2 C_{yp}). \tag{2}$$

$$\text{Bias}(T_{ngp}) = \bar{Y}(\gamma - \gamma_1)C_{yp}.$$

$$\text{MSE}(T_{ngp}) = \bar{Y}^2(\gamma C_y^2 + \gamma_2 C_p^2 + 2\gamma_2 C_{yp}). \tag{3}$$

Later, Singh et al. [17] introduced a double sampling exponential ratio estimator for the population mean, tailored for situations where the auxiliary information is given as an attribute. This approach enhances estimation accuracy by effectively incorporating categorical or binary auxiliary data

$$T_{sgr} = \bar{y} \exp \left(\frac{p_1 - p}{p_1 + p} \right).$$

$$T_{sgp} = \bar{y} \exp \left(\frac{p - p_1}{p_1 + p} \right).$$

The Bias and MSE of estimators T_{sgr} and T_{sgp} is as

$$\begin{aligned} Bias(T_{sgr}) &= \frac{1}{2}\gamma_2\bar{Y}\left(\frac{1}{2}C_p^2 - C_{yp}\right). \\ MSE(T_{sgr}) &= \bar{Y}^2\left(\gamma C_y^2 + \frac{1}{4}\gamma_2 C_p^2 - \gamma_2 C_{yp}\right). \end{aligned} \tag{4}$$

$$\begin{aligned} Bias(T_{sgp}) &= \frac{1}{2}\gamma_2\bar{Y}\left(\frac{1}{2}C_p^2 + C_{yp}\right). \\ MSE(T_{sgp}) &= \bar{Y}^2\left(\gamma C_y^2 + \frac{1}{4}\gamma_2 C_p^2 + \gamma_2 C_{yp}\right). \end{aligned} \tag{5}$$

3. PROPOSED ESTIMATORS IN DOUBLE SAMPLING

The proposed proportion based unbiased exponential ratio-cum-product estimators in double sampling plan are as

$$T_{rp1} = \bar{y} \left[\omega_1 \exp\left(\frac{p_1 - p}{lp_1}\right) + (1 - \omega_1) \exp\left(\frac{p - p_1}{lp_1}\right) \right]. \tag{6}$$

$$T_{rp2} = \bar{y} \left[\omega_2 \exp\left(\frac{p_1 - p}{gp}\right) + (1 - \omega_2) \exp\left(\frac{p - p_1}{gp}\right) \right]. \tag{7}$$

Where $l(\neq 0)$, $g(\neq 0)$, ω_1 and ω_2 are constants. The values of l and g are chosen such that the proposed estimators are unbiased and the values of ω_1 & ω_2 are chosen such that $MSE(T_{rpi})$ ($i = 1, 2$) is minimum.

Theorem 4.1: The Bias and MSE expressions for the double sampling ratio-cum-product type estimators T_{rp1} and T_{rp2} to the first order of approximation are as

$$Bias(T_{rp1}) = \frac{\gamma_2\bar{Y}}{l} \left[C_{yp} - 2\omega_1 C_{yp} + \frac{C_p^2}{2l} \right]. \tag{8}$$

$$Bias(T_{rp2}) = \frac{\gamma_2\bar{Y}}{g} \left[(2\omega_2 - 1)(C_p^2 - C_{yp}) - \frac{C_p^2}{2g} \right]. \tag{9}$$

and

$$MSE(T_{rp1}) = \bar{Y}^2 \left[\gamma C_y^2 + \frac{(2\omega_1 - 1)^2}{l^2} \gamma_2 C_p^2 - \frac{2(2\omega_1 - 1)}{l} \gamma_2 C_{yp} \right]. \tag{10}$$

$$MSE(T_{rp2}) = \bar{Y}^2 \left[\gamma C_y^2 + \frac{(2\omega_2 - 1)^2}{g^2} \gamma_2 C_p^2 - \frac{2(2\omega_2 - 1)}{g} \gamma_2 C_{yp} \right]. \tag{11}$$

Proof: Writing the estimator T_{rp1} and T_{rp2} i.e. (6) and (7) in terms of $e^{l's}$ the expression obtained are as

$$\begin{aligned} T_{rp1} &= \bar{Y}(1 + e_0) \left[\omega_1 \exp\left(\frac{P(1 + e'_\phi) - P(1 + e_\phi)}{lP(1 + e'_\phi)}\right) \right. \\ &\quad \left. + (1 - \omega_1) \exp\left(\frac{P(1 + e_\phi) - P(1 + e'_\phi)}{lP(1 + e'_\phi)}\right) \right] \\ \Rightarrow T_{rp1} &= \bar{Y}(1 + e_0) \left[\omega_1 \exp\left(\frac{e'_\phi - e'^2_\phi - e_\phi + e_\phi e'_\phi + \dots}{l}\right) \right. \\ &\quad \left. + (1 - \omega_1) \exp\left(\frac{e_\phi + e'^2_\phi - e'_\phi - e_\phi e'_\phi + \dots}{l}\right) \right] \end{aligned} \tag{12}$$

$$\begin{aligned}
 T_{rp2} &= \bar{Y}(1 + e_0) \left[\omega_2 \exp \left(\frac{P(1 + e'_\phi) - P(1 + e_\phi)}{gP(1 + e_\phi)} \right) \right. \\
 &\quad \left. + (1 - \omega_2) \exp \left(\frac{P(1 + e_\phi) - P(1 + e'_\phi)}{gP(1 + e_\phi)} \right) \right] \\
 \Rightarrow T_{rp2} &= \bar{Y}(1 + e_0) \left[\omega_2 \exp \left(\frac{e'_\phi + e_\phi^2 - e_\phi - e_\phi e'_\phi + \dots}{g} \right) \right. \\
 &\quad \left. + (1 - \omega_2) \exp \left(\frac{e_\phi - e_\phi^2 - e'_\phi + e_\phi e'_\phi + \dots}{g} \right) \right] \tag{13}
 \end{aligned}$$

Solving (12) & (13) and retaining the terms up to second degree only, we have

$$\begin{aligned}
 T_{rp1} &= \bar{Y}(1 + e_0) \left[1 + \frac{2\omega_1(e'_\phi - e_\phi^2 - e_\phi + e_\phi e'_\phi)}{l} + \frac{e_\phi^2 + e_\phi'^2 - 2e'_\phi e_\phi}{2l^2} \right. \\
 &\quad \left. + \frac{e_\phi + e_\phi'^2 - e'_\phi - e_\phi e'_\phi}{l} \right] \tag{14}
 \end{aligned}$$

$$\begin{aligned}
 T_{rp2} &= \bar{Y}(1 + e_0) \left[1 + \frac{2\omega_2(e'_\phi + e_\phi^2 - e_\phi - e_\phi e'_\phi)}{g} + \frac{e_\phi^2 + e_\phi'^2 - 2e'_\phi e_\phi}{2g^2} \right. \\
 &\quad \left. + \frac{e_\phi - e_\phi^2 - e'_\phi + e_\phi e'_\phi}{g} \right] \tag{15}
 \end{aligned}$$

Further solving (14) & (15) and excluding the terms of degree higher than two, we have

$$\begin{aligned}
 T_{rp1} - \bar{Y} &= \left[e_0 + \frac{2\omega_1(e'_\phi - e_\phi^2 - e_\phi + e_\phi e'_\phi + e'_\phi e_0 - e_0 e_\phi)}{l} + \frac{e_\phi^2 + e_\phi'^2 - 2e'_\phi e_\phi}{2l^2} \right. \\
 &\quad \left. + \frac{e_\phi + e_\phi'^2 - e'_\phi - e_\phi e'_\phi + e_0 e_\phi - e_0 e'_\phi}{l} \right] \tag{16}
 \end{aligned}$$

$$\begin{aligned}
 T_{rp2} - \bar{Y} &= \left[e_0 + \frac{2\omega_2(e'_\phi + e_\phi^2 - e_\phi - e_\phi e'_\phi + e'_\phi e_0 - e_0 e_\phi)}{g} + \frac{e_\phi^2 + e_\phi'^2 - 2e'_\phi e_\phi}{2g^2} \right. \\
 &\quad \left. + \frac{e_\phi - e_\phi^2 - e'_\phi + e_\phi e'_\phi + e_0 e_\phi - e_0 e'_\phi}{g} \right] \tag{17}
 \end{aligned}$$

Taking expectation on both sides of equations (16) and (17), the Bias of the estimator T_{rp1} and T_{rp2} is obtained as

$$\text{Bias}(T_{rp1}) = \frac{\gamma_2 \bar{Y}}{l} \left[C_{yp} - 2\omega_1 C_{yp} + \frac{C_p^2}{2l} \right]. \tag{18}$$

$$\text{Bias}(T_{rp2}) = \frac{\gamma_2 \bar{Y}}{g} \left[(2\omega_2 - 1)(C_p^2 - C_{yp}) - \frac{C_p^2}{2g} \right]. \tag{19}$$

Squaring equations (16) and (17) on both sides and then taking expectation, the mean square error of the proposed estimators is obtained as

$$\text{MSE}(T_{rp1}) = \bar{Y}^2 \left[\gamma C_y^2 + \frac{(2\omega_1 - 1)^2}{\omega_1^2} \gamma_2 C_p^2 - \frac{2(2\omega_1 - 1)}{l} \gamma_2 C_{yp} \right]. \tag{20}$$

$$\text{MSE}(T_{rp2}) = \bar{Y}^2 \left[\gamma C_y^2 + \frac{(2\omega_2 - 1)^2}{g^2} \gamma_2 C_p^2 - \frac{2(2\omega_2 - 1)}{g} \gamma_2 C_{yp} \right]. \tag{21}$$

3.0.1 Unbiased Condition

The double sampling exponential estimators T_{rp1} and T_{rp2} are found unbiased, if

$$l = \frac{C_p}{2(2\omega_1 - 1)\rho_{pb}C_y} \quad \text{and} \quad g = \frac{C_p}{2(2\omega_2 - 1)(\rho_{pb}C_y - C_p)} \quad \text{respectively.}$$

Using the values of l and g in equations (20) and (21) respectively, the expressions obtained are as follows

$$MSE(T_{rp1}) = \bar{Y}^2 \left[\gamma C_y^2 + 4\gamma_3(1 - 2\omega_1)^4 \rho_{pb}^2 C_y^2 - 4\gamma_3(1 - 2\omega_1)^2 \rho_{pb}^2 C_y^2 \right]. \quad (22)$$

$$MSE(T_{rp2}) = \bar{Y}^2 \left[\gamma C_y^2 + 4\gamma_2(1 - 2\omega_2)^4 (\rho_{pb}C_y - C_p)^2 - 4\gamma_2(1 - 2\omega_2)^2 (\rho_{pb}C_y - C_p)\rho_{pb}C_y \right]. \quad (23)$$

3.0.2 Optimum Value of ω_1 and ω_2

Using the differentiation procedure for obtaining the optimal value of ω_1 and ω_2 from equations (22) and (23), we have

$$\omega_{1(opt)} = 0.146, 0.854. \quad (24)$$

$$\omega_{2(opt)} = \frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{\rho_{pb}C_y}{2(\rho_{pb}C_y - C_p)}}. \quad (25)$$

Now substituting the values of $\omega_{1(opt)}$ and $\omega_{2(opt)}$ obtained as equations (24) and (25) in equations (22) and (23) respectively, the minimum value of MSE up to $O(n^{-1})$ for the estimators T_{rpi} ($i = 1, 2$) is obtained as

$$MSE_{min}(T_{rpi}) = \gamma \bar{Y}^2 C_y^2 (1 - \rho_{pb}^2) + \gamma_1 \bar{Y}^2 \rho_{pb}^2 C_y^2. \quad ; i = 1, 2. \quad (26)$$

4. EFFICIENCY COMPARISON

For comparing the MSE of the proposed double sampling estimators with some existing ratio and product type estimators such as mean per unit estimator (t_m), double sampling ratio and product estimator (T_{ngr} & T_{ngp}) of Naik and Gupta [11], double sampling exponential ratio and product estimators (T_{sgr} & T_{sgp}) of Singh *et al.* [17], we first write the expressions of their mean squared error in double sampling up to the first order of approximation

$$Bias(t_m) = 0.$$

$$MSE(t_m) = \gamma \bar{Y}^2 C_y^2. \quad (27)$$

$$Bias(T_{ngr}) = \bar{Y}(\gamma - \gamma_1)(C_p^2 - C_{yp}).$$

$$MSE(T_{ngr}) = \bar{Y}^2(\gamma C_y^2 + \gamma_2 C_p^2 - 2\gamma_2 C_{yp}). \quad (28)$$

$$Bias(T_{ngp}) = \bar{Y}(\gamma - \gamma_1)C_{yp}.$$

$$MSE(T_{ngp}) = \bar{Y}^2(\gamma C_y^2 + \gamma_2 C_p^2 + 2\gamma_2 C_{yp}). \quad (29)$$

$$Bias(T_{sgr}) = \frac{1}{2}\gamma_2 \bar{Y} \left(\frac{1}{2}C_p^2 - C_{yp} \right).$$

$$MSE(T_{sgr}) = \bar{Y}^2 \left(\gamma C_y^2 + \frac{1}{4}\gamma_2 C_p^2 - \gamma_2 C_{yp} \right). \quad (30)$$

$$Bias(T_{sgp}) = \frac{1}{2}\gamma_2 \bar{Y} \left(\frac{1}{2}C_p^2 + C_{yp} \right).$$

$$MSE(T_{sgp}) = \bar{Y}^2 \left(\gamma C_y^2 + \frac{1}{4} \gamma_2 C_p^2 + \gamma_2 C_{yp} \right). \tag{31}$$

The efficiency comparison of the proposed double sampling ratio-cum-product exponential estimators T_{rp1} and T_{rp2} with the other existing estimators is done as

From equations (27), (28), (29), (30) and (31), we get the following conditions under which T_{rp1} and T_{rp2} are more efficient than the estimators t_m , T_{ngp} , T_{ngr} , T_{sgp} and T_{sgr} as

$$\begin{aligned} &MSE_{min}(T_{rpi}) < V(t_m) \\ \Rightarrow &\gamma \bar{Y}^2 C_y^2 (1 - \rho_{pb}^2) + \gamma_1 \bar{Y}^2 \rho_{pb}^2 C_y^2 < \gamma \bar{Y}^2 C_y^2, \text{ if} \\ &\gamma_2 \rho_{pb}^2 \bar{Y}^2 > 0. \end{aligned} \tag{32}$$

$$\begin{aligned} &MSE_{min}(T_{rpi}) < MSE(T_{ngr}) \\ \Rightarrow &\gamma \bar{Y}^2 C_y^2 (1 - \rho_{pb}^2) + \gamma_1 \bar{Y}^2 \rho_{pb}^2 C_y^2 < \bar{Y}^2 (\gamma C_y^2 + \gamma_2 C_p^2 - 2\gamma_2 C_{yp}), \text{ if} \\ &\gamma_2 (C_p - \rho_{pb} C_y)^2 > 0. \end{aligned} \tag{33}$$

$$\begin{aligned} &MSE_{min}(T_{rpi}) < MSE(T_{ngp}) \\ \Rightarrow &\gamma \bar{Y}^2 C_y^2 (1 - \rho_{pb}^2) + \gamma_1 \bar{Y}^2 \rho_{pb}^2 C_y^2 < \bar{Y}^2 (\gamma C_y^2 + \gamma_2 C_p^2 + 2\gamma_2 C_{yp}), \text{ if} \\ &\gamma_2 (C_p + \rho_{pb} C_y)^2 > 0. \end{aligned} \tag{34}$$

$$\begin{aligned} &MSE_{min}(T_{rpi}) < MSE(T_{sgr}) \\ \Rightarrow &\gamma \bar{Y}^2 C_y^2 (1 - \rho_{pb}^2) + \gamma_1 \bar{Y}^2 \rho_{pb}^2 C_y^2 < \bar{Y}^2 \left(\gamma C_y^2 + \frac{1}{4} \gamma_2 C_p^2 - \gamma_2 C_{yp} \right), \text{ if} \\ &\gamma_2 (C_p - 2\rho_{pb} C_y)^2 > 0. \end{aligned} \tag{35}$$

$$\begin{aligned} &MSE_{min}(T_{rpi}) < MSE(T_{sgp}) \\ \Rightarrow &\gamma \bar{Y}^2 C_y^2 (1 - \rho_{pb}^2) + \gamma_1 \bar{Y}^2 \rho_{pb}^2 C_y^2 < \bar{Y}^2 \left(\gamma C_y^2 + \frac{1}{4} \gamma_2 C_p^2 + \gamma_2 C_{yp} \right), \text{ if} \\ &\gamma_2 (C_p + 2\rho_{pb} C_y)^2 > 0. \end{aligned} \tag{36}$$

Which is true in all the cases, therefore the proposed estimators are theoretically efficient .

5. NUMERICAL ILLUSTRATION

The populations P1 and P2 have been considered for comparing the efficiency of the proposed ratio-cum-product double sampling exponential estimators T_{de1} and T_{de2} with the existing estimators. The population P1 is from Sukhatme and Sukhatme [18] where the study variable (Y) is the number of villages in the circles and the auxiliary information (ϕ) is a circle consisting of more than five villages. The population P2 is from Mukhopadhyay [9], in which the study variable (Y) is the household size and the auxiliary attribute (ϕ) is a household that availed an agricultural loan from a bank. A first phase sample of size 45 is drawn from the population P1 and a second phase sample which is nested within the first phase sample of size 23 is drawn. Further, a first phase sample of size 13 is drawn from the population P2 and a second phase sample which is nested within the first phase sample of size 7 is drawn.

Table 1: Characteristics of population data sets

Population	N	η	n	\bar{Y}	p_1	ρ_{pb}	C_y	C_p	$\beta_2(\phi)$	C_{yp}
P1	89	45	23	2.911	0.067	0.586	0.542	3.782	11.433	1.201
P2	25	13	7	9.462	0.385	- 0.396	0.497	1.317	- 2.056	- 0.259

The description of the population given in Table-1 shows that the correlation between study and auxiliary variable for the population P1 is positive, for population P2 is negative.

Table 2: MSE and Bias of various estimators

Estimator	Population	
	P1	
	MSE	Bias
t_m (Sample mean)	0.080	0.000
T_{ngr} (Naik and Gupta, 1996)	2.224	0.811
T_{sgr} (Singh <i>et al.</i> , 2007)	0.508	0.184
T_{rpi} (Proposed)	0.062	0.000

The data of Table-2 clearly shows that the estimators T_{rpi} have the lowest MSE as compared to the other existing estimators t_m , T_{ngr} and T_{sgr} and are also unbiased.

Table 3: PRE of T_{rpi} with respect to the estimators t_m , T_{ngr} and T_{sgr} for P6

Estimator	Population - P1		
	Percent Relative Efficiency w.r.t		
	t_m	T_{ngr}	T_{sgr}
t_m (Sample mean)	100.000	2780.000	635.000
T_{ngr} (Naik and Gupta, 1996)	3.597	100.000	22.842
T_{sgr} (Singh <i>et al.</i> , 2007)	15.748	437.795	100.000
T_{rpi} (Proposed)	129.032	3587.097	819.355

Perusal of Table-3 shows that the PRE of the proposed estimators T_{rpi} for population P1 with respect to t_m , T_{ngr} and T_{sgr} is 129.032, 3587.097 and 819.355 respectively. The highest PRE value is found with respect to the exponential ratio estimator T_{ngr} followed by the PRE with respect to the exponential ratio type estimator T_{sgr} and the sample mean estimator t_m .

Table 4: MSE and Bias of t_m , T_{ngp} , T_{sgp} and T_{rpi}

Estimator	Population	
	P2	
	MSE	Bias
t_m (Sample mean)	2.362	0.000
T_{ngp} (Naik and Gupta, 1996)	9.816	0.162
T_{sgp} (Singh <i>et al.</i> , 2007)	3.431	0.190
T_{rpi} (Proposed)	2.125	0.000

The data of Table-4 clearly shows that the estimators T_{rpi} have the lowest MSE as compared to the other estimators t_m , T_{ngp} and T_{sgp} and are also unbiased.

Table 5: PRE of T_{rpi} with respect to the estimators t_m , T_{ngp} and T_{sgp} for P2

Estimator	Population - P2		
	Percent Relative Efficiency w.r.t		
	t_m	T_{ngp}	T_{sgp}
t_m (Sample mean)	100.000	415.580	145.258
T_{ngp} (Naik and Gupta, 1996)	24.063	100.000	34.953
T_{sgp} (Singh <i>et al.</i> , 2007)	68.843	286.097	100.000
T_{rpi} (Proposed)	111.153	461.929	161.459

Perusal of Table-5 shows that the PRE of the proposed double sampling product type estimators T_{rpi} for population P2 with respect to t_m , T_{ngp} and T_{sgp} is 111.153, 461.929 and 161.459 respectively. The highest PRE value is found with respect to the exponential product estimator T_{ngp} followed by the PRE with respect to the exponential product type estimator T_{sgp} and the sample mean estimator t_m .

It can be observed from Table-3 and Table-5 that the proposed double sampling ratio estimators T_{rpi} have the highest PRE with respect to the existing estimators t_m , T_{ngr} , T_{ngp} , T_{sgp} and T_{sgr} for both the populations P1 and P2 which concludes that the proposed estimators are efficient.

6. CONCLUSION

In this paper, we have introduced two unbiased ratio-cum-product exponential estimators of the population mean, denoted as T_{rp1} and T_{rp2} , within the framework of double sampling. The large sample properties of these estimators have been derived to the first order of approximation and theoretically compared with existing ratio and product type estimators. Through numerical evaluation across various datasets, our comparison demonstrates that the proposed estimators consistently outperform existing ones in terms of efficiency while maintaining unbiasedness. These findings underscore the superior performance of the proposed estimators, making them a valuable contribution to the field.

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