LEHMANN TYPE-II PERK DISTRIBUTION: PROPERTIES AND APPLICATIONS

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Abstract

The Lehmann type-II Perk distribution is a flexible statistical model with a wide range of applications in fields such as reliability analysis, survival modeling, and data fitting. This distribution is notable for its distinct properties, including specific patterns in hazard rates and implications for stochastic ordering. Estimating the distribution parameters is essential for effective model fitting and making inferences. The parameters are estimated using the maximum likelihood estimation method, and confidence intervals are determined using normal approximation. To evaluate the performance of these estimation methods, Monte-Carlo simulation studies are conducted, demonstrating their accuracy and efficiency. The Lehmann type-II Perk distribution provides a robust framework for analyzing complex data sets and deriving reliable statistical conclusions.

Keywords: Lehmann type-II Perk distribution, survival function, hazard function, maximumlikelihood estimates, confidence length, Monte-Carlo simulation.

1. Introduction

Researchers in the scientific community have widely embraced Lehmann's [6] type-I (L-I) and type-II (L-II) lifetime models as straightforward and practical tools. Lehmann's type-I (L-I) model is commonly associated with the Power function (PF) distribution, which has garnered significant attention for its simplicity and utility. Additionally, Gupta et al. [8] applied the L-I model to the exponential distribution. The appeal of the PF distribution's simplicity and versatility has led researchers to explore its various applications, extensions, and generalizations across different scientific domains. Furthermore, Cordeiro and De Castro [9] introduced the Lehmann Type-II (L-II) G class through a dual transformation. Cordeiro et al. [5] developed the Lehman type II distribution as a hybrid of the generalized exponentiated distribution. This distribution's closed-form characteristics facilitate the derivation and examination of numerous properties. Researchers have extensively employed both the L-I and L-II approaches in the literature to investigate novel features of classical and modified models. In recent studies, Arshad et al. [12] delved into the development of the L-II G family, focusing on a bathtub-shaped failure rate model and its application using engineering data. Balogun et al. [14] introduced a potentiated lifetime model exhibiting a bathtub-shaped hazard rate function, termed the Kumaraswamy modified size-biased Lehmann Type-II (Kum–MSBL–II) distribution. Ogunde et al. [1] defined and studied a new generalization of the Frechet distribution, known as the Lehmann Type II Frechet Poisson distribution. Tomazella et al. [23] explored various mathematical properties of the L-II Frechet distribution and its application to aircraft maintenance data, while Awodutire et al. [16] discussed multiple statistical measures of the L-II generalized half logistic distribution, examining its application in sports data. In a different context, Badmus et al. [13] investigated the weighted Weibull distribution via the L-II approach and its application in textile engineering data. Meanwhile, Ogunde et al. [2] extended the Gumbel type-II distribution using the exponentiated L-II G class and applied it to biology data.

The Perks distribution, originally proposed by Perks in [15], is a four-parameter model that also serves as an extension of the Gompertz-Makeham distribution. Researchers in the field have actively explored modifications and generalizations of this model within the literature. Richards, in both [18] and [19], introduced parametric survival models based on the Perks I distribution. Haberman and Renshaw, in [10], conducted a study focused on parametric mortality projection using the Perks distribution as a foundation. Chaudhary and Kumar, in [4], delved into Bayesian analysis of the Perks distribution employing Markov Chain Monte Carlo techniques. In [24], Zeng et al. examined both four and five-parameter Perks mortality equations to model bathtub-shaped failure rates. In another extension of the Perks distribution, Singh and Choudhary [21] introduced the exponentiated Perks distribution, while Chaudhary [3] presented the Perks-II distribution. Most recently, Gonzalez et al. [7] conducted a study on the additive Perks distribution and explored its application in the realm of reliability analysis.

The distribution function of the Lehmann type-II is given by

$$F(x) = 1 - (1 - G(x))^{\alpha}, x > 0, \alpha > 0. \tag{1}$$

The distribtion function of the perk II is given by

$$G(x) = 1 - \left(\frac{1+\alpha}{1+\alpha e^{\beta x}}\right)^{\frac{1}{\beta}}, x > 0, \alpha, \beta > 0. \tag{2}$$

$$F_{LPM}(x) = 1 - \left(\frac{1+\alpha}{1+\alpha e^{\beta x}}\right)^{\frac{\gamma}{\beta}}, x > 0, \alpha, \beta, \gamma > 0. \tag{3}$$

which is a three parameter Lehmann type-II Perk distribution. The corresponding density is given by

$$f_{LPM}(x) = \alpha\gamma\,(1+\alpha)^{\frac{\gamma}{\beta}}\, e^{\beta x}\left[\frac{1}{1+\alpha e^{\beta x}}\right]^{1+\frac{\gamma}{\beta}}, x > 0, \alpha, \beta, \gamma > 0. \tag{4}$$

and $f_{LPM}(x) = 0$ otherwise. A plots of the density function is shown in Figure 1.

2. Properties

This section studies various properties such as the survival and hazard function, quantile function.

2.1. Survival and hazard function

The survival function of the density given in Eq. (5) as

$$S_{LPM}(x) = \left(\frac{1+\alpha}{1+\alpha e^{\beta x}}\right)^{\frac{\gamma}{\beta}} \tag{5}$$

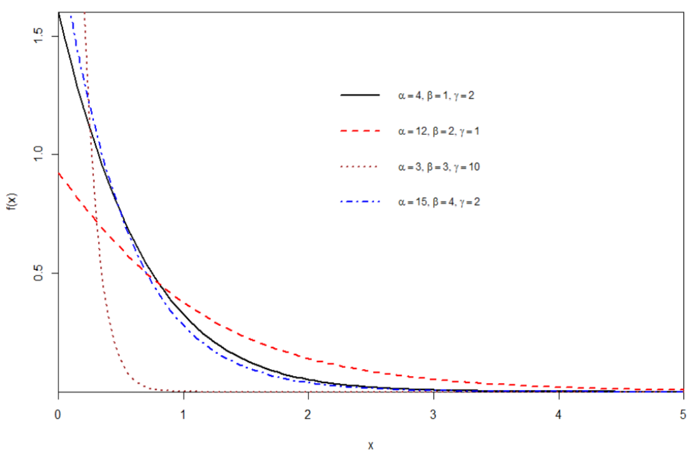


Figure 1: *Few shapes of Lehmann II Perks distribution.*

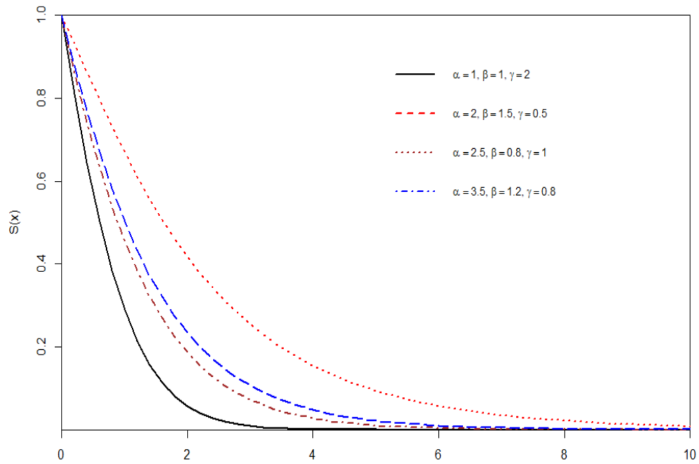


Figure 2: *Survival shapes of Lehmann II Perks distribution*

and the hazard function as $h_{LPM}(x) = \frac{f_{LPM}(x)}{S_{LPM}(x)}$, and hence

$$h_{LPM}(x) = \alpha\gamma \left[\frac{e^{\beta x}}{1 + \alpha e^{\beta x}} \right] \tag{6}$$

The different shapes of the survival and hazard function are given in the Figure 2 and 3.

2.2. The Quantile function

The quantile function $Q(u)$ is defines

$$Q(u) = F^{-1}(u) = \inf\{x : F(x) \geq u\}, \quad 0 \leq u \leq 1 \tag{7}$$

The cumulative distribution function $F(x)$ is defined as $F(x) = 1 - S(x)$. Let's consider $F(x) = u$, resulting in the following expression for x:

$$x = \frac{1}{\beta} \ln \frac{1}{\alpha} \left[\frac{1 + \alpha}{(1 - u)^{\frac{\beta}{\gamma}}} - 1 \right] \tag{8}$$

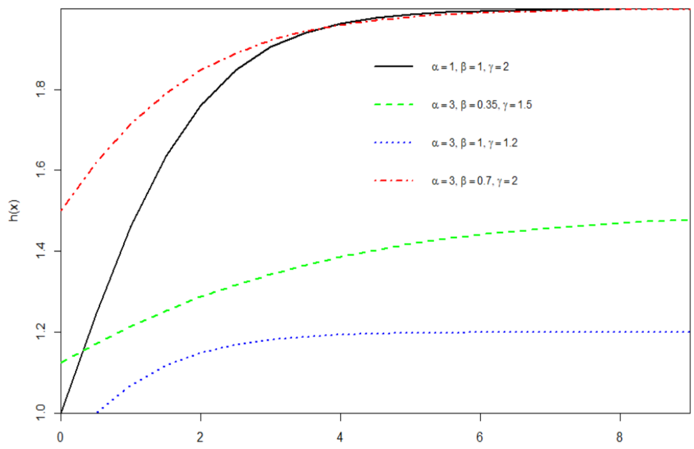


Figure 3: *Hazard shapes of Lehmann II Perks distribution*

2.3. Likelihood ratio ordering

For two random variables X_1 and X_2 where X_1 is larger than X_2 in likelihood ratio ordering, denoted by $X_1 \geq_{lr} X_2$, if $\frac{f_{X_1}(x)}{f_{X_2}(x)}$ is an increasing function in x, where $f_{X_1}(x)$ and $f_{X_2}(x)$ are density function of X_1 and X_2 respectively. Consider that X_1 and X_2 are independent random variables following Lehmann type-II Perks distribution with parameters $(\alpha, \beta, \gamma_1)$ and $(\alpha, \beta, \gamma_2)$ respectively and for $\gamma_1 \leq \gamma_2$. Then

$$\frac{f_{X_1}(x)}{f_{X_2}(x)} = \frac{\gamma_1}{\gamma_2} (1+\alpha)^{\frac{\gamma_1-\gamma_2}{\beta}} \left(1+\alpha e^{\beta x}\right)^{\frac{\gamma_2-\gamma_1}{\beta}}. \tag{9}$$

Further we can have,

$$\frac{\partial}{\partial z}\frac{f_{X_1}(x)}{f_{X_2}(x)} = \alpha\beta e^{\beta x}\frac{\gamma_1}{\gamma_2}\frac{(\gamma_2-\gamma_1)}{\beta} (1+\alpha)^{\frac{\gamma_1-\gamma_2}{\beta}} \left(1+\alpha e^{\beta x}\right)^{\frac{\gamma_2-\gamma_1}{\beta}-1} \tag{10}$$

For $\gamma_1 \leq \gamma_2$, $\frac{f_{X_1}(x)}{f_{X_2}(x)}$ is increasing and hence $X_1 \geq_{lr} X_2$. Shaked and Shanthikumar [20] suggests that likelihood ratio ordering implies hazard rate ordering and which implies stochastic ordering. Thus for $\gamma_1 \leq \gamma_2$, we have $X_1 \geq_{lr} X_2 \Rightarrow X_1 \geq_{hr} X_2 \Rightarrow X_1 \geq_{st} X_2$. This helps to understand one random variable being "bigger" than the other, or in our case when $\gamma_1 \leq \gamma_2$, X_1 is "bigger" than X_2.

2.4. Stress-strength reliability

Suppose X_1 and X_2 are independent random variables following Lehmann type-II Perks distribution with parameters $(\alpha, \beta, \gamma_1)$ and $(\alpha, \beta, \gamma_2)$ respectively. Let $f(x)$ be the density function of X_1 and $g(x)$ be the density function of X_2. Then the stress-strength reliability is

$$\begin{aligned} R &= \Pr\{X_1 > X_2\} = \int_0^\infty f(x_1)\left(\int_0^{x_1} g(x_2)dx_2\right)dx_1 \\ &= 1-\alpha\gamma_1(1+\alpha)^{\frac{\gamma_1+\gamma_2}{\beta}} \int_0^\infty e^{\beta x_1}\left[\frac{1}{1+\alpha e^{\beta x_1}}\right]^{1+\left(\frac{\gamma_1+\gamma_2}{\beta}\right)} dx_1 \\ &= \frac{\gamma_2}{\gamma_1+\gamma_2} \end{aligned} \tag{11}$$

3. Maximum Likelihood Estimation

Let $x_1, x_2, ..., x_n$ be a simple random sample from the density in Eq. (4), then the log-likelihood function is denoted by $L(\alpha, \beta, \gamma)$ is given by

$$\begin{aligned} L(\alpha,\beta,\gamma) &= n\ln\alpha + n\ln\gamma + \frac{n\gamma}{\beta}\ln(1+\alpha) \\ &+ \beta\sum_{i=1}^{n} x_i - \left(1+\frac{\gamma}{\beta}\right)\sum_{i=1}^{n}\ln\left(1+\alpha e^{\beta x_i}\right) \end{aligned} \tag{12}$$

To estimate the parameters α, β and γ by equating the partial derivative of the equation obtained in Eq. (12) with respect to each parameter to zero.

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \frac{n\gamma}{\beta\,(1+\alpha)} - \left(1+\frac{\gamma}{\beta}\right)\sum_{i=1}^{n}\frac{e^{\beta x_i}}{1+\alpha e^{\beta x_i}} \tag{13}$$

$$\frac{\partial L}{\partial \beta} = \frac{-n\gamma}{\beta^2}\ln(1+\alpha) + \sum_{i=1}^{n} x_i - \left(1+\frac{\gamma}{\beta}\right)\sum_{i=1}^{n}\frac{\alpha x_i e^{\beta x_i}}{1+\alpha e^{\beta x_i}} + \frac{\gamma}{\beta^2}\sum_{i=1}^{n}\ln \tag{14}$$

$$\frac{\partial L}{\partial \gamma} = \frac{n}{\gamma} + \frac{n}{\beta}\ln(1+\alpha) - \frac{1}{\beta}\sum_{i=1}^{n}\ln\left(1+\alpha e^{\beta\sum_{i=1}^{n} x_i}\right) \tag{15}$$

Now setting the above three Equations to zero, the maximum likelihood estimates of α, β, γ can be obtained by solving the non-linear equations numerically using analytical methods like Newton-Raphson algorithm. This can be done by using statistical package, R.

Under certain regulatory conditions, the estimator $\hat{\theta} = \left(\hat{\alpha}, \hat{\beta}, \hat{\gamma}\right)$ is asymptotically normally distributed with mean $\theta = (\alpha, \beta, \gamma)$ and variance-covariance matrix $Cov(\hat{\theta}) = I(\theta)^{-1}$ where $I(\theta)$ is the Fisher information matrix or the expected value of $J(\theta) = -\frac{\partial^2 l}{\partial \alpha_i \partial \alpha_j}$ is the (i, j) element of observed information matrix. The normal approximation of the MLEof θ can be used to construct approximate confidence intervals of the parameters. The asymptotic distribution of $\sqrt{n}\left(\hat{\theta} - \theta\right)$ is $N_3\left(0, I^{-1}\left(\hat{\theta}\right)\right)$. The asymptotic multivariate normal distribution can be used to construct asymptotic confidence intervals for α, β and γ. The asymptotic $100\,(1-\eta)\,\%$ confidence intervals for α, β and γ are respectively $\left(\hat{\alpha} \pm z_{\frac{\eta}{2}} S.E(\hat{\alpha})\right)$, $\left(\hat{\beta} \pm z_{\frac{\eta}{2}} S.E(\hat{\beta})\right)$ and $\left(\hat{\gamma} \pm z_{\frac{\eta}{2}} S.E(\hat{\gamma})\right)$, where S.E.(.) is the square root of the diagonal element of $I^{-1}\left(\hat{\theta}\right)$ corresponding to each parameter and $z_{\frac{\eta}{2}}$ is the quantile $\left(1 - \frac{\eta}{2}\right)$ of the standard normal distribution.

4. Simulation study

Here, we used a simulation study to investigate the performance of the accuracy of point estimates of the parameters of the Lehmann type-II Perk model (α, β, γ) distribution. The following steps were followed:

- Specify the sample size n and the values of the parameters α, β, γ.
- Generate $Ui \sim \text{Uniform}(0,1); i = 1, 2, 3...., n$.
- Set
$$x = \frac{1}{\beta}\ln\frac{1}{\alpha}\left[\frac{1+\alpha}{(1-u)^{\frac{\beta}{\gamma}}} - 1\right] \tag{16}$$
- Calculate the MLEs of the three parameters.
- Repeat steps 2-3, N times.
- Calculate the mean squared error (MSE) for each parameter.

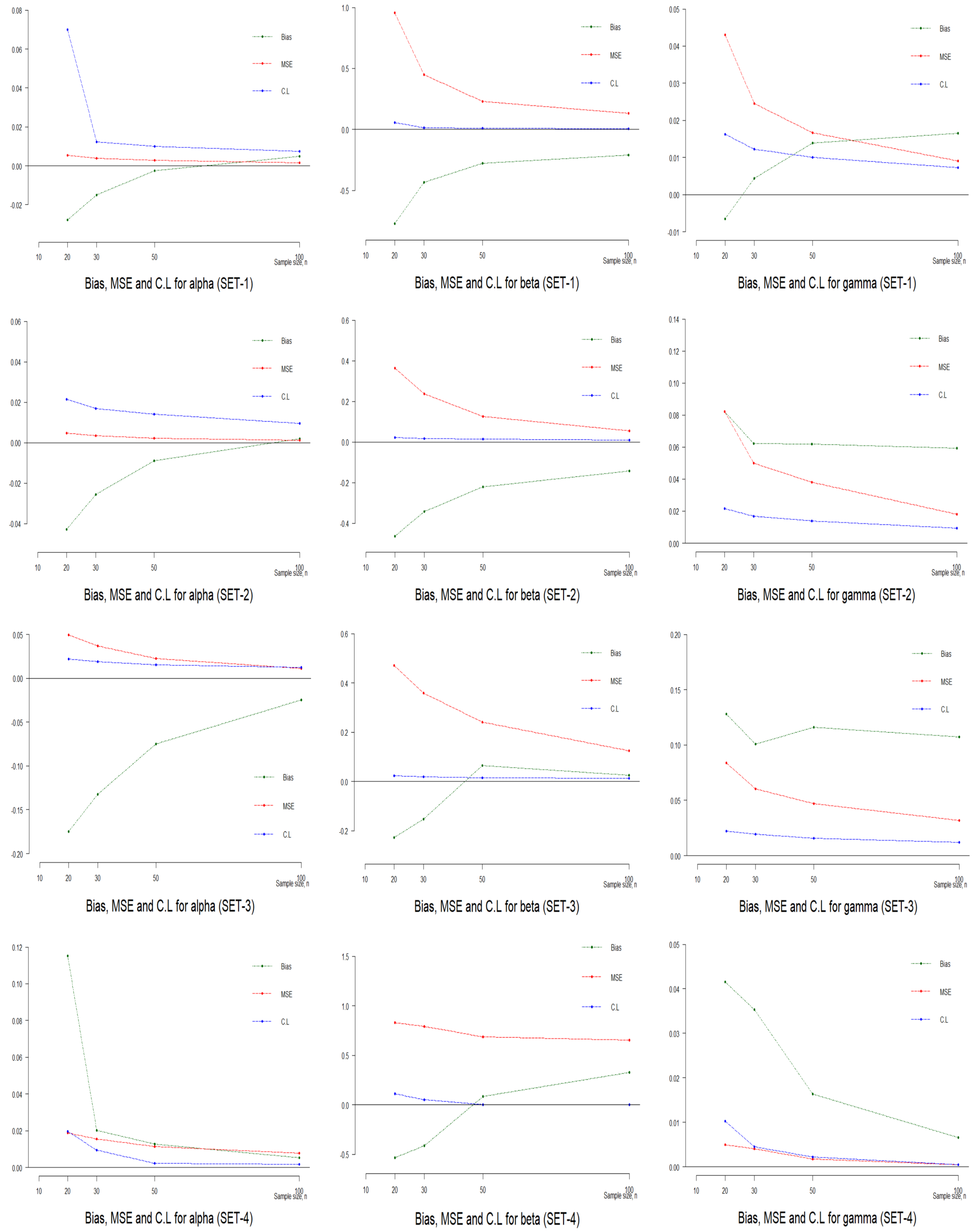


Figure 4: *Bias, mean square error (MSE) and confidence length (C.L)*

Table 1: *Bias, mean square error (MSE) and confidence length (C.L) of different set of parameters*

Sample Size	$\hat{\alpha}$ (Bias)	$\hat{\beta}$ (Bias)	$\hat{\gamma}$ (Bias)	$\hat{\alpha}$ (MSE)	$\hat{\beta}$ (MSE)	$\hat{\gamma}$ (MSE)	$\hat{\alpha}$ (C.L)	$\hat{\beta}$ (C.L)	$\hat{\gamma}$ (C.L)
SET-1									
20	-0.0278	-0.7736	-0.0066	0.0054	0.9552	0.0430	0.0699	0.0544	0.0162
30	-0.0151	-0.4347	0.0043	0.0038	0.4505	0.0245	0.0122	0.0122	0.0122
50	-0.0026	-0.2762	0.0139	0.0027	0.2311	0.0166	0.0100	0.0100	0.0100
100	0.0049	-0.2086	0.0165	0.0016	0.1314	0.0090	0.0073	0.0073	0.0073
SET-2									
20	-0.0428	-0.4639	0.0823	0.0048	0.3628	0.0822	0.0215	0.0215	0.0215
30	-0.0257	-0.3422	0.0623	0.0034	0.2380	0.0500	0.0168	0.0168	0.0168
50	-0.0089	-0.2217	0.0619	0.0021	0.1266	0.0379	0.0140	0.0140	0.0140
100	0.0018	-0.1427	0.0593	0.0011	0.0535	0.0180	0.0094	0.0094	0.0094
SET-3									
20	-0.1749	-0.2284	0.1278	0.0496	0.4697	0.0839	0.0222	0.0222	0.0222
30	-0.1321	-0.1534	0.1008	0.0370	0.3589	0.0605	0.0192	0.0192	0.0192
50	-0.0751	0.0645	0.1161	0.0224	0.2412	0.0468	0.0156	0.0156	0.0156
100	-0.0245	0.0249	0.1072	0.0112	0.1244	0.0319	0.0122	0.0122	0.0122
SET-4									
20	0.1151	-0.5313	0.0416	0.0188	0.8283	0.0049	0.0196	0.1115	0.0102
30	0.0203	-0.4140	0.0353	0.0154	0.7922	0.0040	0.0095	0.0515	0.0045
50	0.0127	0.0826	0.0163	0.0113	0.6858	0.0017	0.0022	0.0022	0.0022
100	0.0052	0.3295	0.0066	0.0076	0.6524	0.0005	0.0016	0.0005	0.0005

The comparison is based on MSEs and C.L. The MSEs and C.L were computed by generating one thousand replications of sample size $n = 20$, 30, 50, 100 from the Lehmann II Perks model with different parameter values. The required results are obtained based on the different combinations of the model parameters place in SET-1 ($\alpha = 0.1$, $\beta = 3$, $\gamma = 1$), SET-2 ($\alpha = 0.1$, $\beta = 2$, $\gamma = 1$), SET-3 ($\alpha = 0.3$, $\beta = 2$, $\gamma = 1$) and SET-4 ($\alpha = 0.1$, $\beta = 3$, $\gamma = 0.2$), which are shown in Tables 1. Figures 4 displays the bias, mean squared error and confidence length for different sample sizes. The assessment based on simulation study is that the MSEs for each parameter decreases with increasing sample size. Moreover, we observe that the confidence lengths also decrease with an increase in the sample size. We used the nlm() package in R to obtain the parameter estimates. All the analyses are conducted using the statistical package, R Studio (version 2023.06.0).

5. Data analysis: Conductors' failure data

This section is devoted to the model comparison between the proposed Lehmann II Perk model (LPM) and some other models. The following data set has been used in order to assess the goodness-offit of the considered models. The data present hours to failure of 59 test conductors of 400 micrometer length, reported by Schafft et al. [19]. All specimens ran to failure at a certain high temperature and current density. All 59 specimens were tested under the same temperature and current density. 6.545, 9.289, 7.543, 6.956, 6.492, 5.459, 8.120, 4.706, 8.687, 2.997, 8.591, 6.129, 11.038, 5.381, 6.958, 4.288, 6.522, 4.137, 7.459, 7.495, 6.573, 6.538, 5.589, 6.087, 5.807, 6.725, 8.532, 9.663, 6.369, 7.024, 8.336, 9.218, 7.945, 6.869, 6.352, 4.700, 6.948, 9.254, 5.009, 7.489, 7.398, 6.033, 10.092, 7.496, 4.531, 7.974, 8.799, 7.683, 7.224, 7.365, 6.923, 5.640, 5.434, 7.937, 6.515, 6.476, 6.071, 10.491, 5.923. We comapre the developed distribution with the following distributions:

1. The Perks distribution (PD) proposed by Perks [15], with density function

$$f_1(x) = \alpha\lambda e^{\lambda x}\frac{(1+\alpha)}{(1+\alpha e^{\lambda x})^2} \tag{17}$$

where $x, \alpha, \lambda > 0$ and $f_1(x) = 0$ otherwise.

2. Exponentiated Perks distribution (EPD) introduced by Sigh B. and Choudhary N. [21], with density function

Table 2: *Goodness of fit for Conductors' failure data.*

Model	α	β	λ	K-S statistic	p value	AIC
LPM	**0.00028**	**1.2427**	**0.9341**	**0.0507**	**0.9962**	**228.74**
PD	0.0052	-	0.7576	0.1134	0.4043	237.20
EPD	0.0029	2.10	0.9990	0.0673	0.9356	230.03
EED	35.0200	-	0.5797	0.1045	0.5065	235.08
EWD	2.72400	2:92	0.1655	0.0648	0.9518	228.59
GRD	6.4000	-	0.2200	0.0719	0.8990	227.74

$$f_2(x) = \alpha\beta\lambda e^{\lambda x} \frac{(1+\alpha)}{(1+\alpha e^{\lambda x})^2} \left[1 - \left(\frac{1+\alpha}{1+\alpha e^{\lambda x}}\right)\right]^{\beta-1} \tag{18}$$

where $x, \alpha, \beta, \lambda > 0$ and $f_2(x) = 0$ otherwise.

3. Exponentiated exponential distribution (EED) introduced by Gupta et.al.[8], with density function

$$f_3(x) = \alpha\lambda \left(1 - e^{-\lambda x}\right)^{\alpha-1} e^{-\lambda x} \tag{19}$$

where $x, \alpha, \lambda > 0$ and $f_3(x) = 0$ otherwise.

4. Exponentiated Weibull distribution (EWD) introduced by Mudholkar and Srivastava [11], with density function

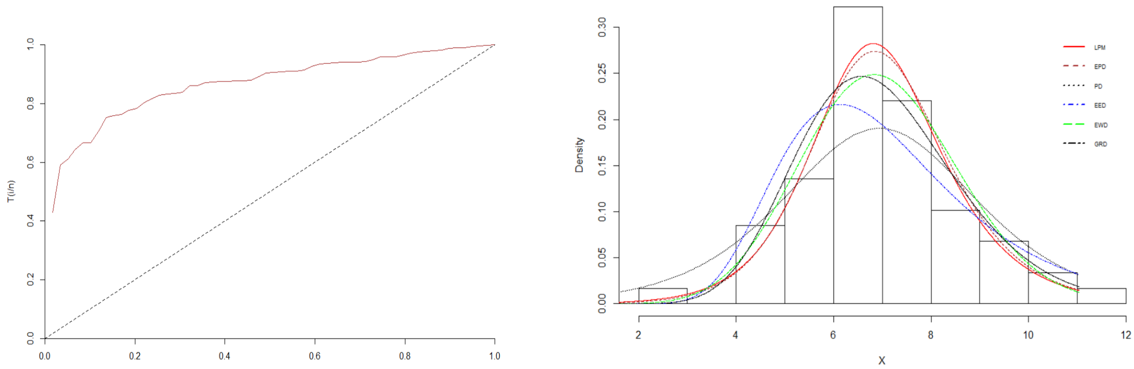
$$f_4(x) = \alpha\beta\lambda^\beta x^{\beta-1} \left[1 - exp(-\lambda x)^\beta\right]^{\alpha-1} exp(-\lambda x)^\beta \tag{20}$$

where $x, \alpha, \beta, \lambda > 0$ and $f_4(x) = 0$ otherwise.

5. Generalized Rayleigh distribution (GRD) introduced by surles and padgett [22], with density function

$$f_5(x) = 2\alpha\lambda^2 x e^{-(\lambda x)^2} \left(1 - e^{-(\lambda x)^2}\right)^{\alpha-1} \tag{21}$$

where $x, \alpha, \lambda > 0$ and $f_5(x) = 0$ otherwise.



(a) *Total time test plot for conductors failure data.* **(b)** *Histogram and fit for Conductors' failure data*

Figure 5: *Comparison of TTT plot and histogram for conductors' failure data.*

The total time on the test plot (TTT) of the data shown in Figure 5a suggests an increasing hazard rate. Now, we demonstrate the fit of LPM in comparison to other considered distributions for the above data sets. For each distribution, the unknown parameters are estimated by the method of maximum likelihood. Table 2 presents the parameter estimates, Kolmogorov-Smirnov statistic, p-value, and the Akaike information criterion value for the models. The best model is the one with the minimum value for the Akaike information criterion and the Kolmogorov-Smirnov statistic. It is observed that the developed distribution provides the best fit with a Kolmogorov-Smirnov statistic of 0.0507 and a p-value of 0.9962. Additionally, the Akaike information criterion value is the lowest for the developed distribution when compared to the models discussed in Singh B and Chaudhary N. [21] (Table 2). This explains the flexibility of the LPM distribution. Figure 5b displays the histogram and the fit plot for the Conductors' failure data.

6. CONCLUSION

We have studied the properties of the Lehmann II Perk model. Using maximum likelihood method we computed the estimates. To assess the performance of the estimates are evaluated through Monte-Carlo simulation. We have compared the fit of the developed distribution with other models in the literature for the conductor's failure data and found it to be better. Furthermore, for future research, we can consider studying more properties of this distribution and explore different methods of estimation, such as the Bayesian method.

References

[1] A. A. Ogunde, G. A. Olalude, O. E. Adeniji1, K. Balogun . Lehmann type II Frechet Poisson distribution: Properties, inference and applications as a life time distribution, International journal of statistics and probability, (2021).

[2] A.A. Ogunde , S. T. Fayose , B. Ajayi, D. O. Omosigho. Extended gumbel type-2 distribution: properties and applications, Journal of applied Mathematics, (2020) .

[3] Chaudhary A. K.. A study of Perks-II distribution via Bayesian paradigm, Pravaha journal, (2018).

[4] Chaudhary A. K, Kumar and V.. A Bayesian analysis of Perks-I distribution via Markov Chain Monte Carlo simulation, Nepal journal of science and technology, 2013.

[5] Cordeiro, G. M., E., Ortega, E. M. M., and Cunha, D. C. C.. The exponentiated generalized class of distributions, Journal of data science, (2013).

[6] E. L., Lehmann. The power of rank tests, The annals of Mathematical Statistics, (1953).

[7] Gonzalez L. C. M., Picon L. A. R., Olguin J. C. P., Garcia V, Cruz D.L.. The additive Perks distribution and its applications in reliability analysis, Quality technology and quantitative management, (2022).

[8] Gupta RC, Gupta PL, Gupta RD. Modeling failure time data by Lehmann alternatives, Communications Statistics theory methods, (1998).

[9] G. M. Cordeiro and M. De Castro. A new family of generalized distributions, Journal of Statistical computation and simulation, (2011).

[10] Haberman, S., Renshaw, A.. A comparative study of parametric mortality projection models, Insurance: Mathematics and Economics, (2011).

[11] Mudholkar GS, Srivastava DK. Exponentiated Weibull family for analyzing bathtub failure-real data, IEEE transactions on reliability, (1993).

[12] M. Z. Arshad , M. Z. Iqbal , A. Anees , Z. Ahmad and O. S. Balogun. A new bathtub shaped failure rate model: properties and applications to engineering sector, Pakistan journal of Statistics, (2021).

[13] N. I. Badmus , T. A. Bamiduro, S. G. Ogunobi. Lehmann Type II weighted Weibull distribution, International journal of Physical sciences, (2014) .

[14] O. S. Balogun , M. Z. Iqbal , M. Z. Arshad , A. Z. Afify, P. E. Oguntunde. A new generalization of Lehmann type-II distribution: Theory, simulation, and applications to survival and failure rate data, Scientific African, (2021).

[15] Perks W.. On some experiments in the graduation of mortality statistics, Journal of the institute of actuaries, (1932).

[16] P. O. Awodutire , E. C. Nduka, M. A. Ijomah. Lehmann type II generalized half logistic distribution: properties and application, Mathematical theory and modeling, (2020).

[17] Richards. Applying survival models to pensioner mortality data, British actuarial journal, (2008).

[18] Richards, S. J.. A handbook of parametric survival models for actuarial use, Scandinavian actuarial journal, (2012).

[19] Schafft HA, Staton TC, Mandel J, Shott JD, Reproducibility of electro migration measurements. IEEE trans electron devices ED, (1989).

[20] Shaked, M., Shanthikumar, J. G.. Stochastic orders, Springer Science and business Media, (2007).

[21] Singh B, Chaudhary N.. The exponentiated Perks distribution, International journal of systems assurance engineering and management, (2016).

[22] Surles JG, Padgett WJ. Inference for reliability and stressstrength for a scaled Burr type X distribution, Lifetime data analysis, (2001).

[23] V. L. D. Tomazella , P. L. Ramos , P. H. Ferreira , A. L. Mota, F. Louzada. The Lehmann type II inverse Weibull distribution in the presence of censored data, Communications in Statistics - simulation and computation, (2020) .

[24] Zeng, H., Lan, T., Chen, Q.. Five and four-parameter lifetime distributions for bathtub-shaped failure rate using Perks mortality equation, Reliability engineering and system safety, (2016).