SINE GENERALIZED ODD LOG-LOGISTIC FAMILY OF DISTRIBUTIONS: PROPERTIES AND APPLICATION TO REAL DATA

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Abstract

In this research, we introduce and analyze a new family of distributions called the sine generalized odd log-logistic-G family. This is driven by the reality that no single distribution can effectively model all types of data across different fields. Consequently, there is a need to develop distributions that possess desirable properties and are flexible enough to accommodate data with diverse characteristics. We explore its statistical properties, including the survival function, hazard function, moments, moment-generating function, and order statistics. A special case of the family of distributions is also presented. The maximum likelihood estimation method is used to obtain estimators of the family of distributions and the performance of the maximum likelihood estimators is evaluated in terms of bias and root mean squared errors through two simulation studies. Additionally, we demonstrate the practicality of this family using two real data sets, where it consistently provides better fits compared to other competitive distributions.

Keywords: Odd Log-Logistic family of distribution, Sine transformation, Maximum likelihood estimation, Monte Carlo Simulation, Breast tumor data

1. Introduction

The generalized odd family has gained attention in the literature because it can capture various distribution shapes, such as unimodality, bimodality, and heavy-tailed behavior [1]. Recent studies have extensively examined the generalized odd log-logistic family of distributions, providing enhanced flexibility for modeling lifetime data. This distribution extends the odd log-logistic family, which is commonly used in modeling lifetime and survival data [1]. A new class of continuous distributions with additional shape parameters was introduced by [2] and [7], leading to improved fits for various datasets. These families include special cases such as the proportional reversed hazard rate and odd log-logistic classes. The odd log-logistic Gompertz distribution was proposed by [3] and can model decreasing, increasing, and bathtub-shaped failure rates. Building on this concept, [4] introduced the odd log-logistic transmuted-G family, which combines the odd log-logistic distribution with the transmuted distribution.

A new family of continuous distributions, called the "exponentiated odd log-logistic family," was proposed by [5]. A new three-parameter lifetime distribution, called the odd log-logistic generalized Lindley (OLLGL) distribution, was introduced by [14]. All of these articles discuss the mathematical properties, parameter estimation methods, and real-world applications of their proposed distributions. These new families have demonstrated superior performance in modeling survival data and reliability problems compared to existing distributions, highlighting their importance in statistical analysis. The Odd Lomax log-logistic distribution (OLLLD) was introduced by [12] as a generalized parental distribution, enhancing flexibility to capture the characteristics of real-world data sets, with parameters estimated using maximum likelihood estimation. The Odd Log-Logistic Generalized Exponential Distribution was developed by [13], enhancing distribution flexibility for survival data analysis, with a focus on statistical properties and parameter estimation methods. The odd log-logistic generalized Lindley distribution, developed by [14], is a three-parameter lifetime model that includes structural properties, estimation methods, a simulation study, and empirical illustrations using real data sets.

Recent advancements in trigonometric distributions have introduced several new families with broad applications in fields such as physics, engineering, and medicine. The Sine Type II Topp-Leone-G family, proposed by [16], offers flexibility and efficient data fitting. The sine-G family expanded by [18] to include the alpha-sine-G family, which adds a parameter for enhanced statistical inference. The exponentiated sine-G family was introduced by [19] for lifetime studies, emphasizing properties such as density, reliability, and entropy. The Sin-G family was extended by jamal2021beyond with a transformed version that enhances modeling. The sine Kumaraswamy-G family was developed by [21], exploring its statistical applications. Meanwhile, the sine-G family, an extension of the exponentiated Burr XII distribution for heavy-tailed datasets in insurance, was proposed by [15]. .

2. Methods

This paper presents a comprehensive study of the theoretical properties and applications of the sine generalized odd log-logistic distribution. By introducing the trigonometric function in the sine generalized odd log-logistic distribution, researchers and practitioners can now model a wide range of real-world phenomena with greater flexibility, making it a versatile tool for working with lifetime and survival data. The cdf of generalized log-logistic families of distributions by [9] is given as;

$$P(x) = \frac{G(x)^{\alpha}}{G(x)^{\alpha} + \bar{G}(x)^{\alpha}} \tag{1}$$

Where $\bar{G}(x) = 1 - G(x)$. And the cdf anf pdf of generalized odd log-logistic family by [11] is given by;

$$F(x) = \frac{G(x)^{\alpha\theta}}{G(x)^{\alpha\theta} + (1 - G(x)^{\theta})^{\alpha}} \tag{2}$$

$$f(x) = \frac{\alpha\theta g(x) G(x)^{\alpha\theta-1}(1 - G(x)^{\theta})^{\alpha-1}}{(G(x)^{\alpha\theta} + (1 - G(x)^{\theta})^{\alpha})^2} \tag{3}$$

2.1. Sine Generalized Odd Log-logistic Family

The cumulative distribution function and Probability density function of SGOLL-G is derived as;

$$F(x) = Sin\left[\frac{\pi G(x)^{\alpha\theta}}{2(G(x)^{\alpha\theta} + (1 - G(x)^{\theta})^{\alpha})}\right] \tag{4}$$

The corresponding pdf of SGOLL-G is,

$$f(x) = \frac{\pi\alpha\theta g(x) G(x)^{\alpha\theta-1}(1 - G(x)^{\theta})^{\alpha-1} Cos\left[\frac{\pi G(x)^{\alpha\theta}}{2(G(x)^{\alpha\theta}+(1-G(x)^{\theta})^{\alpha})}\right]}{2(G(x)^{\alpha\theta} + (1 - G(x)^{\theta})^{\alpha})^2} \tag{5}$$

2.2. Mathematical Properties of SGOLL-G

In this part, we derive some properties of SGOLL-G Distribution like Survival function, hazard rate function, and quantile function.

$$S(x) = 1 - Sin\left[\frac{\pi G(x)^{\alpha\theta}}{2(G(x)^{\alpha\theta} + (1 - G(x)^{\theta})^{\alpha})}\right] \tag{6}$$

$$H(x) = \frac{\pi\alpha\theta g(x) G(x)^{\alpha\theta - 1}(1 - G(x)^{\theta})^{\alpha - 1} Cos\left[\frac{\pi G(x)^{\alpha\theta}}{2(G(x)^{\alpha\theta} + (1 - G(x)^{\theta})^{\alpha})}\right]}{2(G(x)^{\alpha\theta} + (1 - G(x)^{\theta})^{\alpha})^2 \left[1 - Sin\left[\frac{\pi G(x)^{\alpha\theta}}{2(G(x)^{\alpha\theta} + (1 - G(x)^{\theta})^{\alpha})}\right]\right]} \tag{7}$$

$$x_u = G^{-1}\left[\frac{\left(\frac{2arcsinU}{\pi(1 - \frac{2}{\pi}arcsinU)}\right)^{\frac{1}{\alpha}}}{1 + \left(\frac{2arcsinU}{\pi(1 - \frac{2}{\pi}arcsinU)}\right)^{\frac{1}{\alpha}}}\right]^{\frac{1}{\theta}} \tag{8}$$

Where$G^{-1}(.)$ is the inverse of the baseline cdf. hence U is a uniform random variable on (0,1), then has SDOLL-G distribution.

However, other properties are entropy, moment, moment generating function, and order statistics were derived below;

2.2.1 Renyi's Entropy of SGOLL-G

The definition of Renyi's Entropy is given as,

$$RE = I_x = \frac{1}{1 - k} log \int_{-\infty}^{\infty} f(x)^k dx \tag{9}$$

Substituting the PDF of SGLLO-G we have,

$$= \frac{1}{1 - k} log \int_{-\infty}^{\infty} \left(\frac{\pi\alpha\theta g(x) G(x)^{\alpha\theta - 1}(1 - G(x)^{\theta})^{\alpha - 1} Cos\left[\frac{\pi G(x)^{\alpha\theta}}{2(G(x)^{\alpha\theta} + (1 - G(x)^{\theta})^{\alpha})}\right]}{2(G(x)^{\alpha\theta} + (1 - G(x)^{\theta})^{\alpha})^2}\right)^k dx \tag{10}$$

$$= \frac{1}{1 - k}\left[log(\pi\alpha\theta)^k - klog2 + log \int_{-\infty}^{\infty} \left(\frac{g(x) G(x)^{\alpha\theta - 1}(1 - G(x)^{\theta})^{\alpha - 1} Cos\left[\frac{\pi G(x)^{\alpha\theta}}{2(G(x)^{\alpha\theta} + (1 - G(x)^{\theta})^{\alpha})}\right]}{(G(x)^{\alpha\theta} + (1 - G(x)^{\theta})^{\alpha})^2}\right)^k dx\right] \tag{11}$$

$$= \frac{1}{1 - k}\left[log(\pi\alpha\theta)^k - klog2 + log\beta\right] \tag{12}$$

$$= \frac{1}{1 - k}\left[klog(\pi\alpha\theta) - klog2 + log\beta\right] \tag{13}$$

Where

$$\beta = \int_{-\infty}^{\infty} \left(\frac{g(x) G(x)^{\alpha\theta - 1}(1 - G(x)^{\theta})^{\alpha - 1} Cos\left[\frac{\pi G(x)^{\alpha\theta}}{2(G(x)^{\alpha\theta} + (1 - G(x)^{\theta})^{\alpha})}\right]}{(G(x)^{\alpha\theta} + (1 - G(x)^{\theta})^{\alpha})^2}\right)^k dx \tag{14}$$

2.2.2 Moment of SGOLL-G

The definition of Moment is given as'

$$\mu^r = E(x^r) = \int_{-\infty}^{\infty} x^r f(x) dx \tag{15}$$

$$\mu^r = \frac{\pi\alpha\theta}{2} \int_0^{\infty} \frac{x^r g(x) G(x)^{\alpha\theta-1}(1 - G(x)^{\theta})^{\alpha-1} Cos\left[\frac{\pi G(x)^{\alpha\theta}}{(G(x)^{\alpha\theta}+(1-G(x)^{\theta})^{\alpha})}\right]}{2(G(x)^{\alpha\theta} + (1 - G(x)^{\theta})^{\alpha})^2} dx \tag{16}$$

$$\mu^r = \eta\tau \tag{17}$$

Where,

$$\tau = \int_0^{\infty} \frac{x^r g(x) G(x)^{\alpha\theta-1}(1 - G(x)^{\theta})^{\alpha-1} Cos\left[\frac{\pi G(x)^{\alpha\theta}}{(G(x)^{\alpha\theta}+(1-G(x)^{\theta})^{\alpha})}\right]}{2(G(x)^{\alpha\theta} + (1 - G(x)^{\theta})^{\alpha})^2} \tag{18}$$

and,

$$\eta = \frac{\pi\alpha\theta}{2} \tag{19}$$

2.2.3 Moment Generating function of SGOLL-G

The definition of Moment Generating function is given,

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx \tag{20}$$

Substituting the PDF of SGOLL-G we have,

$$M_x(t) = \frac{\pi\alpha\theta}{2} \int_0^{\infty} \frac{e^{tx} g(x) G(x)^{\alpha\theta-1}(1 - G(x)^{\theta})^{\alpha-1} Cos\left[\frac{\pi G(x)^{\alpha\theta}}{(G(x)^{\alpha\theta}+(1-G(x)^{\theta})^{\alpha})}\right]}{2(G(x)^{\alpha\theta} + (1 - G(x)^{\theta})^{\alpha})^2} dx \tag{21}$$

Using taylor series expansion,

$$e^{tx} = \sum \frac{t^j x^j}{j!} \tag{22}$$

implies that,

$$M_x(t) = \frac{\pi\alpha\theta t^j}{2j!} \sum_{j=0}^{\infty} \int_0^{\infty} \frac{x^j g(x) G(x)^{\alpha\theta-1}(1 - G(x)^{\theta})^{\alpha-1} Cos\left[\frac{\pi G(x)^{\alpha\theta}}{(G(x)^{\alpha\theta}+(1-G(x)^{\theta})^{\alpha})}\right]}{2(G(x)^{\alpha\theta} + (1 - G(x)^{\theta})^{\alpha})^2} dx \tag{23}$$

$$\Rightarrow M_x(t) = \sum_{j=0}^{\infty} \varphi\vartheta \tag{24}$$

Where,

$$\vartheta = \int_0^{\infty} \frac{x^j g(x) G(x)^{\alpha\theta-1}(1 - G(x)^{\theta})^{\alpha-1} Cos\left[\frac{\pi G(x)^{\alpha\theta}}{(G(x)^{\alpha\theta}+(1-G(x)^{\theta})^{\alpha})}\right]}{2(G(x)^{\alpha\theta} + (1 - G(x)^{\theta})^{\alpha})^2} dx \tag{25}$$

and,

$$\varphi = \frac{\pi\alpha\theta t^j}{2j!} \tag{26}$$

2.2.4 Order Statistics of SGOLL-G

The definition of Order Statistics is,

$$f_{(i,n)}(x) = \frac{n!}{(i-1)(n-i)} f(x)[F(x)]^{i-1}[1-F(x)]^{n-i} \tag{27}$$

Now using power series expansion;

$$[1-F(x)]^{n-1} = \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} [F(x)]^j \tag{28}$$

$$f_{(i,n)}(x) = \frac{n!}{(i-1)(n-i)} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j f(x)[F(x)]^{i+j-1} \tag{29}$$

$$f_{(i,n)}(x) = \frac{n!(-1)^j}{(i-1)(n-i-j)} \sum_{j=0}^{n-i} \binom{n-i}{j} f(x)[F(x)]^{i+j-1} \tag{30}$$

Now, $f_{(i,n)}(x) = \sum_{j=0}^{n-i} \binom{n-i}{j} \left[\frac{n!(-1)^j \pi\alpha\theta g(x) G(x)^{\alpha\theta-1}(1-G(x)^\theta)^{\alpha-1} Cos\left[\frac{\pi G(x)^{\alpha\theta}}{2(G(x)^{\alpha\theta}+(1-G(x)^\theta)^\alpha)}\right]}{2(i-1)!(n-i-j)(G(x)^{\alpha\theta}+(1-G(x)^\theta)^\alpha)^2} \right] \times$

$$\left[Sin\left[\frac{\pi G(x)^{\alpha\theta}}{2(G(x)^{\alpha\theta}+(1-G(x)^\theta)^\alpha)}\right]\right]^{i+j-1} \tag{31}$$

2.2.5 Maximum Likelihood Estimation

Let $x_1, x_2, ..., x_n$ be a random sample from the SGOLL-G family of distribution with pdf in equation (5) with $\bar{\kappa} = (\alpha, \lambda, \theta)^T$, the SGOLL-G's n sample log-likelihood is driven as:

$LL(\kappa) = nlog\pi + nlog\theta + \sum logg(x) + \sum logG(x)^{\alpha\theta-1} - \sum log(1-G(x)^\theta) - nlog2 - 2\alpha\theta \sum logG(x) + .$

$$\sum logCos\left[\frac{\pi G(x)^{\alpha\theta}}{2(G(x)^{\alpha\theta}+(1-G(x)^\theta)^\alpha)}\right] \tag{32}$$

$$\frac{dLL(\kappa)}{d\alpha} = \frac{n}{\alpha} - \theta \sum logG(x) + \sum \frac{\pi\theta G(x)^{\alpha\theta}(1-G(x)^\theta)^\alpha (1-log(1-G(x)) tan\left[\frac{\pi G(x)^{\alpha\theta}}{2(G(x)^{\alpha\theta}+(1-G(x)^\theta)^\alpha)}\right]}{2(G(x)^{\alpha\theta}+(1-G(x)^\theta)^\alpha)^2} \tag{33}$$

$$\frac{dLL(\kappa)}{d\theta} = \frac{n}{\theta} - \alpha \sum logG(x) + \sum \frac{G(x)^\theta logG(x)}{(1-G(x)^\theta)} + \sum \frac{\frac{d}{d\theta} Cos\left[\frac{\pi G(x)^{\alpha\theta}}{2(G(x)^{\alpha\theta}+(1-G(x)^\theta)^\alpha)}\right]}{Cos\left[\frac{\pi G(x)^{\alpha\theta}}{2(G(x)^{\alpha\theta}+(1-G(x)^\theta)^\alpha)}\right]} \tag{34}$$

$$\frac{dLL(\kappa)}{d\psi} = \sum \frac{g'(x)}{g(x)} - \sum \frac{g(x)}{G(x)} - \alpha\theta \sum \frac{g(x)}{G(x)} + \sum \frac{\theta g(x) G(X)^{\theta-1}}{(1-G(x)^\theta)} + \sum \frac{\frac{d}{d\psi} Cos\left[\frac{\pi G(x)^{\alpha\theta}}{2(G(x)^{\alpha\theta}+(1-G(x)^\theta)^\alpha)}\right]}{Cos\left[\frac{\pi G(x)^{\alpha\theta}}{2(G(x)^{\alpha\theta}+(1-G(x)^\theta)^\alpha)}\right]} \tag{35}$$

2.3. Special member of the SGOLL-G family

In this section family member sine generalized odd log-logistic weibul were introduced as a special member of SGOLL-G, which were used for the simulation and application to real data. Considering Weibull distribution as a baseline ditribution we derive the Sine Generalized Odd Log-Logistic Weibul (SGOLL-W)Distribution, with CDF given by;

$$F(x,\alpha,\theta,\lambda,\omega) = Sin\left[\frac{\pi(1-e^{-(\frac{x}{\lambda})^{\omega}})^{\alpha\theta}}{2((1-e^{-(\frac{x}{\lambda})^{\omega}})^{\alpha\theta}+(1-(1-e^{-(\frac{x}{\lambda})^{\omega}})^{\theta})^{\alpha})}\right] \tag{36}$$

The corresponding pdf of SGOLL-W is;

$$f(x) = \frac{\pi\alpha\theta\omega x^{\omega-1}e^{-(\frac{x}{\lambda})^{\omega}}(1-e^{-(\frac{x}{\lambda})^{\omega}})^{\alpha\theta-1}(1-(1-e^{-(\frac{x}{\lambda})^{\omega}})^{\theta})^{\alpha-1}Cos\left[\frac{\pi(1-e^{-(\frac{x}{\lambda})^{\omega}})^{\alpha\theta}}{2((1-e^{-(\frac{x}{\lambda})^{\omega}})^{\alpha\theta}+(1-(1-e^{-(\frac{x}{\lambda})^{\omega}})^{\theta})^{\alpha})}\right]}{2\lambda^{\omega}((1-e^{-(\frac{x}{\lambda})^{\omega}})^{\alpha\theta}+(1-(1-e^{-(\frac{x}{\lambda})^{\omega}})^{\theta})^{\alpha})^{2}} \tag{37}$$

$$S(x) = 1 - Sin\left[\frac{\pi(1-e^{-(\frac{x}{\lambda})^{\omega}})^{\alpha\theta}}{2((1-e^{-(\frac{x}{\lambda})^{\omega}})^{\alpha\theta}+(1-(1-e^{-(\frac{x}{\lambda})^{\omega}})^{\theta})^{\alpha})}\right] \tag{38}$$

$$H(x) = \frac{\pi\alpha\theta\omega x^{\omega-1}e^{-(\frac{x}{\lambda})^{\omega}}(1-e^{-(\frac{x}{\lambda})^{\omega}})^{\alpha\theta-1}(1-(1-e^{-(\frac{x}{\lambda})^{\omega}})^{\theta})^{\alpha-1}Cos\left[\frac{\pi(1-e^{-(\frac{x}{\lambda})^{\omega}})^{\alpha\theta}}{2((1-e^{-(\frac{x}{\lambda})^{\omega}})^{\alpha\theta}+(1-(1-e^{-(\frac{x}{\lambda})^{\omega}})^{\theta})^{\alpha})}\right]}{2\lambda^{\omega}((1-e^{-(\frac{x}{\lambda})^{\omega}})^{\alpha\theta}+(1-(1-e^{-(\frac{x}{\lambda})^{\omega}})^{\theta})^{\alpha})^{2}\left[1-Sin\left[\frac{\pi(1-e^{-(\frac{x}{\lambda})^{\omega}})^{\alpha\theta}}{2((1-e^{-(\frac{x}{\lambda})^{\omega}})^{\alpha\theta}+(1-(1-e^{-(\frac{x}{\lambda})^{\omega}})^{\theta})^{\alpha})}\right]\right]} \tag{39}$$

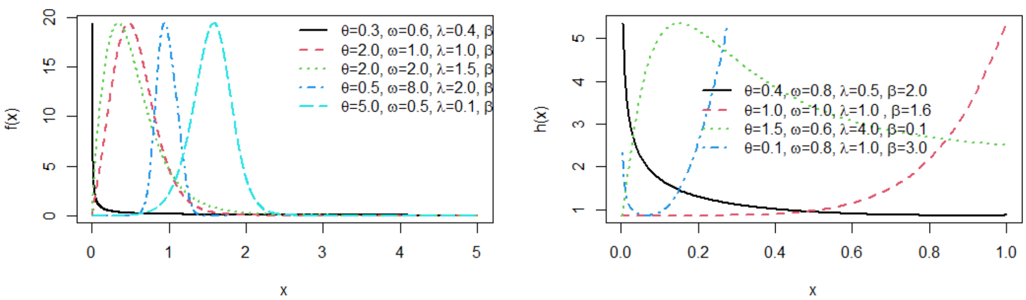

Figure 1: *The pdf and the hazard plot for some values of parameters of SGOLL-W*

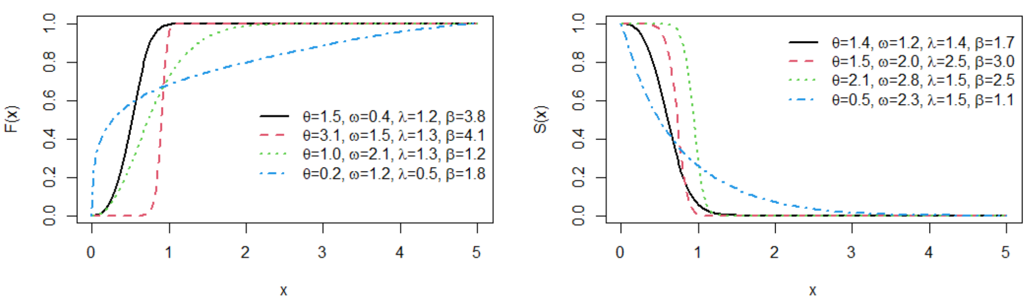

Figure 2: *The CDF and Survival plot for some values of parameters of SGOLL-W*

Figures 1 and 2 present PDF and HR, CDF and hazard plots of the SGOLLW distribution for the specified parameter values. The diagrams in Figure 1 demonstrate that the HR feature of the SGOLLW distribution can exhibits increasing, decreasing, bathtub shapes. While the PDF can have revered-j shape or right-skewed. This can be considered an advantage of the SGOLLW distribution because of its ability to model phenomena with increasing and decreasing shapes or bathtub failure rates, making SGOLLW more versatile for analyzing lifetime data.

3. Results

3.1. simulations Study

The behavior of the maximum likelihood of SGOLL-W for certain parameter values in the first trial (i.e. $\alpha = 0.8, \theta = 2.5, \omega = 3.4, \lambda = 4.5$) and second trial (i.e. $\alpha = 1.5, \theta = 1.3, \omega = 3.8, \lambda = 2.8$) was investigated using a finite sample. The random numbers for the SGOLL-W were generated using the quantile function. For 1000 repeats, samples with sizes of n=20, 50, 100, 250, 500, and 1000 were created. The Means, Bias, and RMSE were then calculated.Table 1 presents the outcomes of the simulation. We conclude that our model yields consistent results when predicting parameters for the mode based on the results of the Monte Carlo simulation.

Table 1: *Result of simulation for different values of parameters*

		$\alpha = 0.8$	$\theta = 2.5$	$\omega = 3.4$	$\lambda = 4.5$	$\alpha = 1.5$	$\theta = 1.3$	$\omega = 3.8$	$\lambda = 2.7$
n=20	Means	0.8415	2.6521	3.2461	4.9523	1.6197	1.3773	3.9759	2.7684
	Bias	0.0415	0.1521	0.2461	0.4523	0.1197	0.0773	0.1759	0.0684
	RMSE	0.2592	1.1345	0.7826	1.2290	0.4204	0.4576	0.4977	0.5142
n=50	Means	0.7953	2.6518	3.2039	4.8429	1.5253	1.3359	3.9630	2.7728
	Bias	-0.0047	0.1518	0.2039	0.3429	0.0253	0.0359	0.1630	0.0728
	RMSE	0.1759	0.9456	0.6222	1.0789	0.2406	0.3157	0.4162	0.3885
n=100	Means	0.7811	2.6235	3.1533	4.7364	1.4928	1.3362	3.9529	2.7533
	Bias	-0.0189	0.1235	0.1533	0.2364	-0.0072	0.0362	0.1529	0.0533
	RMSE	0.1247	0.7014	0.4316	0.8227	0.1652	0.2272	0.3190	0.2669
n=250	Means	0.7756	2.5626	3.1093	4.6981	1.4784	1.3170	3.9331	2.7568
	Bias	-0.0244	0.0626	0.1093	0.1981	-0.0216	0.0170	0.1331	0.0568
	RMSE	0.0969	0.4986	0.3363	0.6252	0.1120	0.1599	0.2783	0.2053
n=500	Means	0.7733	2.5596	3.1050	4.6583	1.4758	1.3119	3.9164	2.7477
	Bias	-0.0267	0.0596	0.1050	0.1583	-0.0242	0.0119	0.1164	0.0477
	RMSE	0.0881	0.3698	0.3023	0.4949	0.0904	0.1168	0.2648	0.1758
n=1000	Means	0.7741	2.5453	3.0949	4.6305	1.4752	1.3056	3.9074	2.7463
	Bias	-0.0259	0.0453	0.0949	0.1305	-0.0248	0.0056	0.1074	0.0463
	RMSE	0.0713	0.2629	0.2457	0.3592	0.0735	0.0837	0.2368	0.1347

3.2. Application

The sub-model were compared with some standard models using the waiting time and bladder tumors Data sets, we used methods of goodness of fit and information criteria, which comprises the log-likelihood function evaluated at the MLEs, the Akaike information criterion (AIC), the Bayesian information criterion (BIC), the corrected Akaike information criterion (CAIC), Hannan Quinn information criterion (HQIC), Anderson-Darling (A), Cramer-von Mises (W), and Kolmogorov-Smirnov (Ks). The competitors model were generalized odd log-logistic weibul, odd log-logistic weibul, topp-leone odd log-logistic weibul. Therefore, the better the fit is the one with smaller values of information criteria. aryal2017topp.

$$AIC = -2L + 2k \tag{40}$$

$$BIC = -2L + klog(n) \tag{41}$$

$$CAIC = -2L + \frac{2kn}{n-k-1} \tag{42}$$

$$HQIC = -2L_{max} + 2klog(log(n)) \tag{43}$$

3.2.1 Application to waiting time data

In this research, we have considered real data that contains 100 observations on minutes waiting time before a client receives the desired service in a bank: the data was used by zeineldin2021generalized and the data is given as:

0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 2.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23, 27, 31.6, 33.1, 38.5

Table 2: *Information criteria measure for the fitted models using waiting time data*

Distribution	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\omega}$	$\hat{\lambda}$	$\hat{LL}$	AIC	CAIC	BIC	HQIC
SGOLLW	2.13	26.68	2.46	0.16	317.00	642.00	642.43	652.42	646.22
GOLLW	66.66	59.78	4.41	0.01	319.27	646.54	646.96	656.96	650.76
OLLW	7.59	0.45	0.21	—	318.04	642.09	642.34	649.91	645.25
TLOLLW	5.88	28.65	O.71	0.09	325.88	659.78	660.20	670.20	663.99

Table 3: *Goodness of fit measure for the fitted models using waiting time data*

Distribution	KS	A	W	P-value
SGOLLW	0.042081	0.15976	0.021594	0.9944
GOLLW	0.0656	0.3507	0.0505	0.7825
OLLW	0.0456	0.2555	0.0365	0.9853
TLOLLW	0.3532	11.9933	2.1270	<0.000

Tables 2 displays the maximum likelihood estimates (MLEs) along with their standard errors and information criteria for four fitted models. These results pertain to the SGOLLW distribution and three competing models. All parameter estimates are significant across the fitted models, with the SGOLLW distribution demonstrating the lowest values in all information criteria. Therefore, the SGOLLW model emerges as a strong alternative to the other fitted models. Table 3 presents the gooness of fit statistics values specifically the Cram©r-von Mises (W*), Anderson"Darling (A*), and Kolmogorov"Smirnov (KS) along with its corresponding P values, for competing models fitted to the waiting time data sets. the SGOLLW distribution achieves the highest P value and the lowest distances for the Kolmogorov"Smirnov (KS), W*, and A* values. Figure 3 displays the empirical PDF and CDF plots for the SGOLLW distribution based on waiting time data. These findings suggest that the SGOLLW model offers a better fit compared to other distributions.

3.2.2 Application to breast tumors data

The second set of real data was obtained at Benha University Hospital in Egypt from June to October 2014. This data represents the ages of 155 patients with breast tumors in the early detection unit for breast cancer. The data are 46, 32, 50, 46, 44, 42, 69, 31, 25, 29, 40, 42, 24, 17, 35, 48, 49, 50, 60, 26, 36, 56, 65, 48, 66, 44, 45, 30, 28, 40, 40, 50, 41, 39, 36, 63, 40, 42, 45, 31, 48, 36, 18, 24, 35, 30, 40, 48, 50, 60, 52, 47, 50, 49, 38, 30, 52, 52, 12, 48, 50, 45, 50, 50, 50, 53, 55, 38, 40, 42, 42, 32, 40, 50, 58, 48, 32, 45, 42, 36, 30, 28, 38, 54, 90, 80, 60, 45, 40, 50, 50, 40, 50, 50, 50, 60, 39, 34, 28,

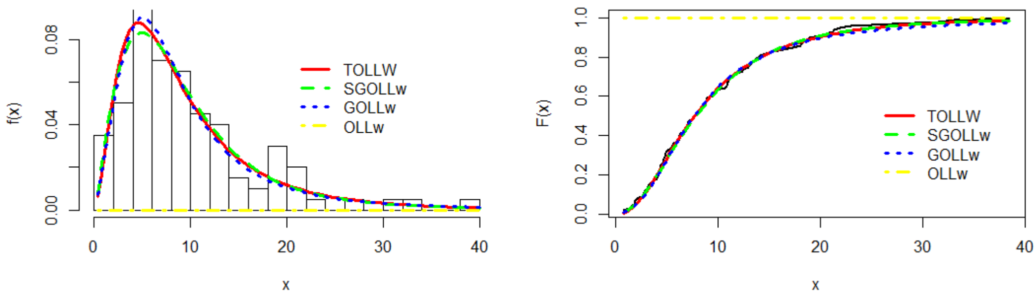

Figure 3: *Estimated pdf and cdf of the SGOLL-W and other distributions for waiting time data*

18, 60, 50, 20, 40, 50, 38, 38, 42, 50, 40, 36, 38, 38, 50, 50, 31, 59, 40, 42, 38, 40, 38, 50, 50, 50, 40, 65, 38, 40, 38, 58, 35, 60, 90, 48, 58, 45, 35, 38, 32, 35, 38, 34, 43, 40, 35, 54, 60, 33, 35, 36, 43, 40, 45, 56´.

Table 4: *Table 4: Information criteria measure for the fitted models using breast tumors data*

Distribution	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\omega}$	$\hat{\lambda}$	$\hat{LL}$	AIC	CAIC	BIC	HQIC
SGOLLW	18.53	90.55	4.15	0.04	599.76	1207.532	1207.79	1219.71	1212.47
GOLLW	55.48	90.37	4.57	0.02	601.62	1211.24	1211.50	1223.41	1216.18
OLLW	0.0509	6.2462	0.3948	—	958.16	1922.31	1922.47	1931.44	1926.02
TLOLLW	0.14	50.14	O.14	1.23	602.59	1213.19	1213.46	1225.37	1218.14

Table 5: *Goodness of fit measure for the fitted models using breast tumors data*

Distribution	KS	A	W	P-value
SGOLLW	0.0801	0.8190	0.1431	0.2874
GOLLW	0.0818	1.0636	0.1758	0.2509
OLLW	0.7211	1.3358	0.2117	<0.001
TLOLLW	0.0917	1.1823	0.1902	0.1474

Table 4 presents the maximum likelihood estimates (MLEs), their standard errors, and information criteria for four fitted models, including the SGOLLW distribution and three competing models. All parameter estimates across the fitted models are significant, with the SGOLLW distribution yielding the lowest values for all information criteria. As a result, the SGOLLW model is identified as a strong alternative to the other fitted models. Table 5 provides the goodness-of-fit statistics, specifically the Cram©r-von Mises (W*), Anderson-Darling (A*), and Kolmogorov-Smirnov (KS) values, along with their corresponding P values for the competing models fitted to the waiting time data sets. The SGOLLW distribution stands out with the highest P value and the lowest distances in the Kolmogorov-Smirnov (KS), W*, and A* statistics. Figure 4 illustrates the empirical PDF and CDF plots for the SGOLLW distribution based on the breast tumors data. These results indicate that the SGOLLW model offers a superior fit compared to the other distributions.

4. Discussion

In this paper, we introduce the Sine generalized odd log-logistic family of distributions. We derive several properties of this distribution, including the quantile, entropy, hazard rate function, survival function, moments, moment generating function, and order statistics. Additionally, we estimate the parameters of the SGOLL-G distribution using maximum likelihood estimation. We examine the generalized Sine odd log-logistic Weibull distribution using two data sets. Through

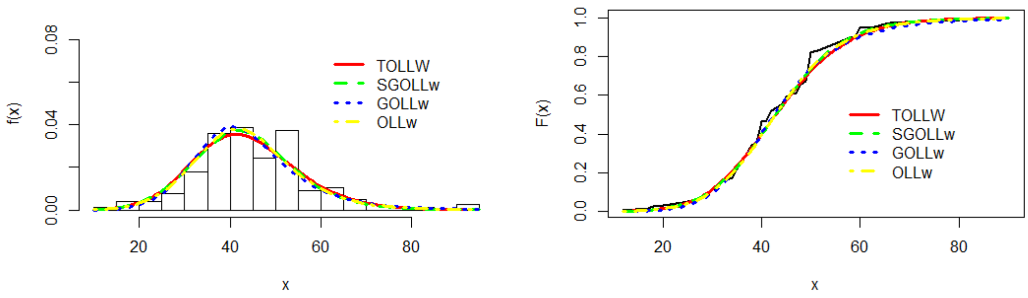

Figure 4: *Estimated pdf and cdf of the SGOLL-W and other distributions for breast tumors data*

analysis and simulation, we demonstrate the performance of the proposed distribution compared to other underlying models. The goodness-of-fit measures and information criteria, such as AIC, CAIC, and BIC, indicate that the SGOLLW distribution outperforms its competitors.

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