SIMULATIONS AND BAYESIAN ESTIMATION OF TRUNCATED EXPONENTIAL LOG-TOPP-LEONE GENERALIZED FAMILY WITH APPLICATION TO SURVIVAL TIME DATA

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Abstract

Due to the requirements for the flexible statistical model to fit the lifetime data, we extended the truncated exponential topp-leone family due to its bounded interval, and introduced a truncated exponential log topp-leone generalized family of distributions. we examine some properties including survival function, hazard rate function, residual lifetime, reverse residual lifetime, moment, moment generating function, Shannon entropy, quantile, and parameter estimation using maximum likelihood, maximum product spacing, and bayesian estimation. Two simulation studies were conducted to investigate the properties (i.e. mean, variance, skewness, and kurtosis), and behavior of the maximum likelihood estimate using mean, bias, and RMSE. Finally, we apply the data on the survival times of breast cancer patients and suggest that the family of the proposed distribution outperforms other standard distributions based on information criteria and goodness of fit.

Keywords: Truncated Exponential Log Topp-Leone-G; Statistical Properties; Estimation; Simulation; Application.

I. Introduction

Lifetime data modeling has become a significant area in many fields, including actuarial science, economics, life science, engineering, business, and industries, among other [11] scientific fields, as data sets get more diverse and complicated [21]. This attracts much literature, which requires an appropriate model for accurate data realization [21]. To overcome such an issue, [22] developed a new distribution of empirical data with J-shaped histograms. It is a bounded support continuous distribution that can be used to simulate a distribution's lifetime. There has been little discussion prior to its discovery by [15], who examined a few of its properties, including moments and central moments. The flexibility of its hazard rate function makes Topp-leone a suitable distribution for modeling lifetime data [14].

In order to extend the existing truncated exponential topp-leone distribution by [6] with a bounded interval, there is need to modify the existing work of [12] on generalized topp-leone distribution using the concept of log transformation on log topp-leone [23] to become the log topp-leone generalized distribution, and incorporate it with truncated exponential as in [6] to provide a distribution with an unbounded interval.

Many researchers have introduced a generalization of distribution in relation to topp-leone including Topp-Leone Generalized Family of Distribution by [12], A New Topp-Leone Generalized Family of Distribution by [10], Topp-Leone Exponentiated-Generalized by [20], Sin Topp-Leone Generalized Distribution by [**?**], Exponentiated Topp-Leone Exponentiated-Generalized Distibution by [18], Frechet Topp-Leone Generalized Distribution by [18], Transmuted Topp-Leone Generalized by [24], New Power Topp-Leone Generated Distribution by [9], Poisson Topp-Leone Generator of Distribution by [13], truncated exponential topp-leone exponential by [6], truncated exponential topp-leone rayleigh by [5], among others.

In this paper, we conducted a Monte Carlo simulation to examine the behavior and consistency of the maximum likelihood estimate on the family of the Truncated exponential log-topp-leone-G introduced in [1] and also in [2]. The bayesian estimation would also be introduced in this paper, where the prior, conditional, and posterior distributions are to be discussed, while some properties were derived in including residual, reverse residual, Shannon entropy, and other methods of estimation parameters (see [1]), while other properties were already derived in [2].

II. Methods

1. Truncated Exponential Log-topp-leone Generalized Family of Distributions

The proposed Truncated exponential Log Topp-leone Generalized Family of distributions is drive from the cdf of truncated exponential distribution in [3] and the cdf of Log Topp-Leone-G distribution (which is an extension to the work of [23]) by integrating the pdf of truncated exponential distribution in equation with limit from 0 to the cdf of log topp-leone generalized family, and is drive as follows.

$$F_{TELTL-G}(y,\beta) = \int_0^{(1-e^{-2H(y,\psi)})^\theta} \frac{\beta e^{-\beta y}}{1-e^{-\beta}} dy = \frac{1-e^{-\beta(1-e^{-2H(y,\psi)})^\theta}}{1-e^{-\beta}} \tag{1}$$

Therefore

$$F(y,\beta,\theta) = \frac{1-e^{-\beta(1-e^{-2H(y,\psi)})^\theta}}{1-e^{-\beta}} \tag{2}$$

And we can find the probability density function by differentiating the Cdf using quotient rule. Let The cumulative distribution and probability density function of the truncated exponential Log Topp-leone-G family are given by

$$f_{TELTL-G}(y,\beta,\theta,\psi) = \frac{2\beta\theta h(y,\psi) e^{-2H(y,\psi)} (1-e^{-2H(y,\psi)})^{\theta-1} e^{-\beta(1-e^{-2H(y,\psi)})^\theta}}{1-e^{-\beta}} \qquad y,\ \theta,\beta > 0 \tag{3}$$

2. Mathematical properties of TELTL-G

In this part, we discussed some mathematical properties of TELTL-G as follows:

2.1 Survival function and Hazard rate function of TELTL-G

The Survival function S(Y) of a TELTL-G as one of the important tools for measuring the failure time of a system is given by

$$S(y) = \frac{e^{-\beta(1-e^{-2H(y,\psi)})^\theta} - e^{-\beta}}{1-e^{-\beta}} \tag{4}$$

The hazard rate function H(Y) of a TELTL-G is given by

$$H(y) = \frac{2\beta\theta g(t,\psi) e^{-2H(y,\psi)} (1-e^{-2H(y,\psi)})^{\theta-1} e^{-\beta(1-e^{-2H(y,\psi)})^\theta}}{e^{-\beta(1-e^{-2H(y,\psi)})^\theta} - e^{-\beta}} \tag{5}$$

2.2 Residual and Reverse residual of TELTL-G

one of the tools with application in acturial science, biometry, risk management, survival analysis is residual and reverse residual life. The residual and reverse residual life denoted by $r_t(y)$ and $\bar{r}_t(y)$ in the following equation.

$$r_t(y) = \frac{S(y+t)}{S(t)} = \frac{e^{-\beta(1-e^{-2H(y+t,\psi)})^{\theta}} - e^{-\beta}}{e^{-\beta(1-e^{-2H(t,\psi)})^{\theta}} - e^{-\beta}} \tag{6}$$

$$\bar{r}_t(y) = \frac{S(y-t)}{S(t)} = \frac{e^{-\beta(1-e^{-2H(y-t,\psi)})^{\theta}} - e^{-\beta}}{e^{-\beta(1-e^{-2H(t,\psi)})^{\theta}} - e^{-\beta}} \tag{7}$$

2.3 Moment and moment generating function

Moments is a crucial part of any statistical study [10]. They may be used to characterize key distributional features and forms, such as dispersion and spread as determined by mean and variance and peakiness of the distribution as determined by kurtosis. They can also be used to look at the symmetry of the distribution's shape as determined by skewness. Using the pdf from (3), the rth moment of a TELTL-Gdistribution is given by,

$$E(y^r) = \mu^r \int_{-\infty}^{\infty} y^r f(y,\beta,\theta,\psi)dy \tag{8}$$

$$\Rightarrow \mu^r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{e=0}^{\infty}\int_0^{\infty} \kappa y^r h(y,\psi)(H(y,\psi))^{j+p}dy \tag{9}$$

$$\Rightarrow \mu^r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{e=0}^{\infty} \kappa\Delta \tag{10}$$

Where

$$\kappa = \binom{\theta-1}{i}\binom{k\theta}{m}\frac{(-1)^{i+j+k+p}2^{i+P+1}(i+1)^j\beta^{k+1}\theta m^p t^e}{j!k!p!e!(1-e^{-\beta})} \tag{11}$$

And

$$\Delta = \int_0^{\infty} y^r h(y,\psi)(H(y,\psi))^{j+p}dy \tag{12}$$

The MGF of the random variable that follows TELTL-G having pdf in equation (3) is drive as.

$$E(e^{ty}) = M_y(t) = \int_{-\infty}^{\infty} e^{ty} f(y,\beta,\theta,\psi)dy \tag{13}$$

$$M_y(t) = \int_0^{\infty} \tau y^{e+1} h(y,\psi)(H(y,\psi))^{j+p}dy \tag{14}$$

$$\Rightarrow M_y(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{e=0}^{\infty} \tau\varrho \tag{15}$$

where

$$\tau = \binom{\theta-1}{i}\binom{k\theta}{m}\frac{(-1)^{i+j+k+p}2^{i+P+1}(i+1)^j\beta^{k+1}\theta m^p t^e}{j!k!p!e!(1-e^{-\beta})} \tag{16}$$

And

$$\varrho = \int_0^{\infty} y^{e+1} h(y,\psi)(H(y,\psi))^{j+p}dy \tag{17}$$

2.4 Quantile function and Shannon Entropy

The Quantile of TELTL-G is derived as,

$$\Rightarrow y_u = -\frac{1}{2}\ln\left\{1-\left[\frac{-log(1-u(1-e^{-\beta}))}{\beta}\right]^{\frac{1}{\beta}}\right\} \tag{18}$$

Shannon entropy is a concept from information theory, introduced by [19]. It measures the amount of uncertainty or randomness in a probability distribution. The Shannon entropy is calculated as

$$\gamma_y = E(-log(f(y))) \tag{19}$$

Implies that

$$log(f(y)) = log\left[\frac{2\beta\theta h(y,\psi)e^{-2H(y,\psi)}(1-e^{-2H(y,\psi)})^{\theta-1}e^{-\beta(1-e^{-2H(y,\psi)})^{\theta}}}{1-e^{-\beta}}\right] \tag{20}$$

$$= log\frac{2\beta\theta}{1-e^{-\beta}} + log\left(\frac{h(y,\psi)e^{-2H(y,\psi)}(1-e^{-2H(y,\psi)})^{\theta-1}e^{-\beta(1-e^{-2H(y,\psi)})^{\theta}}}{1-e^{-\beta}}\right) \tag{21}$$

$$= log\frac{2\beta\theta}{1-e^{-\beta}} + loge^{-2H(y,\psi)} + log\left(h(y,\psi)(1-e^{-2H(y,\psi)})^{\theta-1}\right) - \beta(1-e^{-2H(y,\psi)})^{\theta} \tag{22}$$

$$= log\frac{2\beta\theta}{1-e^{-\beta}} - 2H(y,\psi) + log(h(y,\psi)) + log(1-e^{-2H(y,\psi)})^{\theta-1} - \beta(1-e^{-2H(y,\psi)})^{\theta} \tag{23}$$

$$E(-log(f(y))) = E\left[-log\tfrac{2\beta\theta}{1-e^{-\beta}}\right] + 2E\left[H(y,\psi)\right] - E\left[log(h(y,\psi))\right] - E\left[log(1-e^{-2H(y,\psi)})^{\theta-1}\right] + \beta E\left[(1-e^{-2H(y,\psi)})^{\theta}\right] \tag{24}$$

2.5 Order statistics

Let $y_1, y_2..., y_n$ be a random sample from the TELTL-G distribution and let y(1), . . . , y(n) be the corresponding order statistics. The pdf of nth order statistic can be written as

$$f_{(i,n)}(y) = \frac{n!(-1)^j}{(i-1)(n-i-j)!j!}\sum_{j=0}^{n-i}\binom{n-i}{j}f(y)[F(y)]^{i+j-1} \tag{25}$$

$$f_{(i,n)}(y) = \sum_{j=0}^{n-i}\binom{n-i}{j}\frac{n!(-1)^j\eta(1-e^{-\beta(1-e^{-2H(y,\psi)})^{\theta}})^{j+i-1}}{(i-1)(n-i-j)!j!(1-e^{-\beta})^{j+i}} \tag{26}$$

where $\eta = 2\beta\theta h(y,\psi)e^{-2H(y,\psi)}(1-e^{-2H(y,\psi)})^{\theta-1}e^{-\beta(1-e^{-2H(y,\psi)})^{\theta}}$

3. Estimation of Parameters

3.1 Maximum likelihood estimation (MLE)

Let $y_1, y_2, ..., y_n$ be a random sample from the TELTL-G family of distribution with pdf in equation (3) with $\bar{\omega} = (\beta, \theta, \psi)$, the TELTL-G's n sample log-likelihood is drive as:

$$l = logl(y/\bar{\omega}) = log\prod_{i=1}^{n}f(y/\bar{\omega}) \tag{27}$$

$$l(\bar{\omega}) = \prod_{i=1}^{n} \frac{2\beta\theta h(y,\psi)e^{-2H(y,\psi)}(1-e^{-2H(y,\psi)})^{\theta-1}e^{-\beta(1-e^{-2H(y,\psi)})^{\theta}}}{1-e^{-\beta}} \tag{28}$$

$$l(\bar{\omega}) = nlog(2) + nlog(\beta) + nlog(\theta) + \textstyle\sum_{i=1}^{n} logh(y,\psi) - 2\sum_{i-1}^{n} H(y,\psi) +$$

$$(n\theta - n)log(1-e^{-2H(y,\psi)}) - \beta\sum_{i=1}^{n}(1-e^{-2H(y,\psi)})^{\theta} + nlog(1) - nlog(1-e^{-\beta}) \tag{29}$$

By differentiating the log likelihood with respect to β, θ, and ψ, we have:

$$\frac{dl}{d\beta} = \frac{n}{\beta} - \frac{ne^{-\beta}}{(1-e^{-\beta})} - \sum_{i=1}^{n}(1-e^{-2H(y,\psi)})^{\theta} \tag{30}$$

$$\frac{dl}{d\theta} = \frac{n}{\theta} + nlog(1-e^{-2H(y,\psi)}) - \theta\beta\sum_{i=1}^{n}\left\{1-e^{-2H(y,\psi)}\right\}^{\theta-1} \tag{31}$$

$$\frac{dl}{d\psi} = \textstyle\sum_{i=1}^{n} \frac{dh(y,\psi)/d\psi}{h(y,\psi)} + \frac{2(n\theta-n)h(y,\psi)e^{-2H(y,\psi)}}{(1-e^{-2H(y,\psi)})} - 2\sum_{i=1}^{n} h(y,\psi) -$$

$$2\theta\beta\sum_{i=1}^{n}\left\{1-e^{-2H(y,\psi)}\right\}^{\theta-1} h(y,\psi)e^{-2H(y,\psi)} \tag{32}$$

Where $\frac{dH(y,\psi)}{d\psi} = h(y,\psi)$

The 3x3 observed information matrix $J(\omega)$ will be obtain for the interval estimation of β,θ, and ψ and test of hypothesis for the parameters.

$$J(\omega) = \begin{pmatrix} J_{\beta\beta}(\omega) & J_{\beta\theta}(\omega) & J_{\beta\psi}(\omega) \\ J_{\theta\beta}(\omega) & J_{\theta\theta}(\omega) & J_{\beta\psi}(\omega) \\ J_{\psi\beta}(\omega) & J_{\psi\theta}(\omega) & J_{\psi\psi}(\omega) \end{pmatrix} \tag{33}$$

Where the element are

$$J_{\beta\beta} = \frac{-n}{\beta^2} - \frac{e^{-\beta}}{(1-e^{-\beta})^2} \tag{34}$$

$$J_{\theta\theta} = \frac{-n}{\theta^2} - \beta\sum(1-e^{-2H(y,\psi)})^{\theta-1} - \beta\theta(\theta-1)\sum(1-e^{-2H(y,\psi)})^{\theta-1} \tag{35}$$

$$J_{\psi\psi} = \frac{h(y,\psi)h''(y,\psi)-(h'(y,\psi))^2}{(h(y,\psi))^2} + 2(n\theta-n)\frac{d}{d\psi}\left(\frac{h(y,\psi)e^{-2H(y,\psi)}}{1-e^{-2H(y,\psi)}} - 2\sum h'(y,\psi)\right) -$$

$$2\theta\beta\frac{d}{d\psi}\left(\sum(1-e^{-2H(y,\psi)})^{\theta-1}h(y,\psi)e^{-2H(y,\psi)}\right) \tag{36}$$

$$J_{\beta\theta} = -\sum(1-e^{-2H(y,\psi)})^{\theta-1}log(1-e^{-2H(y,\psi)}) \tag{37}$$

$$J_{\beta\psi} = -2\theta h(y,\psi)e^{-2H(y,\psi)}\sum(1-e^{-2H(y,\psi)})^{\theta-1} \tag{38}$$

$$J_{\theta\psi} = \frac{-2nh(y,\psi)e^{-2H(y,\psi)}}{1-e^{-2H(y,\psi)}} - 2\beta\theta(\theta-1)\sum h(y,\psi)e^{-2H(y,\psi)}(1-e^{-2H(y,\psi)})^{\theta-2} \tag{39}$$

3.2 Maximum Product Spacing

The method of maximum product spacing (MPS) is a technique used in statistics and data analysis to estimate the parameters of a distribution. It is a alternative method to maximum likelihood estimation (MLE) and method of moments. In MPS, the goal is to find the parameter values that maximize the product of the spacings between the order statistics of the data. Order statistics are the values in the data set arranged in increasing order. The MPS estimate can be obtained by maximizing the geometric mean of the spacings between the order statistics, rather than the product of the spacings.

$$GM = \left(\prod_{j=1}^{n+1} K_j \right)^{\frac{1}{n+1}} \qquad j = 1, 2, 3...., n+1 \tag{40}$$

Where the jth difference K_j is define as

$$K_j = \int_{y(j-1)}^{y(j)} f(y, \beta, \theta, \psi) dy \tag{41}$$

Where $f(y(0), \beta, \theta, \psi) = 0$ and $f(y(n+1), \beta, \theta, \psi) = 1$. Therefore, the mps of $\beta, \theta, and \psi$ are the result of maximizing the GM of difference.

$$logGM = \frac{1}{n+1} \sum_{j=1}^{n+1} log \left[H(y(j); \beta, \theta, \psi) - H(y(j-1); \beta, \theta, \psi) \right] \tag{42}$$

$$\frac{dlogGM}{d\beta} = \frac{1}{n+1} \sum_{j=1}^{n+1} \frac{d \left[H(y(j); \beta, \theta, \psi) - H(y(j-1); \beta, \theta, \psi) \right] / d\beta}{\left[H(y(j); \beta, \theta, \psi) - H(y(j-1); \beta, \theta, \psi) \right]} \tag{43}$$

$$\frac{dlogGM}{d\theta} = \frac{1}{n+1} \sum_{j=1}^{n+1} \frac{d \left[H(y(j); \beta, \theta, \psi) - H(y(j-1); \beta, \theta, \psi) \right] / d\theta}{\left[H(y(j); \beta, \theta, \psi) - H(y(j-1); \beta, \theta, \psi) \right]} \tag{44}$$

$$\frac{dlogGM}{d\psi} = \frac{1}{n+1} \sum_{j=1}^{n+1} \frac{d \left[H(y(j); \beta, \theta, \psi) - H(y(j-1); \beta, \theta, \psi) \right] / d\psi}{\left[H(y(j); \beta, \theta, \psi) - H(y(j-1); \beta, \theta, \psi) \right]} \tag{45}$$

3.3 Bayesian Estimation of TELTL-G

Supposed that $\beta \sim \Gamma(a, b)$, $\theta \sim \Gamma(c, d)$, and $\psi \sim \Gamma(m, l)$ respectively, where a,b,c,d,l,m are positive constant. The Gamma prior of β, θ, ψ take the forms

$$\tau_1(\beta) = \frac{b^a}{\Gamma(a)} \beta^{a-1} e^{-b\beta} \qquad \beta, a, b > 0 \tag{46}$$

$$\tau_2(\theta) = \frac{d^c}{\Gamma(c)} \theta^{c-1} e^{-d\theta} \qquad \theta, c, d > 0 \tag{47}$$

$$\tau_3(\psi) = \frac{l^m}{\Gamma(m)} \psi^{m-1} e^{-l\psi} \qquad \psi, m, l > 0 \tag{48}$$

The joint density function were given based on conditional distribution of $\beta, \theta, and \psi$ given by

$$f(\beta/\theta, \psi, y) \propto \beta^{a+n-1} \prod_{i=1}^{n} (2h(y, \psi)(1 - e^{-\beta})^{-1} (1 - e^{-2H(y,\psi)})^{\theta}) e^{-b\beta - 2\sum H(y,\psi) - \beta \sum (1 - e^{-2H(y,\psi)})^{\theta}} \tag{49}$$

$$f(\theta/\beta, \psi, y) \propto \theta^{c+n-1} \prod_{i=1}^{n} (2h(y, \psi)(1 - e^{-\beta})^{-1} ((1 - e^{-2H(y,\psi)})^{\theta}) e^{-d\theta - 2\sum H(y,\psi) - \beta \sum (1 - e^{-2H(y,\psi)})^{\theta}} \tag{50}$$

$$f(\psi/\beta, \theta, y) \propto \psi^{m-1} \prod_{i=1}^{n} (2h(y, \psi)(1 - e^{-\beta})^{-1} (1 - e^{-2H(y,\psi)})^{\theta}) e^{-l\psi - 2\sum H(y,\psi) - \beta \sum (1 - e^{-2H(y,\psi)})^{\theta}} \tag{51}$$

And the posterior distribution is given by

$$\tau * (\beta, \theta, \psi | y) = \propto \tau(\beta, \theta, \psi) \prod_{i=1}^{n} f(y; \beta, \theta, \psi) \tag{52}$$

$$f(\omega/y) \propto \beta^{a+n-1} \theta^{c+n-1} \psi^{m-1} \prod_{i=1}^{n} \left(2h(y, \psi)(1 - e^{-\beta})^{-1} (1 - e^{-2H(y,\psi)})^{\theta} \right) \times$$
$$e^{-b\beta - d\theta - l\psi - 2\sum H(y,\psi) - \beta \sum (1 - e^{-2H(y,\psi)})^{\theta}} \tag{53}$$

4. Sub-model of TELTL-G

The TELTL-G family's unique sub-models, the Truncated exponential log topp-leone Pareto Distribution (TELTL-PD) and the Truncated exponential log topp-leone lomax Distribution (TELTL-LD), are addressed in this section.

4.1 Truncated exponential log topp-leone pareto distribution

Let $H(y;\psi)$ be cdf of the pareto random variable given by $H(y;\psi) = 1 - (\frac{\omega}{y})^{\alpha}, \quad y, \alpha, \omega > 0$, and $h(y;\psi) = \frac{\alpha\omega^{\alpha}}{y^{\alpha+1}}$. Then, the cdf, pdf, and quantile of the TELTL-P distribution is given as,

$$F(y,\beta,\theta,\alpha,\omega) = \frac{1 - e^{-\beta(1-e^{-2(1-(\frac{\omega}{y})^{\alpha})})^{\theta}}}{1 - e^{-\beta}} \tag{54}$$

$$f(y,\beta,\theta,\alpha,\omega) = \frac{2\alpha\beta\theta\omega^{\alpha}y^{-\alpha-1}(1-e^{-2(1-(\frac{\omega}{y})^{\alpha})})^{\theta-1}e^{-2(1-(\frac{\omega}{y})^{\alpha})}e^{-\beta(1-e^{-2(1-(\frac{\omega}{y})^{\alpha})})^{\theta}}}{1 - e^{-\beta}} \quad y,\theta,\beta,\alpha,\omega > 0 \tag{55}$$

$$y_u = \left(\frac{1}{\omega^{\alpha}}\left(1 + \frac{1}{2}log\left(1 - \left(\frac{-log(1-u(1-e^{-\beta}))}{\beta}\right)\right)\right)^{\frac{1}{\theta}}\right)^{\frac{-1}{\alpha}} \tag{56}$$

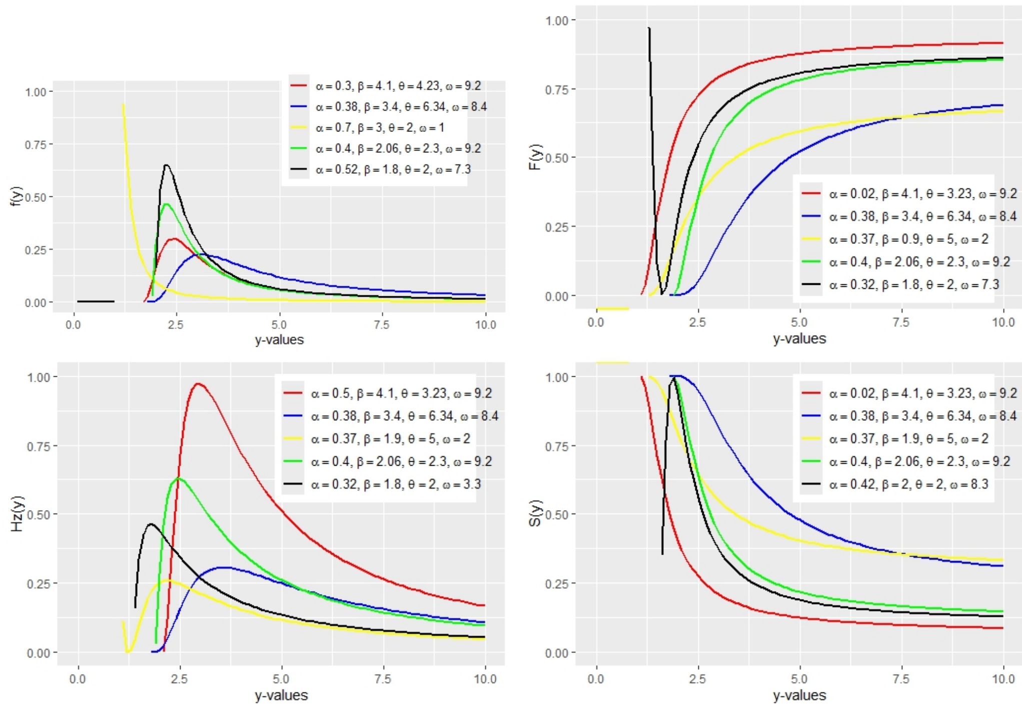

Figure 1: *CDF, PDF, survival and Hazard plot of TELTL-Pareto for values of parameters*

4.2 Truncated exponential log top-leone lomax Distribution

Let $H(y;\delta,\sigma)$ be cdf of the lomax random variable given by $H(y;\delta,\sigma) = 1 - (\frac{\delta}{\delta+y})^{\sigma}, \quad y,\delta,\sigma > 0$, and $h(y;\delta,\sigma) = \frac{\sigma\delta^{\sigma}}{(\delta+y)^{\sigma}}$. Then, the cdf, pdf, and quantile of the TELTL-L distribution is given as,

$$F(y,\beta,\theta,\delta,\sigma) = \frac{1 - e^{-\beta(1-e^{-2(1-(\frac{\delta}{\delta+y})^{\sigma}))})^{\theta}}}{1 - e^{-\beta}} \tag{57}$$

$$f(y,\beta,\theta,\delta,\sigma) = \frac{2\sigma\beta\theta\delta^{\sigma}\delta(1-e^{-2(1-(\frac{\delta}{\delta+y})^{\sigma})})^{\theta-1}e^{-2(1-(\frac{\delta}{\delta+y})^{\sigma})}e^{-\beta(1-e^{-2(1-(\frac{\delta}{\delta+y})^{\sigma})})^{\theta}}}{1-e^{-\beta}} \quad y,\theta,\beta,\delta,\sigma > 0 \tag{58}$$

$$y_u = \delta - \left(\frac{1}{\delta^{\sigma}}\left(1+\frac{1}{2}log\left(1-\left(\frac{-log(1-u(1-e^{-\beta}))}{\beta}\right)\right)\right)^{\frac{1}{\theta}}\right)^{\frac{-1}{\sigma}} \tag{59}$$

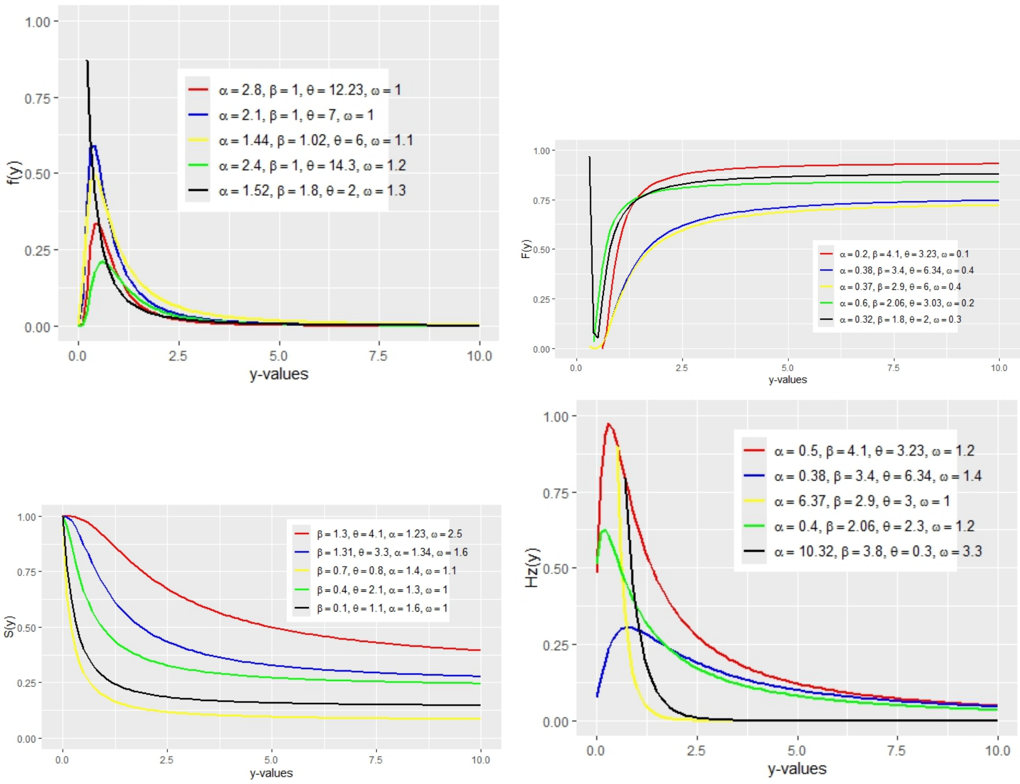

Figure 2: *CDF, PDF, survival and Hazard rate plot of TELTL-Lomax for some values of parameters*

III. Results

1. Simulation study

The simulations on TELTL-Pr and TELTL-Lmx were conducted to examine the behavior of the distribution. The consistency of the maximum likelihood of TELTL-Pr and TELTL-Lmx for set values (i.e. $\alpha = 1.9, \beta = 2.2, \theta = 3.3, \omega = 4.1$), while for TELTL-Lmx with four parameters ($\beta = 3.6, \theta = 1.2, \delta = 2.5, \sigma == 4.1$) were investigated using a finite sample of n=20, 50, 150, 250, 500, and 1000 were created. The random numbers for the TELTL-Pr were generated using the quantile function. For 1000 repeated samples. The Means, Bias, and RMSE were then calculated. Table 2 and 3 presents the outcomes of the simulation. It is concluded that the family member TELTL-G yields consistent results when predicting parameters for the mode based on the results of the monte-carlo simulation. Likewise, some properties including the mean, variance, skewness, and kurtosis were obtained from the simulation for a specific values of parameters as in Table 1. The bias and root mean square error (RMSE) are given by the following equation.

$$Bias(\hat{\theta}) = \frac{\sum_{i=1}^{N}\hat{\theta}_i}{N} - \theta \tag{60}$$

$$RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2}{N}} \tag{61}$$

Table 1: *Properties from Simulation result*

Distribution	$\alpha, \beta, \theta, \omega$	Mean	Variance	Skeweness	Kurtosis
TPr	(1, 2, 2, 3)	3.057956	2.479738e-03	1.0395213	3.329671
TLmx	(1, 2, 1, 3)	-5.57991e-02	2.270049e-03	-1.03586576	3.369155
TETL	(1, 2, 2)	-0.643513	0.07888371	0.7992454	2.510671
TGPr	(1, 2, 2)	0.5286839	0.04769423	-0.1457797	1.92661
NHTLmx	(1, 2, 2)	0.2232752	0.01735384	0.9802952	4.097517
RP	(1, 2, 2)	1.804450	4.221214e+01	-0.56238231	2.394527

Table 2: *Simulation result for first set of parameters of TELTL-Pr*

Sample	Properties	$\alpha = 1.9$	$\beta = 2.2$	$\theta = 3.3$	$\omega = 4.1$
n=20	Means	2.3348	4.6704	2.3998	1.3606
	Bias	0.4348	2.4704	-0.9002	-2.7394
	RMSE	0.6240	2.5323	0.9787	2.7587
n=50	Means	2.2570	4.5514	2.5102	1.2743
	Bias	0.3570	2.3514	-0.7898	-2.8257
	RMSE	0.4916	2.3813	0.8390	2.8346
n=150	Means	2.1654	4.5406	2.5857	1.2525
	Bias	0.2654	2.3406	-0.7143	-2.8475
	RMSE	0.3382	2.3540	0.7318	2.8510
n=250	Means	2.1622	4.5524	2.5897	1.2454
	Bias	0.2622	2.3524	-0.7103	-2.8546
	RMSE	0.3227	2.3672	0.7205	2.8580
n=500	Means	2.1495	4.6217	2.5829	1.2478
	Bias	0.2495	2.4217	-0.7171	-2.8522
	RMSE	0.3171	2.4407	0.7241	2.8558
n=1000	Means	0.7741	2.5453	3.0949	4.6305
	Bias	-0.0259	0.0453	0.0949	0.1305
	RMSE	0.0713	0.2629	0.2457	0.3592

2. Application to survival times of breast cancer patients data

The dataset was collected from 1929 to 1938, which represents the survival times of breast cancer patients. The data was used by [16]. The observations are given as 0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0,

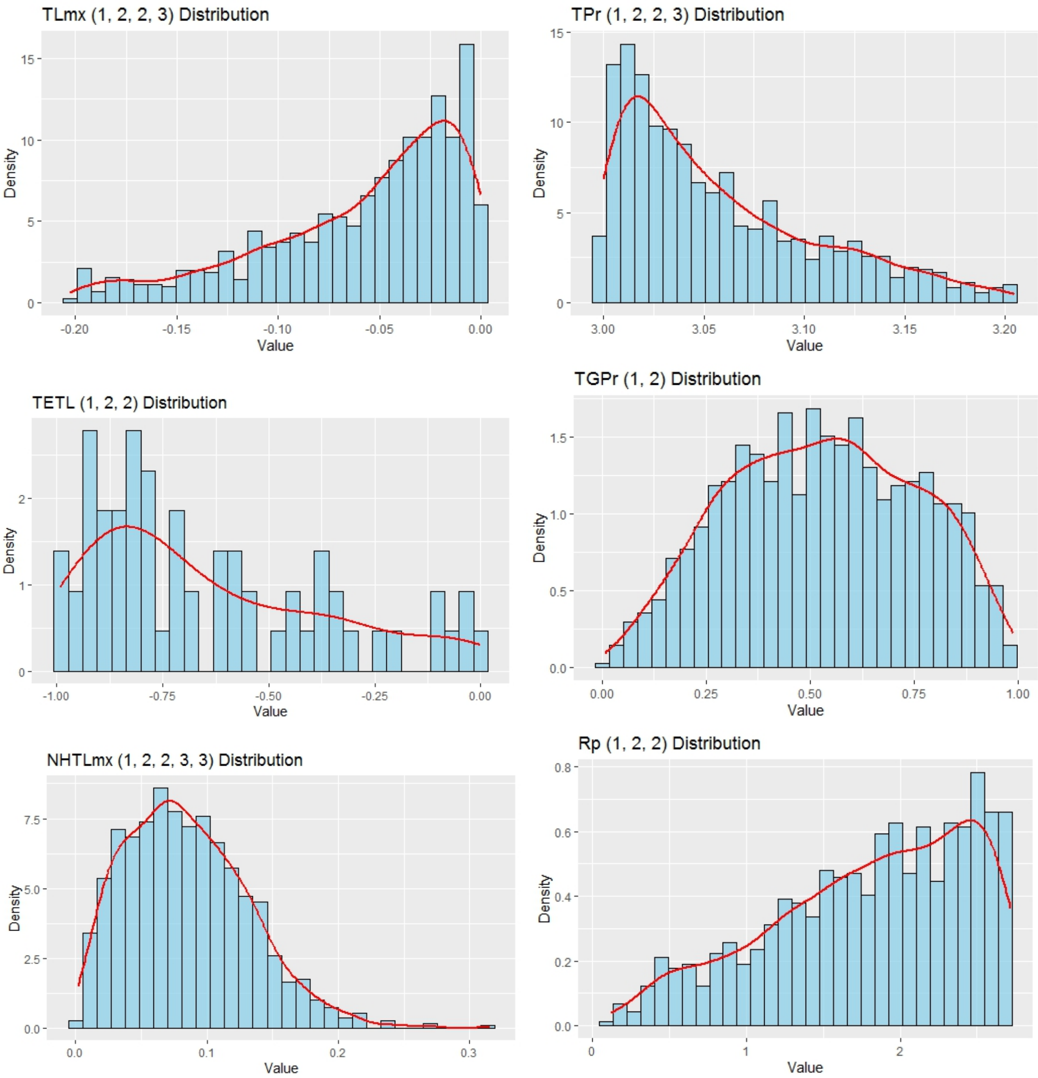

Figure 3: *Skewness and kurtosis for proposed and existing models*

Table 3: *Simulation result for second set of parameters of TELTL-Lmx*

Sample	Properties	$\beta = 3.6$	$\theta = 1.2$	$\delta = 2.5$	$\sigma == 4.1$
n=20	Means	5.8301	4.1297	2.2794	1.9739
	Bias	2.2301	2.9297	-0.2206	-2.1261
	RMSE	2.2301	2.9297	-0.2206	-2.1261
n=50	Means	2.2301	2.9297	-0.2206	-2.1261
	Bias	1.9216	2.6165	-0.2111	-2.2393
	RMSE	2.2505	2.9445	0.2468	2.2915
n=150	Means	5.1789	3.4406	2.2681	1.8987
	Bias	1.5789	2.2406	-0.2319	-2.2013
	RMSE	1.7777	2.4579	0.2542	2.2292
n=250	Means	5.0407	3.2331	2.2515	1.9199
	Bias	1.4407	2.0331	-0.2485	-2.1801
	RMSE	1.6257	2.2035	0.2679	2.2058
n=500	Means	4.9989	3.0595	2.2294	1.8953
	Bias	1.3989	1.8595	-0.2706	-2.2047
	RMSE	1.5425	1.9954	0.2842	2.2245
n=1000	Means	5.0224	2.9238	2.2215	1.8293
	Bias	1.4224	1.7238	-0.2785	-2.2707
	RMSE	1.5313	1.8206	0.2889	2.2893

89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0,126.0, 127.0, 129.0, 129.0, 139.0, 154.0.

Calculations were made to compare the fitted models using the goodness-of-fit metrics, which include the kolmogrov smirnov, crimer von-mises, and Andason darling. The information log-likelihood and the information criteria were also examine using Akaike information criteria (AIC), Bayesian information criteria (BIC), and Akaike information corrected criteria (AICc). The model with smaller value of information criteria is the model with best fit [8].

Table 4: *Competitors Distributions*

Distributions	Author(s)	Year	Citation
TETLE	Al-noor and Hilal	2021	[?]
TGPr	Al-quraishy et al	2022	[?]
NHTLmx	Reyad et al	2019	[?]
Rp	Al-kadim and Muhammad	2018	[?]

Table 5: *MLE of the parameter(s) using Survival time data*

Distributions	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\gamma}$
TPr	0.92342	1.1666	7.3870	0.3609	-
TLmx	0.06907	1.643	41.81	19.84	-
TETL	-7.8510	0.01562	0.251287	-	-
TGPr	1.7716	0.5148	-	-	-
NHTLmx	20.002471	16.485334	0.123569	0.02324	0.3241
Rp	672.80819	-80.6895	0.0044	-	-

Table 6: *Information Criteria for the fitted models*

Distributions	$\hat{LL}$	AIC	AICc	BIC
TPr	-184.1018	374.2036	374.4087	392.5910
TLmx	-194.5029	397.0059	397.3507	408.1891
TETLE	-578.8903	1163.781	1163.986	1172.168
TGPr	-152.1718	411.2006	412.8517	418.1152
NHTLmx	-491.124	924.418	923.181	931.201
RP	-989.0158	1972.032	1971.826	1979.644

Table 7: *Goodness of fit test for the fitted models*

Distributions	KS	A	W	P-value
TPr	1.0	0.3242	0.34343	<0.00
TLmx	0.96138	5.6062	0.97944	<0.00
TETL	0.06105	0.38433	0.05142	0.7579
TGPr	0.10742	5.6358	0.10444	0.2132
NHTLmx	0.28944	1.5209	0.26993	<0.00
RP	0.11552	0.41323	0.058538	0.0914

IV. Discussion

The bayesian estimate derived in Section 2.7 gives the prior, conditional, and posterior distributions for the parameters of TELTL-G, which is an alternative estimation method of parameters. The first simulation conducted was to observe the behavior of the distribution, and it shows that the accuracy of the estimate is better as the sample size increases for the TELTL-Pr and for the TELTL-LMx; the accuracy is decreasing as the sample size is increasing. Likewise, Table 2 shows the properties for the family of TELTL-G, which includes the mean, variance, skewness, and kurtosis. As illustrated in the table, the TELTL-P is more skewed positively than competitors distributions with a value of 1.0395213, but the kurtosis of NHTLmx (4.097517) is greater than that of the proposed distribution. The TELTL-Lmx is negatively skewed with a value of -1.0358657 and a kurtosis of 3.369155, and these show that the proposed distributions have more skewness and kurtosis than the existing TETL-E distribution with a skewness of 0.799245 and a kurtosis of 2.520671. The plots illustrated in Figure 3 show the shape of the proposed and competitors distributions, while Figures 1 and 2 are the pdf, cdf, survival, and hazard rate functions for the different values of parameters of the TELTL-Pr and TELTL-Lmx distributions, respectively.

Conclusion

In this paper, we introduce the simulation and bayesian estimation of a truncated exponential log topp-leone generalized family of distributions with additional properties and estimation methods, since the properties and other characteristics were examined in our previous study, see [1]. The family of the proposed distribution demonstrates outstanding performance, and shows how it is good and flexible in term of fit compared to other standard distributions. Finally, we suggested that the TELTL-G is an alternative distributions in modeling heavy tail or skewed data set.

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Ethical Consideration

No any Ethical clearance required before the commencement of these research.

Data availability statement

All data used in this work can be found within the manuscript.

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