MARSHALL-OLKIN EXPONENTIATED NADARAJAH HAGHIGHI DISTRIBUTION AND ITS APPLICATIONS

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Abstract

In this paper, we introduce a new generalization of exponentiated Nadarajah Haghighi distribution, namely Marshall-Olkin exponentiated Nadarajah Haghighi (MOENH) distribution and study its properties. The stress-strength parameter estimation is also taken into account. Characterizations of the new distribution are obtained. The unknown parameters of the distribution are estimated using the maximum likelihood method. It is established how important this distribution is to the research of the minification process. Simulation studies are done, and sample path properties are explored. A real data set is fitted to the new distribution to demonstrate the model's adaptability and effectiveness.

Keywords: Marshall-Olkin family, Exponentiated Nadarajah Haghighi distribution, Stress-Strength reliability, Maximum likelihood, Minification process.

1. INTRODUCTION

Marshall and Olkin introduced a new family of distributions in 1997 with one additional parameter α, known as the Marshall-Olkin Family of Distributions. The distribution function of the family is given by,

$$G(x) = \frac{F(x)}{\alpha + (1-\alpha)F(x)}, -\infty < x < \infty, \alpha > 0 \tag{1}$$

Here, $F(x)$ represents the distribution function of a random variable X. The Marshall-Olkin family of distributions is a class of continuous probability distributions that are used to model lifetime data. It has the advantage of being flexible and able to fit a wide range of data sets. Cordeiro and Lemonte [1] discussed some mathematical properties of the Marshall-Olkin extended Weibull distribution and estimation of the model parameters by the maximum likelihood method. Ristić and Kundu [14] introduced a third shape parameter to the two-parameter generalized exponential distribution, adopted from the Marshall-Olkin method so that the hazard function of the proposed model can have all the four major shapes, namely increasing, decreasing, bathtub or inverted bathtub types. Recent developments in the Marshall-Olkin family of distributions can be given in George and Thobias [4], Gillariose and Tomy [5], etc.

The exponential distribution is a versatile and widely used probability distribution that has many important applications in various fields, particularly in modeling time-related events. The generalized exponential distribution, sometimes referred to as the exponentiated exponential (EE) distribution, failure rate function that might be either increasing, decreasing, or constant, see Gupta, et al. [8]. According to Gupta and Kundu [9], the failure rate function of the EE distribution has similar behaviour to that of the gamma distribution and in many cases, it can be utilized as an alternate distribution to the gamma and Weibull distributions. Nadarajah and Haghighi [13] proposed a new generalization of the exponential distribution as an alternative to the gamma, Weibull, and EE distributions. Lemonte [11] defined a new three-parameter exponential-type distribution family, exponentiated Nadarajah Haghighi, that can be used to model survival data and reliability issues. A three-parameter distribution called Exponentiated Nadarajah and Haghighi (ENH) with cdf is given in (2)

$$F(x) = \begin{cases} (1 - e^{1-(1+\lambda x)^{\gamma}})^{\beta} & ; x > 0, \gamma, \beta, \lambda > 0 \\ 0 & ; \text{Otherwise}, \end{cases} \tag{2}$$

where the parameters γ, β control the shape of the distribution and parameter λ is the scale parameter. The major goal of the present study is to create a new model of the four-parameter distribution, in the expectation that, in some instances, the new distribution will "fit better" than the exponential, ENH, Nadarajah-Haghighi (NH), and Marshall-Olkin-Nadarajah-Haghighi Distribution (MONH) distributions.

The rest of the paper is organized as follows: In Section 2, we propose a new generalization of ENH distribution, namely MOENH distribution. Various structural properties of the MOENH distribution, such as moments, quantile function, order statistics and stress-strength reliability are studied in Section 3. Characterizations of MOENH distribution are obtained in Section 4. In Section 5, we study the estimation of parameters of the MOENH distribution using the method of Maximum Likelihood. In Section 6, we discuss the MOENH Minification process and the corresponding sample paths. In Section 7, we have fitted the model to real-life data to show the flexibility of the new distribution. Concluding remarks are presented in Section 8.

2. MOENH distribution

In this section, we discuss the MOENH distribution introduced by using (1) and (2) we have the distribution function of MOENH distribution as follows:

$$G(x) = \begin{cases} \frac{\left(1-e^{1-(1+\lambda x)^{\gamma}}\right)^{\beta}}{\alpha+(1-\alpha)\left(1-e^{1-(1+\lambda x)^{\gamma}}\right)^{\beta}} & ; x > 0, \alpha, \gamma, \beta, \lambda > 0 \\ 0 & ; \text{ Otherwise}. \end{cases} \tag{3}$$

where the parameters β, γ are shapes of the distribution, and α, λ are the scale parameter. If $\beta = 1$, the $MOENH$ distribution reduces to $MONH$ distribution. We have the NH distribution when $\alpha = 1$, $\beta = 1$. For $\alpha = 1$, $\beta = 1$, $\gamma = 1$, we obtain the exponential distribution. The probability density function of MOENH distribution is,

$$g(x) = \begin{cases} \frac{\alpha\beta\gamma\lambda(1+\lambda x)^{\gamma-1}\left(e^{1-(1+\lambda x)^{\gamma}}\right)}{\left(1-e^{1-(1+\lambda x)^{\gamma}}\right)^{1-\beta}\left(\alpha+(1-\alpha)\left(1-e^{1-(1+\lambda x)^{\gamma}}\right)^{\beta}\right)^{2}} & ; x > 0, \alpha, \gamma, \beta, \lambda > 0 \\ 0 & ; \text{ Otherwise}. \end{cases} \tag{4}$$

The survival function and failure rate function of MOENH distribution are respectively as,

$$S(x) = 1 - \frac{\left(1 - e^{1-(1+\lambda x)^{\gamma}}\right)^{\beta}}{\alpha + (1-\alpha)\left(1 - e^{1-(1+\lambda x)^{\gamma}}\right)^{\beta}} = \frac{\alpha - \alpha(1 - e^{1-(1+\lambda x)^{\gamma}})^{\beta}}{\alpha + (1-\alpha)(1 - e^{1-(1+\lambda x)^{\gamma}})^{\beta}}, \tag{5}$$

$$\begin{aligned} h(x) = & \frac{\alpha\beta\gamma\lambda(1+\lambda x)^{\gamma-1}e^{1-(1+\lambda x)^{\gamma}}}{\left[\left(1 - e^{1-(1+\lambda x)^{\gamma}}\right)\left(\alpha + (1-\alpha)\left(1 - e^{1-(1+\lambda x)^{\gamma}}\right)^{\beta}\right)\right]} \\ & \times \frac{1}{\left[\left(\left(1 - e^{1-(1+\lambda x)^{\gamma}}\right)^{-\beta}\left(\alpha + (1-\alpha)\left(1 - e^{1-(1+\lambda x)^{\gamma}}\right)^{\beta}\right)\right) - 1\right]}. \end{aligned}$$

It can demonstrate that the distribution exhibits increasing, decreasing, bathtub-shaped, inverse bathtub-shaped, and constant hazard functions. In Figure 1, we can see the plots of pdf and hazard function of MOENH distribution for different values of the parameters.

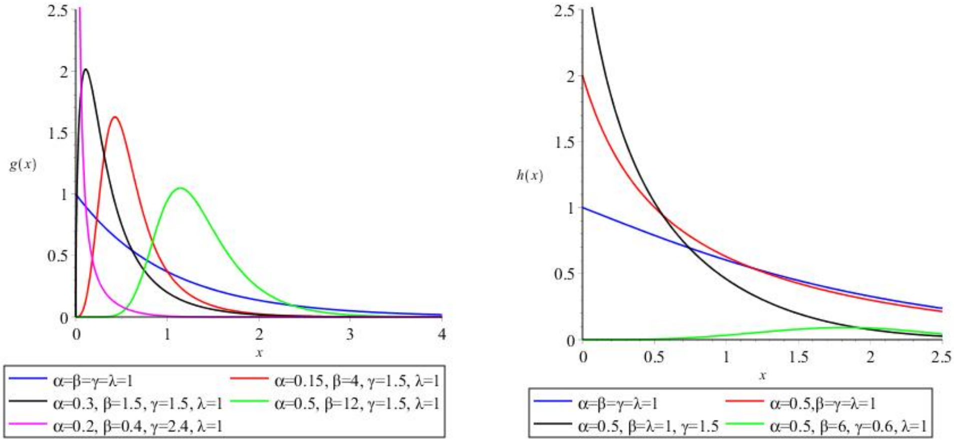

Figure 1: *Plots of pdf(left) and hazard function (right) for different values of parameters.*

3. Statistical properties

In this section, some statistical properties of MOENH distribution are discussed.

3.1. Moment Generating Function

The moment generating function of MOENH distribution is given by

$$\begin{aligned} M_X(t) &= \sum_{h=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{h+k+j}\frac{(t)^h}{h!}\binom{h}{k}\binom{k/r}{m}\frac{(n+1)\beta}{\lambda^h\alpha j!} \\ &\times\left(\frac{\alpha-1}{\alpha}\right)^n\frac{\Gamma(\beta n+\beta)\Gamma(m+1)}{\Gamma(\beta n+\beta-j)(j+1)^{m+1}}. \end{aligned}$$

3.2. Quantile Function

The quantile function has a number of applications. It can be used to obtain median, skewness, and kurtosis and can also be used to generate random variables. The quantile function of MOENH distribution is obtained as,

$$X = \frac{\left(1-\ln\left(1-e^{\frac{1}{\beta}\ln\left(\frac{p\alpha}{1-p+p\alpha}\right)}\right)\right)^{\frac{1}{\gamma}}-1}{\lambda}, 0<p<1, \alpha,\gamma,\beta,\lambda>0. \tag{6}$$

3.3. Order Statistic

Order statistic makes their appearance in many areas of statistical theory. Let $X_1, X_2, \ldots, X_n$ be a random sample from the MOENH family of distributions, and let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the corresponding order statistic. The pdf of i^{th} order statistic, say $X_{i:n}$, can be written as

$$\begin{aligned} g_{i:m}(x) &= \frac{m!g(x)}{(i-1)!\,(m-i)!}\left[G(x)\right]^{i-1}\left[1-G(x)\right]^{m-i} \\ &= \frac{m!g(x)}{(i-1)!\,(m-i)!}\sum_{j=0}^{m-i}(-1)^j\binom{m-i}{j}\left[G(x)\right]^{j+i-1} \end{aligned} \tag{7}$$

by using binomial expansion $[1-G(x)]^{m-i} = \sum_{j=0}^{m-i}(-1)^j\binom{m-i}{j}\sum_{j=0}^{m-i}(-1)^j\binom{m-i}{j}[G(x)]^j$ The pdf of the 1^{st} and n^{th} ordered statistic will be,

$$
\begin{aligned}
g_{1:m}(x) = & \frac{\sum_{j=0}^{m-1}(-1)^j\begin{pmatrix} m-1 \\ j \end{pmatrix} m!}{(m-1)!} \\
& \times \frac{\alpha\beta\gamma\lambda(1+\lambda x)^{\gamma-1}e^{1-(1+\lambda x)^\gamma}}{\left(1-e^{1-(1+\lambda x)^\gamma}\right)^{1-\beta}\left(\alpha+(1-\alpha)\left(1-\mathrm{e}^{1-(1+\lambda \mathrm{x})^\gamma}\right)^\beta\right)^2} \\
& \times \frac{\left(1-\mathrm{e}^{1-(1+\lambda \mathrm{x})^\gamma}\right)^\beta}{\alpha+(1-\alpha)\left(1-\mathrm{e}^{1-(1+\lambda \mathrm{x})^\gamma}\right)^\beta}, \\
g_{n:m}(x) = & \frac{\sum_{j=0}^{m-n}(-1)^j\begin{pmatrix} m-n \\ j \end{pmatrix} m!}{(m-n)!(n-1)!} \\
& \times \frac{\alpha\beta\gamma\lambda(1+\lambda x)^{\gamma-1}\mathrm{e}^{1-(1+\lambda \mathrm{x})^\gamma}}{\left(1-\mathrm{e}^{1-(1+\lambda \mathrm{x})^\gamma}\right)^{1-\beta}\left(\alpha+(1-\alpha)\left(1-e^{1-(1+\lambda x)^\gamma}\right)^\beta\right)^2} \\
& \times \frac{\left(1-\mathrm{e}^{1-(1+\lambda \mathrm{x})^\gamma}\right)^\beta}{\alpha+(1-\alpha)\left(1-\mathrm{e}^{1-(1+\lambda \mathrm{x})^\gamma}\right)^\beta}.
\end{aligned}
$$

3.4. Stress-strength reliability

In order to estimate the stress-strength parameter, considering two random variables X and Y with $MOENH(\alpha_1,\beta,\gamma,\lambda)$ and $MOENH(\alpha_2,\beta,\gamma,\lambda)$ distributions, respectively, with the same baseline parameters β,γ,λ. We assume that X and Y are independent random variables. Then the stress-strength parameter is obtained in the form

$$
\begin{aligned}
R &= P(Y<X) = \int_0^\infty \left[\int_0^x g_Y(y)\right] g_X(x)dx \\
&= \int_0^\infty G_Y(x)g_X(x)dx = -\alpha_1\left[\frac{\alpha_2\left(ln\left(\alpha_1\right)-ln(\alpha_2)+1\right)+\alpha_1}{(\alpha_1-\alpha_2)^2}\right].
\end{aligned} \tag{8}
$$

4. Characterization

This section deals with the characterization of the MOENH distribution based on the ratio of two truncated moments. To present the characterization of the distribution, consider the theorem presented in Glänzel [7].

Theorem 1. Let (Ω, G, P) be a given probability space and let $H=[a,b]$ be an interval for some $a<b$ ($a=-\infty, b=\infty$ might be allowed). Let $X:\Omega\to H$ be a continuous random variable with the distribution function G and let q_1 and q_2 be two real functions defined on H that

$$E[q_2(x)|X\geq x] = E[q_1(x)|X\geq x]\xi(x), x\in H$$

are defined with some real function ξ. Assume that q_1 and $q_2 \in C^1(H)$, $\xi \in C^2(H)$ and G is a twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\xi q_1 = q_2$ has no real solution in the interior of H. Then G is uniquely determined by the function q_1, q_2 and ξ, particularly

$$G(x) = \int_a^x C \left[\frac{\xi'(u)}{\xi(u) q_1(u) - q_2(u)} \right] exp\left(-S(u)\right) du,$$

where the function is a solution of the differential equation $S' = \frac{\xi' q_1}{\xi q_1 - q_2}$ and C is the normalization constant, such that $\int_H dG = 1$ and $\xi(x) = \frac{E[q_2(x)|X \geq x]}{E[q_1(x)|X \geq x]}$.

Proposition 1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and

$$q_1(x) = (\alpha + (1-\alpha)(1 - e^{1-(1+\lambda x)^\gamma})^\beta)^2 (1 - e^{1-(1+\lambda x)^\gamma})^{1-\beta}$$

and

$$q_2 = q_1(x) e^{1-(1+\lambda x)^\gamma}, x > 0.$$

Then the random variable X has pdf (4) if and only if the function ξ defined in the Theorem 1 is of the form

$$\xi(x) = \frac{e^{1-(1+\lambda x)^\gamma}}{2}. \tag{9}$$

Proof. Suppose the random variable X has pdf (4), then

$$(1 - G(x)) E\left[q_1(X) \mid X \geq x\right] = -C e^{1-(1+\lambda x)^\gamma}$$
$$(1 - G(x)) E\left[q_2(X) \mid X \geq x\right] = -\frac{C}{2} \mathrm{e}^{2(1-(1+\lambda x)^\gamma)}$$

where, $C = \alpha\beta$. Further,

$$\xi(x) q_1(x) - q_2(x) = -\frac{1}{2} q_1(x) \mathrm{e}^{1-(1+\lambda \mathrm{x})^\gamma} \neq 0, x > 0.$$

Conversely, if ξ is of the above form, then

$$S'(x) \quad = \quad \frac{\xi'(x) q_1(x)}{\xi(x) q_1(x) - q_2(x)} = \gamma\lambda(1+\lambda x)^{\gamma-1}$$

and hence,

$$S(x) \quad = \quad (1+\lambda x)^\gamma.$$

Now, in view of Theorem 1, X has density (4). ■

Corollary 1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and $q_1(x)$ be as in Proposition 1. The pdf of X in (4) if and only if there exist functions q_2 and ξ defined in Theorem 1 satisfying the differential equation

$$\frac{\xi'(x) q_1(x)}{\xi(x) q_1(x) - q_2(x)} \quad = \quad \gamma\lambda(1+\lambda x)^{\gamma-1}, x > 0.$$

The general solution of the differential equation given in Corollary 1 is,

$$\xi(x) = \mathrm{e}^{(1+\lambda x)^\gamma} \left[-\int \gamma\lambda(1+\lambda \mathrm{x})^{\gamma-1} \mathrm{e}^{-(1+\lambda x)^\gamma} \left(\mathrm{q}_1(\mathrm{x})\right)^{-1} \mathrm{q}_2(\mathrm{x}) \mathrm{dx} + \mathrm{D} \right],$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 1 with $D = 0$. However, it should be noted that there are other triplets (q_1, q_2, ξ) satisfying the conditions of Theorem 1.

Proposition 2. The MOENH density function is log-convex for fixed α say, $\alpha = 1$ and if $\gamma < 1$ and $\beta < 1$, and it is log-concave if $\gamma > 1$ and $\beta > 1$.

Proof. Let $z = (1 + \lambda x)^\gamma$, which implies that $z > 1$ for $x > 0$. We have $x = \left(z^{1/\gamma} - 1\right)/\lambda$. The MOENH density is now rewritten as a function of $z, \xi(z)$ say, we obtain

$$\xi(z) = f\left(\left(z^{1/\gamma} - 1\right)/\lambda\right) = \gamma\lambda\beta\frac{z^{(\gamma-1)/\gamma}\mathrm{e}^{(1-z)}}{[1 - \mathrm{e}^{(1-z)}]^{1-\beta}}, \quad z > 1.$$

The result follows by noting that the second derivative of $\log[\xi(z)]$ is

$$\frac{\mathrm{d}^2 \log[\xi(z)]}{\mathrm{d}z^2} = -\left[\frac{(\gamma-1)}{\gamma z^2} + \frac{(\beta-1)\mathrm{e}^{1-z}}{[1 - \mathrm{e}^{1-z}]^2}\right].$$

■

Proposition 3. For any $\lambda > 0, \alpha = 1$, the MOENH distribution has an increasing failure rate function if $\gamma > 1$ and $\beta > 1$, and it has a decreasing failure rate function if $\gamma < 1$ and $\beta < 1$. The failure rate function is constant if $\gamma = \beta = 1$.

Proof. Using the log-convexity of the density function, the conclusion is valid. ■

5. Estimation of parameters

There are several methods in the literature for estimating unknown parameters. In this section, maximum likelihood method of estimation is used for estimating the parameters of MOENH distribution. Let us consider $x_1, x_2, \ldots, x_n$ be the random variables having MOENH distribution. Then the likelihood function is given by,

$$L(x_i, \alpha, \beta, \gamma, \lambda) = \prod_{i=1}^{n} \frac{\alpha\beta\gamma\lambda(1+\lambda x)^{\gamma-1}\mathrm{e}^{1-(1+\lambda x)^\gamma}}{\left(1 - \mathrm{e}^{1-(1+\lambda x)^\gamma}\right)^{1-\beta}\left(\alpha + (1-\alpha)\left(1 - \mathrm{e}^{1-(1+\lambda x)^\gamma}\right)^\beta\right)^2}$$

The log-likelihood function is given by,

$$\begin{aligned} l =& n\ln(\alpha\beta\gamma\lambda) + \sum_{i=1}^{n} \ln(1+\lambda x)^{\gamma-1} + \sum_{i=1}^{n} 1 - (1+\lambda x)^\gamma \\ & - 2\sum_{i=1}^{n} \ln\left(\alpha + (1-\alpha)\left(1 - \mathrm{e}^{1-(1+\lambda x)^\gamma}\right)^\beta\right) - \sum_{i=1}^{n}(1-\beta)\ln\left(1 - \mathrm{e}^{1-(1+\lambda x)^\gamma}\right) \end{aligned} \tag{10}$$

Partial derivatives of (10) with respect to the unknown parameters $\alpha, \beta, \gamma, \lambda$

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - 2\sum_{i=1}^{n} \frac{1 - (1 - e^{1-(1+\lambda x)^\gamma})^\beta}{(\alpha + (1-\alpha)(1 - e^{1-(1+\lambda x)^\gamma})^\beta)^2} \tag{11}$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} - 2\sum_{i=1}^{n} \frac{\beta(1-\alpha)(1 - e^{1-(1+\lambda x)^\gamma})^{(\beta-1)}}{\alpha + (1-\alpha)(1 - e^{1-(1+\lambda x)^\gamma})^\beta} \tag{12}$$

$$\begin{aligned} \frac{\partial l}{\partial \gamma} &= \frac{n}{\gamma} + \sum_{i=1}^{n} \ln(1+\lambda x) - \sum_{i=1}^{n} (1+\lambda x)^\gamma \ln(1+\lambda x) - 2(1-\alpha)\beta \\ &\times \sum_{i=1}^{n} \frac{e^{1-(1+\lambda x)^\gamma}(1+\lambda x)^\gamma \ln(1+\lambda x)(1 - e^{1-(1+\lambda x)^\gamma})^{(\beta-1)}}{\alpha - (\alpha-1)(1 - e^{1-(1+\lambda x)^\gamma})^\beta} \\ &-\sum_{i=1}^{n} \frac{(1-\beta)\ln(1+\lambda x)e^{1-(1+\lambda x)^\gamma}}{e^{1-(1+\lambda x)^\gamma}} \end{aligned} \tag{13}$$

$$\begin{aligned}\frac{\partial l}{\partial \lambda} &= \frac{n}{\lambda}+\sum_{i=1}^{n}\frac{(\gamma-1)x}{1+\lambda x}-\sum_{i=1}^{n}\gamma x(1+\lambda x)^{\gamma-1}\\ &-2\sum_{i=1}^{n}\frac{x\gamma\beta(1-\alpha)e^{1-(1+\lambda x)^{\gamma}}(1+\lambda x)^{\gamma-1}(1-e^{1-(1+\lambda x)^{\gamma}})^{\beta-1}}{\alpha-(\alpha-1)(1-e^{1-(1+\lambda x)^{\gamma}})^{\beta}}\\ &-\sum_{i=1}^{n}\frac{\gamma x(1-\beta)(1+\lambda x)^{\gamma-1}e^{1-(1+\lambda x)^{\gamma}}}{1-e^{1-(1+\lambda x)^{\gamma}}}\end{aligned} \tag{14}$$

To obtain the maximum likelihood estimates of the unknown parameters, we equate,

$$\frac{\partial l}{\partial \alpha}=0;\frac{\partial l}{\partial \beta}=0;\frac{\partial l}{\partial \gamma}=0;\frac{\partial l}{\partial \lambda}=0 \tag{15}$$

The ML estimators are found through the solution of the nonlinear system. Hence, using R or MATLAB, a numerical approximation of the software's solution to this system of equations is possible.

6. Autoregressive Time Series Modelling

The autoregressive model is a stochastic process used in statistical modeling in which future values are forecasted based on a weighted sum of past values. The idea behind an autoregressive process is that the values of the past have an impact on the values of the present. A first-order autoregressive time series model with exponential stationary marginal distribution was developed by Gaver and Lewis [3]. In recent years, many authors Jayakumar and Babu [10] and Gillariose and Tomy [6] have developed various autoregressive models with minification structures. In this section, we develop various Autoregressive models of order 1 with Marshall-Olkin Exponentiated Nadarajah Haghighi as marginals, namely MIN AR (1) Model I and Model II and MAX - MIN AR (1) Model I and Model II, and explore some properties.

6.1. MIN AR(1) Model - I with MOENH Marginal Distribution

Consider an AR(1) structure,

$$X_n=\begin{cases}\epsilon_n; & \text{with probability} \quad \delta\\ \min(X_{n-1},\epsilon_n); & \text{with probability} \quad 1-\delta\end{cases} \tag{16}$$

where $\{\epsilon_n\}$ is a sequence of iid random variables independent of $\{X_n\}$ and $\delta\in(0,1)$. Then the process is Stationary Markovian with MOENH Distribution.

Theorem 2. In an AR(1) process with structure (16), $\{X_n\}$ is Stationary Markovian with MOENH distribution with parameters δ, γ, β, λ iff $\{\epsilon_n\}$ is distributed as ENH(γ, β, λ)

Proof. Let $\epsilon_n \sim ENH(\gamma,\beta,\lambda)$ From(16)

$$\bar{G}_{X_n}(x)=\delta\bar{G}_{\epsilon_n}(x)+(1-\delta)\bar{G}_{X_{n-1}}(x)\bar{G}_{\epsilon_n}(x).$$

Under stationary equilibrium,

$$\bar{G}_X(x)=\frac{\delta\bar{G}_{\epsilon}(x)}{1-(1-\delta)\bar{G}_{\epsilon}(x)},$$

and hence

$$\bar{G}_{\epsilon}(x)=\frac{\bar{G}_X(x)}{\delta+(1-\delta)\bar{G}_X(x)}.$$

If $\epsilon_n \sim ENH(\gamma, \beta, \lambda)$

$$\bar{G}_\epsilon(x) = 1 - (1 - e^{1-(1+\lambda x)^\gamma})^\beta.$$

Thus

$$\begin{aligned}\bar{G}_X(x) &= \frac{\delta\{1 - (1 - e^{1-(1+\lambda x)^\gamma})^\beta\}}{1 - (1-\delta)\{1 - (1 - e^{1-(1+\lambda x)^\gamma})^\beta\}} \\ &= \frac{\delta - \delta(1 - e^{1-(1+\lambda x)^\gamma})^\beta}{\delta + (1-\delta)(1 - e^{1-(1+\lambda x)^\gamma})^\beta},\end{aligned}$$

which is the survival function of MOENH(δ). Conversely, if

$$\bar{G}_X(x) = \frac{\delta - \delta(1 - e^{1-(1+\lambda x)^\gamma})^\beta}{\delta + (1-\delta)(1 - e^{1-(1+\lambda x)^\gamma})^\beta},$$

then $\bar{G}_{\epsilon_n}(x)$ is distributed as ENH(γ, β, λ) and the process is stationary. In order to establish stationarity, assume that $X_{n-1} \sim MOENH(\delta)$ and $\epsilon_n \sim ENH(\gamma, \beta, \lambda)$ then,

$$\bar{G}_X(x) = \frac{\delta - \delta(1 - e^{1-(1+\lambda x)^\gamma})^\beta}{\delta + (1-\delta)(1 - e^{1-(1+\lambda x)^\gamma})^\beta}.$$

This means that X_n is distributed as MOENH(δ). ■

Remark 1. If X_0 has an arbitrary distribution G_{X_0}, the minification process is asymptotically stationary with MOENH($\delta, \gamma, \beta, \lambda$).
Since

$$\begin{aligned}\bar{G}_{X_n}(x) &= \delta \bar{G}_\epsilon(x) \sum_{i=0}^{n-1} (1-\delta)^i \bar{G}_\epsilon^i(x) + (1-\delta)^n \bar{G}_{X_0}(x) \bar{G}_\epsilon^n(x) \\ &= \frac{\delta \bar{G}_\epsilon(x)}{1 - (1-\delta)\bar{G}_\epsilon(x)} \\ &= \frac{\delta - \delta(1 - e^{1-(1+\lambda x)^\gamma})^\beta}{\delta + (1-\delta)(1 - e^{1-(1+\lambda x)^\gamma})^\beta},\end{aligned}$$

the survival function of $MOENH(\delta, \gamma, \beta, \lambda)$.

6.2. MIN AR(1) Model - II with MOENH Distribution

Here we discuss a more general structure which allows probabilistic selection of process values, innovations and combinations of both. Consider the AR(1) structure given by

$$X_n = \begin{cases} X_{n-1}; & \text{with probability } \delta_1 \\ \epsilon_n; & \text{with probability } \delta_2 \\ \min(X_{n-1}, \epsilon_n); & \text{with probability } 1 - \delta_1 - \delta_2, \end{cases} \tag{17}$$

where $\delta_1, \delta_2 > 0, \quad \delta_1 + \delta_2 < 1$ and $\{\epsilon_n\}$ is a sequence of iid random variables independent of $\{X_n\}$. Then the process is stationary with Marshall-Olkin Exponentiated Nadaraja Haghighi distribution.

Theorem 3. In an AR(1) process with structure (17), $\{x_n\}$ is stationary Markovian with MOENH distribution with parameters $\tau, \gamma, \beta,$ and λ iff $\{\epsilon_n\}$ is distributed as ENH with parameters γ, β and λ, where $\tau = \frac{\delta_2}{1-\delta_1}$.

Proof. Let $\epsilon_n \sim ENH(\gamma, \beta, \lambda)$. From (17)

$$\bar{G}_{X_n}(x) = \delta_1 \bar{G}_{X_{n-1}}(x) + \delta_2 \bar{G}_{\epsilon_n}(x) + (1 - \delta_1 - \delta_2)\, \bar{G}_{X_{n-1}}(x) \bar{G}_{\epsilon_n}(x)$$

Under stationary equilibrium we have,

$$\begin{aligned}\bar{G}_X(x) &= \frac{\tau\bar{G}_\epsilon(x)}{1-(1-\tau)\bar{G}_\epsilon(x)}\\ &= \frac{\tau\left[1-\left(1-e^{1-(1+\lambda x)^\gamma}\right)^\beta\right]}{1-(1-\tau)\left[1-\left(1-e^{1-(1+\lambda x)^\gamma}\right)^\beta\right]}\\ &= \frac{\frac{\delta_2}{1-\delta_1}\left[1-\left(1-e^{1-(1+\lambda x)^\gamma}\right)^\beta\right]}{1-\left(1-\frac{\delta_2}{1-\delta_1}\right)\left[1-\left(1-e^{1-(1+\lambda x)^\gamma}\right)^\beta\right]},\end{aligned}$$

where $\tau = \frac{\delta_2}{1-\delta_1}$, which is in the Marshall-Olkin form. Now let us assume that $\{X_n\} \sim$ MOENH$(\tau,\gamma,\beta,\lambda)$. From (17) under stationarity,

$$\bar{G}_\epsilon(x) = \frac{(1-\delta_1)\,\bar{G}_X(x)}{\delta_2+(1-\delta_1-\delta_2)\,\bar{G}_X(x)}.$$

Now by using X_n as MOENH $(\tau,\gamma,\beta,\lambda)$, we have

$$\begin{aligned}\bar{G}_\epsilon(x) &= \frac{(1-\delta_1)\left[\frac{\frac{\delta_2}{1-\delta_1}\left[1-\left(1-e^{1-(1+\lambda x)^\gamma}\right)^\beta\right]}{1-\left(1-\frac{\delta_2}{1-\delta_1}\right)\left[1-\left(1-e^{1-(1+\lambda x)^\gamma}\right)^\beta\right]}\right]}{\delta_2+(1-\delta_1-\delta_2)\left[\frac{\frac{\delta_2}{1-\delta_1}\left[1-\left(1-e^{1-(1+\lambda x)^\gamma}\right)^\beta\right]}{1-\left(1-\frac{\delta_2}{1-\delta_1}\right)\left[1-\left(1-e^{1-(1+\lambda x)^\gamma}\right)^\beta\right]}\right]}\\ &= 1-\left(1-e^{1-(1+\lambda x)^\gamma}\right)^\beta.\end{aligned}$$

Which is the Survival function of Exponentiated Nadaraja Haghighi distribution with parameters γ,β and λ. ■

6.3. MAX-MIN AR(1) Model - I with MOENH Distribution

Consider the AR(1) structure given by,

$$X_n = \begin{cases}\max\left(X_{n-1},\epsilon_n\right); & \text{with probability } \delta_1\\ \min\left(X_{n-1},\epsilon_n\right); & \text{with probability } \delta_2\\ X_{n-1}; & \text{with probability } 1-\delta_1-\delta_2,\end{cases} \tag{18}$$

where $0<\delta_1, \quad \delta_2>1, \quad \delta_2<\delta_1, \quad \delta_1+\delta_2<1$ and $\{\epsilon_n\}$ is a sequence of iid random variables independent of $\{X_n\}$. Then the process is Stationary Markovian with Marshall-Olkin Exponentiated Nadaraja Haghighi distribution.

Theorem 4. In an $AR(1)$ MAX-MIN process with structure (18), $\{X_n\}$ is a stationary Markovian AR(1) MAX-MIN process with MOENH stationary distribution with parameters τ,γ,β and λ iff $\{\epsilon_n\}$ is distributed as ENH with parameters γ,β and λ, where $\tau = \frac{\delta_1}{\delta_2}$.

Proof. Let $\epsilon_n \sim$ ENH(γ,β,λ). From (18) we have,

$$\bar{G}_{X_n}(x) = \delta_1\left[1-\left(1-\bar{G}_{X_{n-1}}(x)\right)\left(1-\bar{G}_{\epsilon_n}(x)\right)\right]+\delta_2\bar{G}_{X_{n-1}}(x)\bar{G}_{\epsilon_n}(x)+(1-\delta_1-\delta_2)\,\bar{G}_{X_{n-1}}(x).$$

Under stationary equilibrium,

$$\begin{aligned}\bar{G}_{X_n}(x) &= \frac{\tau\bar{G}_{\epsilon(x)}}{1-(1-\tau)\bar{G}_{\epsilon}(x)}\\ &= \frac{\tau[1-\left(1-e^{1-(1+\lambda x)^{\gamma}}\right)^{\beta}]}{1-(1-\tau)[1-\left(1-e^{1-(1+\lambda x)^{\gamma}}\right)^{\beta}]}\\ &= \frac{\frac{\delta_1}{\delta_2}[1-\left(1-e^{1-(1+\lambda x)^{\gamma}}\right)^{\beta}]}{1-(1-\frac{\delta_1}{\delta_2})[1-\left(1-e^{1-(1+\lambda x)^{\gamma}}\right)^{\beta}]}.\end{aligned}$$

Where $\tau = \frac{\delta_1}{\delta_2}$ and $\bar{G}_{X_n}(x)$ is in the form of Marshall-Olkin distribution. Now Let $X_n \sim$ MOENH$(\tau, \gamma, \beta, \lambda)$. Then from (18), under stationarity,

$$\bar{G}_{\epsilon}(x) = \frac{\delta_2\bar{G}_{X_n}(x)}{\delta_1+(\delta_2-\delta_1)\bar{G}_{X_n}(x)}.$$

Thus, after simplification it can be written as

$$\begin{aligned}\bar{G}_{\epsilon}(x) &= \frac{\delta_2\left[\frac{\frac{\delta_1}{\delta_2}\left[1-\left(1-e^{1-(1+\lambda x)^{\gamma}}\right)^{\beta}\right]}{1-\left(1-\frac{\delta_1}{\delta_2}\right)\left[1-\left(1-e^{1-(1+\lambda x)^{\gamma}}\right)^{\beta}\right]}\right]}{\delta_1+(\delta_2-\delta_1)\left[\frac{\frac{\delta_1}{\delta_2}\left[1-\left(1-e^{1-(1+\lambda x)^{\gamma}}\right)^{\beta}\right]}{1-\left(1-\frac{\delta_1}{\delta_2}\right)\left[1-\left(1-e^{1-(1+\lambda x)^{\gamma}}\right)^{\beta}\right]}\right]}\\ &= 1-(1-e^{1-(1+\lambda x)^{\gamma}})^{\beta},\end{aligned}$$

which is the survival function of Exponentiated Nadaraja Haghighi distribution with parameters γ, β and λ. ■

6.4. MAX-MIN AR(1) Model - II with MOENH Distribution

Consider the more general MAX-MIN process that includes minimum, maximum innovations and the process values, The AR(1) structure is given by

$$X_n = \begin{cases}\max\left(X_{n-1},\epsilon_n\right); & \text{with probability } \delta_1\\ \min\left(X_{n-1},\epsilon_n\right); & \text{with probability } \delta_2\\ \epsilon_n; & \text{with probability } \delta_3\\ X_{n-1}; & \text{with probability } 1-\delta_1-\delta_2-\delta_3,\end{cases} \tag{19}$$

where $0 < \delta_1, \quad \delta_2, \quad \delta_3 < 1, \quad \delta_1+\delta_2+\delta_3 < 1$ and $\{\epsilon_n\}$ is a sequence of iid random variables independent of $\{X_n\}$. Then the process is stationary Markovian with Marshall-Olkin Exponentiated Nadaraja Haghighi distribution.

Theorem 5. AR(1) MAX-MIN process $\{X_n\}$ with structure (19) is a stationary Markovian AR(1) MAX-MIN process with MOENH distribution $(\tau, \gamma, \beta, \lambda)$ if $\{\epsilon_n\}$ is distributed as ENH with parameters γ, β and λ where $\tau = \frac{\delta_1+\delta_3}{\delta_2+\delta_3}$.

Proof. Let $\epsilon_n \sim$ ENH(γ, β, λ). From (19) we have,

$$\begin{aligned}\bar{G}_{Xn}(x) &= \delta_1[1-(1-\bar{G}_{X_{n-1}}(x))(1-\bar{G}_{\epsilon_n}(x))]+\delta_2\bar{G}_{X_{n-1}}(x)\bar{G}_{\epsilon_n}(X)+\delta_3\bar{G}_{\epsilon_n}(X)\\ &\quad+(1-\delta_1-\delta_2-\delta_3)\bar{G}_{X_{n-1}}(x)\end{aligned}$$

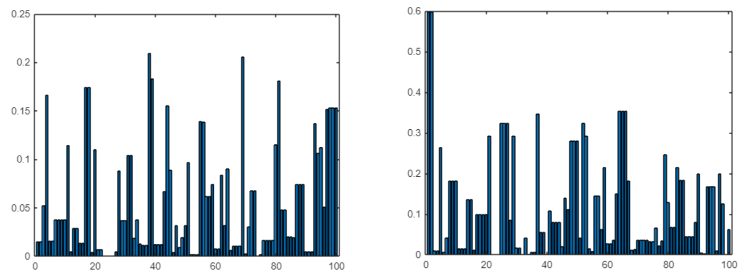

Figure 2: *Sample path for AR(1) Minification Model-I for* $p = 0.6, 0.7$, *β=0.5, γ=1.2 and λ=1.*

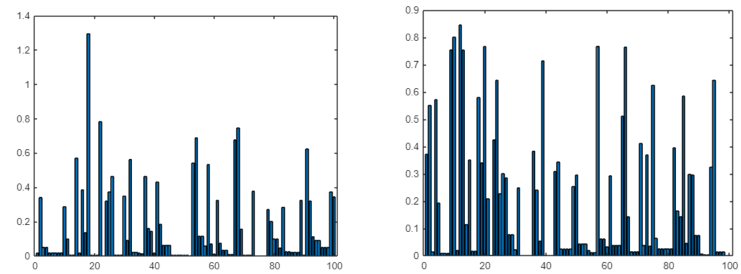

Figure 3: *Sample path of AR(1) Minification Model-II for different sets of* $(p_1, p_2) = (0.3, 0.4), (0.2, 0.5), \beta = 0.5, \gamma = 1.5,$ *and* $\lambda = 1$.

Under stationary equilibrium it gives,

$$\begin{aligned}
\bar{G}_{X_n}(x) &= \frac{\tau \bar{G}_\epsilon(x)}{1-(1-\tau)\bar{G}_\epsilon(x)} \\
&= \frac{\frac{\delta_1+\delta_3}{\delta_2+\delta_3}[1-\left(1-e^{1-(1+\lambda x)^\gamma}\right)^\beta]}{1-(1-\frac{\delta_1+\delta_3}{\delta_2+\delta_3})[1-\left(1-e^{1-(1+\lambda x)^\gamma}\right)^\beta]},
\end{aligned}$$

where $\tau = \frac{\delta_1+\delta_3}{\delta_2+\delta_3}$, which is in the form of Marshall-Olkin distribution. Now let $X_n \sim$ MOENH$(\tau, \gamma, \beta, \lambda)$. Then from (19), we have

$$\bar{G}_\epsilon(x) = \frac{(\delta_2+\delta_3)\,\bar{G}_X(x)}{(\delta_1+\delta_3)+(\delta_2-\delta_1)\,\bar{G}_X(x)}$$

By simplifying, we get

$$\begin{aligned}
\bar{G}_\epsilon(x) &= \frac{(\delta_2+\delta_3)\left[\frac{\frac{\delta_1+\delta_3}{\delta_2+\delta_3}\left[1-\left(1-e^{1-(1+\lambda x)^\gamma}\right)^\beta\right]}{1-\left(1-\frac{\delta_1+\delta_3}{\delta_2+\delta_3}\right)\left[1-\left(1-e^{1-(1+\lambda x)^\gamma}\right)^\beta\right]}\right]}{(\delta_1+\delta_3)+(\delta_2-\delta_1)\left[\frac{\frac{\delta_1+\delta_3}{\delta_2+\delta_3}\left[1-\left(1-e^{1-(1+\lambda x)^\gamma}\right)^\beta\right]}{1-\left(1-\frac{\delta_1+\delta_3}{\delta_2+\delta_3}\right)\left[1-\left(1-e^{1-(1+\lambda x)^\gamma}\right)^\beta\right]}\right]} \\
&= 1-\left(1-e^{1-(1+\lambda x)^\gamma}\right)^\beta.
\end{aligned}$$

Which is the Survival function of Exponentiated Nadaraja Haghighi distribution with parameters γ, β and λ. Figure 2-5 displays the sample path features of the four AR(1) models created in this section and how these measures change with different parameter settings. ■

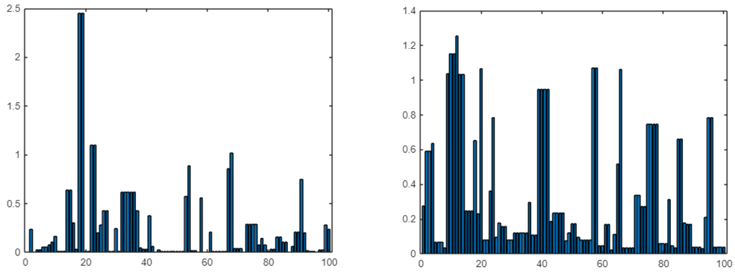

Figure 4: *Sample path for AR(1) MAX-MIN Model -I for various combinations of* $(p_1, p_2) = (0.2, 0.1), (0.3, 0.2), \beta = 0.25, \gamma = 1.2,$ *and* $\lambda = 1$.

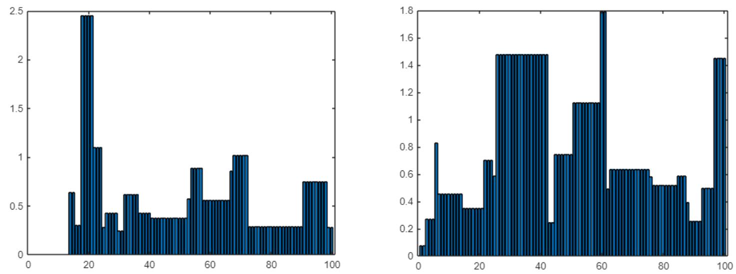

Figure 5: *Sample path for AR(1) MAX-MIN Model -II for* $(p_1, p_2, p_3) = (0.4, 0.2, 0.2), (0.5, 0.2, 0.1), \beta = 0.25, \gamma = 1.2,$ *and* $\lambda = 1$.

7. Numerical illustration

7.1. Simulation study

A simulation study is conducted to evaluate the effectiveness of the MLEs for estimating the parameters of MOENH distribution. For this, we take into account, $\alpha = 0.5, \beta = 1.2, \gamma = 0.8$, and $\lambda = 0.09$. For different sample sizes of n = 1000, n = 2000, and n = 4000, we simulate data from the MOENH model and determine the MLEs by maximizing the likelihood function. We carry out the procedure, 10000 times, and the results show that bias and root-mean-square error (RMSE) decreases as sample size increases. The results are in the Table 1.

Table 1: *Simulation*

n	Parameters	Estimate	Bias	RMSE
	α	0.6352	0.0135	0.1687
1000	β	1.2098	0.0009	0.0246
	γ	0.9561	0.0156	0.1306
	λ	0.1352	0.0045	0.0512
	α	0.5895	0.0089	0.1295
2000	β	1.2074	0.0007	0.0168
	γ	0.9000	0.0100	0.0995
	λ	0.1193	0.0029	0.0373
	α	0.5509	0.0051	0.0881
4000	β	1.2027	0.0002	0.0113
	γ	0.8511	0.0051	0.0652
	λ	0.1058	0.0015	0.0242

7.2. Data illustration

We take into account the following DARWin data set for numerical illustration. DARWin is a project, within the Swedish industry organization Swedish Energy, which collects and annually presents outage data from most of the Swedish electricity system operators. The annual reports from Swedish Energy are open-accessible and can be downloaded from the Swedish Energy website. The unplanned events from 2012 divided into voltage level (12 k V) and failure causes are given by Ekstedt et al. [2]. The parameters are estimated using the maximum likelihood method. Akaike information criteria (AIC), Bayesian information criteria (BIC), Kolmogrov-Smirnov (K-S), and $p - value$ are the goodness-of-fit metrics that we take into consideration.

Table 2: *Parameter estimates and goodness of fit statistics for models fitted to the data*

Model	MLE	-Log L	AIC	BIC	KS	p-VALUE
MOENH	$\alpha = 2.000$ $\beta = 0.2777$ $\gamma = 1.2189$ $\lambda = 0.00015$	73.6221	155.24	156.03	0.10819	0.9994
ENH	$\beta = 0.2777$ $\gamma = 1.2189$ $\lambda = 0.00013$	78.6904	163.38	163.9725	0.2306	0.6449
NH	$\gamma = 1.2189$ $\lambda = 0.0003$	79.124	162.249	132.6435	0.2370	0.6122
Exp	$\lambda = 0.0004$	78.8909	159.78	159.982	0.2240	0.6782

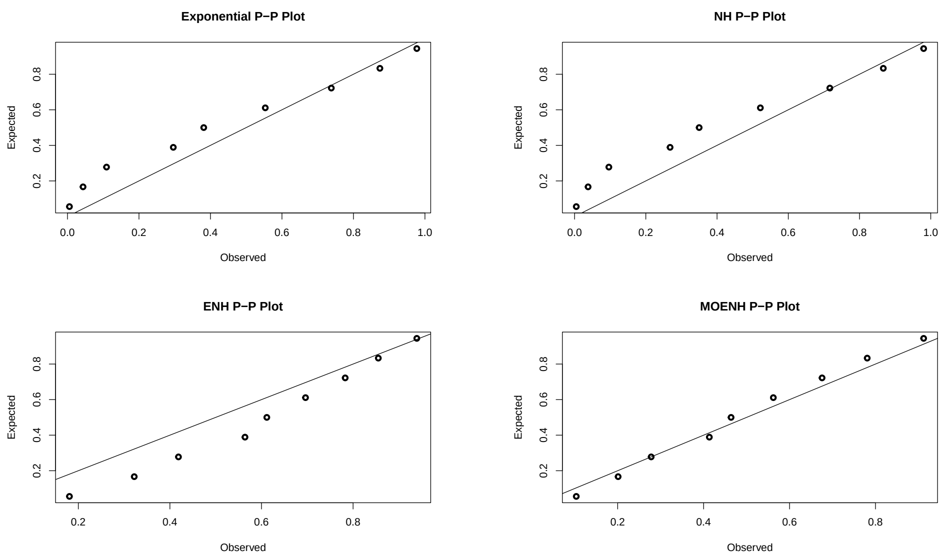

Figure 6: *pp-plots*

Table 2 lists the parameter estimates and goodness of fit statistics for DARWin voltage level (12 k V) failure data. The MOENH model is more suitable for this data since the values of $-\log L$, AIC, BIC, K-S and p-value for the MOENH distribution are lower than those of the other competing models. Figure 6 represents the pp-plot for the fitted models.

7.3. Testing of Hypothesis

In this section, we present the likelihood ratio test procedure for testing the significance of the parameters of the MOENH model. We consider LR statistics to check if the fitted MOENH distribution for a given data set is statistically superior to the fitted exponential, NH, ENH distributions. In any case, the hypothesis test of the type $H_0 : \theta = \theta_0 \; vs \; H_1 : \theta \neq \theta_0$ using the generalized likelihood ratio test. The test statistic is,

$$-2\ln\lambda(x) = 2[\ln L(\hat{\Theta}; x) - \ln L(\hat{\Theta}^*; x)] \tag{20}$$

where $\hat{\Theta}$ is the maximum likelihood estimator with no restriction, and $\hat{\Theta}^*$ is the maximum likelihood estimator with restriction. The test statistic follows a Chi-square distribution with degrees of freedom $(df = df_{alt} - df_{null})$. So here we consider the following likelihood ratio tests.

1. $H_{01} : \alpha = \beta = \gamma = 1$, the sample is from $Exp(\lambda)$
   $H_{11} : \alpha \neq \beta \neq \gamma \neq 1$, the sample is from $MOENH(\alpha, \beta, \gamma, \lambda)$
2. $H_{02} : \alpha = \beta = 1$, the sample is from NH
   $H_{12} : \alpha \neq \beta \neq 1$, the sample is from $MOENH(\alpha, \beta, \gamma, \lambda)$
3. $H_{03} : \alpha = 1$, the sample is from ENH
   $H_{13} : \alpha \neq 1$, the sample is from $MOENH(\alpha, \beta, \gamma, \lambda)$

Table 3: *Likelihood ratio test*

Model	Hypothesis	Test statistic	p-value
Exp vs $MOENH$	$H_{01} : \alpha, \beta, \gamma = 1$ $H_{11} : H_{01}$ is false	10.5406	0.0145
NH vs $MOENH$	$H_{02} : \alpha, \beta = 1$ $H_{12} : H_{02}$ is false	11.0046	0.0041
ENH vs $MOENH$	$H_{03} : \alpha = 1$ $H_{13} : H_{03}$ is false	10.1365	0.0014

The test statistic $-2\ln\lambda(x)$ given in (20) is asymptotically distributed as χ^2 with three degrees of freedom for test 1, 2 degrees of freedom for test 2, and 1 degree of freedom for test 3. The computed values of the test statistic in the case of the DARWin data set are listed in Table 3. From Table 3, we can see that $p - value$ is less than the significant level of 0.05.LR tests reject the three sub-models in favour of the MOENH distribution. Since the critical values at the significance level 0.05 and degree of freedom three, two, and one for the two-tailed tests are 9.348, 7.378, and 5.024 respectively the null hypothesis is rejected in all cases, which shows the appropriateness of the MOENH distribution to the DARWin data.

8. Conclusion

In this paper, as a generalization of the exponential distribution, the MOENH distribution is introduced. Statistical properties, characterization properties, and autoregressive time series models of MOENH distribution are obtained. It is shown that the new model is a competitor to the exponential distribution for modeling certain types of data sets. Also, the generation of random variables from the new model is simple. The new model may attract the attention of researchers as a viable competitor to the exponential distribution.

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