A $MAP/PH_1, PH_2/2$ INVENTORY QUEUEING SYSTEM WITH TWO COMMODITY, MULTIPLE VACATION, SERVER FEEDBACK, WORKING BREAKDOWN, REPAIR AND EMERGENCY REPLENISHMENT

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Abstract

We investigate a continuous review inventory queuing system in the present study that has two heterogeneous servers: Server-2, which is reliable, and Server-1, which is unreliable. An exponentially distributed random time is used to describe the repair process when server-1 has an interruption. On the other hand, server-2 is completely dependable, but it goes on vacation when the system is empty. These two goods can be reordered under ordering regulations. To ensure customer satisfaction, an emergency replenishment of one item with no lead time occurs when the on-hand inventory level falls to zero. We use the matrix analytic approach for the QBD process under a steady-state probability vector. We also take into account the overall cost and the busy time. Furthermore, numerical data shows the benefits of the suggested approach in a range of random circumstances.

Keywords: Markovian arrival process, PH-distribution, multiple vacation, two commodity, working breakdown, emergency replenishment.

AMS Subject Classification (2010): 60K25, 68M30, 90B22.

1. Introduction

Substitution methods are essential for reducing client losses in an inventory-based organization. While the demand-driven item's stock-out time, emergency replenishment may be utilized. Stockouts may be managed and frequently avoided by employing excellent inventory management methods such as accurate demand forecasts, establishing reorder points, utilizing buffers and safety stock, and discovering stockout trends and emergency orders using inventory management software. In circumstances where stockouts are beyond a retailer's control, it's necessary to take efforts to save expenses and prevent dissatisfied consumers, such as proposing or inventing a product substitute or finding an alternate supplier.

The study [8] examined a $(0, S)$ in a service facility with multiple server vacations and impatient clients.The combined probability distribution of the inventory level and the number of customers in the waiting area is calculated under steady-state scenarios. A few system performance metrics are obtained, and the total estimated cost rate is computed. The two-commodity (TC) inventory system, in which commodities are split into primary and supplemental products, was examined by Jeganathan et al. [13]. They examined an individual reordering strategy for a single commodity inventory with no replenishment period and another commodity utilised for the (s,Q) policy. An inventory queuing system with Poisson arrival, randomly distributed service times, zero lead times, requiring one item per client, and limited waiting room capacity was studied by Berman and Sapna [3].

Kalyanaraman and Senthilkumar [7] studied two heterogeneous server Markovian queues, with the first server's service mode altering at a threshold. If both servers are idle, the consumer will be served by the faster server. Using MAP arrival and group services, Chakravarthy et al. [4] investigated the multi-server finite capacity model. They proved that the invariant distribution of the sojourn time is phase-type if the inter-arrival durations are as well. They also developed mathematical approaches for determining the invariant density of waiting queue size. Yang et al. [15] examined a Markovian queue with two servers, malfunctions, and leaves of absence. They used the usual particle swarm optimization technique for numerical analysis, developed a heuristic cost model, and investigated their system using the matrix approach.

In a recent article, Laxmi and Soujanya [11] discussed QIS with multiple server working vacations. They consider both working vacation and multiple vacation, and numerically analyze the system to determine the best strategy. An (s, Q) replenishment technique was used by Manikandan and Nair [12] to assess a single server QIS while taking breaks and interruptions into consideration. Performance data are examined, the steady-state probability vector is calculated, and the stability condition of the system is established. A busy period analysis and a derived stationary waiting time distribution for the queue are included in the study.

Yadavalli et al. [16] used a finite source QIS with interruption to conduct research on two heterogeneity servers. Jeganathan et al. [6] studied a Markovian inventory system with two queues and customers jockeying. Yadavalli and Jeganathan [17] investigate the impact of two servers and partial vacations on inventory systems. Suganya et al. [14] proposed an inventory system with numerous server vacations, as opposed to partial ones.They did, however, take into account the Markovian Arrival Process (MAP) of clients. A queueing model subject to a single server that can offer two types of heterogeneous services, closedown, vacation, setup, immediate feedback, breakdown, and repair was determined by Ayyappan and Gowthami [2]. Ayyappan and Karpagam [1] have discussed a classical model with unreliable server, vacation and immediate feedback. They have assumed that the unsatisfied customers will be given re-service immediately without any delay. They have found the PGF of the size of the waiting line and some interesting performance measures.

2. Model Description

We investigate the two server inventory queueing model, where customers arrive at the system based on MAP. The parameter matrices D_0 and D_1 indicate that no customers have arrived at the system and that customers have arrived, respectively. Dimension m is used by the matrices D_0 and D_1. We examine two different kinds of heterogeneous servers here: One among them does not go on vacation and the other one goes for a vacation after service completion. That is, the server-1 will always be there in the system but the server-2 will move forward to vacation after service completion to the customers. The service rendering by the server-1 during normal mode with representations (γ_1, U_1) of dimension n_1 such that $U_1^0 + U_1 e = 0$. Likewise, The service rendering by the server-2 during normal mode is follows the PH-type with representation (γ_2, U_2) and of dimension n_2 such that $U_2^0 + U_2 e = 0$. During the server-1 providing service to the customers who may struck with breakdown and the system has an option: continue delivering slow service to current customer. During the working breakdown period, the server provide slow service to current customer and service times are phase type distributed with notation $(\gamma_1, \theta_1 U_1)$; $0 < \theta_1 < 1$ of order n_1 and rate is $\mu_{bd} = [\gamma_1(-\theta_1 U_1)^{-1} e]^{-1}$.

After service completion during this working breakdown period, the system automatically enters a repair phase. The server will begin a rejuvenating service for the customer, after the repair process. After service completion, no customer in the system server-2 go for multiple vacation. At the end of a vacation, when the customers are staying in the system then the server-2 is provided normal service. Otherwise, server-2 takes another vacation immediately. After received service from the service station, the satisfied customer would left out of the system with probability p_i, for $i = 1, 2$ and if the customer is not satisfied with probability q_i, for i=1,2 then they will get feedback immediately that is, the instantaneous feedback will offer by the same server. The breakdown and repair times of server-1 follows an exponentially distributed with parameter ψ and τ respectively. We analyse a stochastic inventory system with two distinct goods in stock: server-1 (I-commodity) and server-2 (II-commodity). The maximal storage capacity for the commodity i^{th} is S_i $(i = 1, 2)$. The initial commodity with zero lead time follows a $(0, S_1)$ ordering strategy. When the on-hand inventory level of the second commodity falls below a predetermined level, s_2, an order for S_2 units is made. The lead time for this order is exponentially distributed with parameter β. Furthermore, suppose an emergency replenishment of one item with zero lead time occurs when the on-hand inventory level falls to zero. Emergency replenishment is included into the system to ensure client happiness. The server-2 does not offer feedback service during emergency replenishment. The schematic picture of this model is provided in Figure 1.

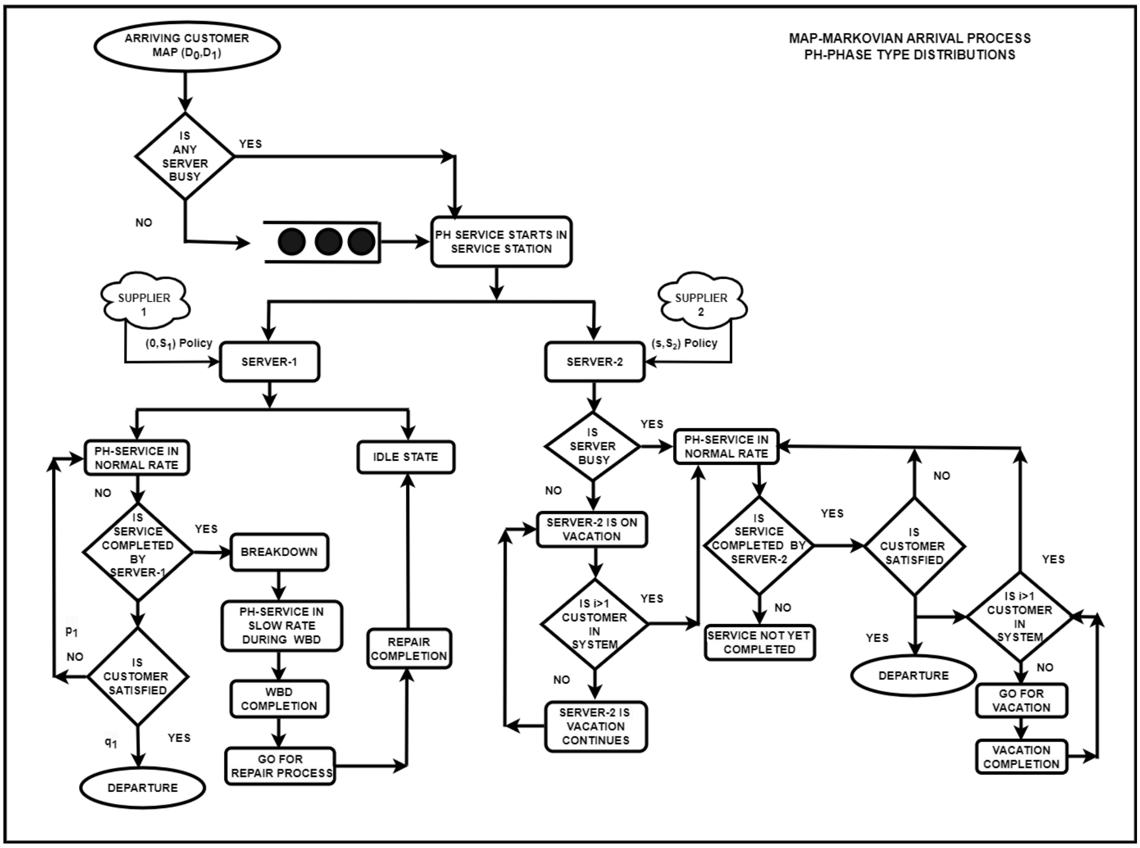

Figure 1: *Schematic representation*

3. Analysis

In the following section, we establish the queueing-inventory system's transition rate matrix. Assume that $N(t), J_1(t), J_2(t), I_1(t), I_2(t), S_1(t), S_2(t), M(t)$ described total customers in the system, status of server-1, status of server-2, stock level for commodity I, stock level for commodity II, service phase for server-1, service phase for server-2, arrival phases, respectively.

$$J_1(t) = \begin{cases} 0, & \text{server-1 is idle ,} \\ 1, & \text{server-1 is busy ,} \\ 2, & \text{server-1 is busy in WBD mode,} \\ 3, & \text{server-1 is repair,} \end{cases}$$

$$J_2(t) = \begin{cases} 0, & \text{server-2 is vacation,} \\ 1, & \text{server-2 is busy in normal mode,} \end{cases}$$

Consider $X(t) = \{N(t), J_1(t), J_2(t), I_1(t), I_2(t), S_1(t), S_2(t), M(t)\}$ is a $CTMC$ with state space

$$\Phi = \phi(0) \bigcup \phi(1) \bigcup_{i=1}^{\infty} \phi(i). \tag{1}$$

where

$$\begin{aligned} \phi(0) = &\{(0,0,0,u_1,u_2,u_5) :\ 1 \le u_1 \le S_1,\ 1 \le u_2 \le S_2,\ 1 \le u_5 \le m\} \\ &\cup \{(0,3,0,u_1,u_2,u_5) :\ 1 \le u_1 \le S_1,\ 1 \le u_2 \le S_2,\ 1 \le u_5 \le m\} \end{aligned}$$

$$\begin{aligned} \phi(1) = &\{(1,0,1,u_1,u_2,u_4,u_5) :\ 1 \le u_1 \le S_1,\ 1 \le u_2 \le S_2,\ 1 \le u_4 \le n_2,\ 1 \le u_5 \le m\} \\ &\cup \{(1,1,0,u_1,u_2,u_3,u_5) :\ 1 \le u_1 \le S_1,\ 1 \le u_2 \le S_2,\ 1 \le u_3 \le n_1,\ 1 \le u_5 \le m\} \\ &\cup \{(1,2,0,u_1,u_2,u_3,u_5) :\ 1 \le u_1 \le S_1,\ 1 \le u_2 \le S_2,\ 1 \le u_3 \le n_1,\ 1 \le u_5 \le m\} \\ &\cup \{(1,3,0,u_1,u_2,u_5) :\ 1 \le u_1 \le S_1,\ 1 \le u_2 \le S_2,\ 1 \le u_5 \le m\} \\ &\cup \{(1,3,1,u_1,u_2,u_4,u_5) :\ 1 \le u_1 \le S_1,\ 1 \le u_2 \le S_2,\ 1 \le u_4 \le n_2,\ 1 \le u_5 \le m\} \end{aligned}$$

and for $i \ge 2$,

$$\begin{aligned} \phi(i) = &\{(i,1,0,u_1,u_2,u_3,u_5) :\ 1 \le u_1 \le S_1,\ 1 \le u_2 \le S_2,\ 1 \le u_3 \le n_1,\ 1 \le u_5 \le m\} \\ &\cup \{(i,1,1,u_1,u_2,u_3,u_4,u_5) :\ 1 \le u_1 \le S_1,\ 1 \le u_2 \le S_2,\ 1 \le u_3 \le n_1,\ 1 \le u_4 \le n_2,\ 1 \le u_5 \le m\} \\ &\cup \{(i,2,0,u_1,u_2,u_3,u_5) :\ 1 \le u_1 \le S_1,\ 1 \le u_2 \le S_2,\ 1 \le u_3 \le n_1,\ 1 \le u_5 \le m\} \\ &\cup \{(i,2,1,u_1,u_2,u_3,u_4,u_5) :\ 1 \le u_1 \le S_1,\ 1 \le u_2 \le S_2,\ 1 \le u_3 \le n_1,\ 1 \le u_4 \le n_2,\ 1 \le u_5 \le m\} \\ &\cup \{(i,3,0,u_1,u_2,u_5) :\ 1 \le u_1 \le S_1,\ 1 \le u_2 \le S_2,\ 1 \le u_5 \le m\} \\ &\cup \{(i,3,1,u_1,u_2,u_4,u_5) :\ 1 \le u_1 \le S_1,\ 1 \le u_2 \le S_2,\ 1 \le u_4 \le n_2,\ 1 \le u_5 \le m\} \end{aligned}$$

Notations:

- $\otimes$ - Kronecker product of two matrices of different dimensions.
- $\oplus$ - Kronecker sum of two matrices of different dimensions.
- e - Column vector has an suitable size with each of its entries as 1.
- e_0 - $e_{2S_1S_2m}$.
- e_1 - $e_{2S_1S_2n_1m+2S_1S_2n_2m+S_1S_2m}$.
- e_2 - $e_{2S_1S_2n_1m+2S_1S_2n_1n_2m+S_1S_2n_2m+S_1S_2m}$.
- I_j - Square matrix with jxj size with diagonal entries as 1.
- $\mathbf{0}$ - It denotes zero matrices in the suitable order.

3.1. Construction of the QBD process for our Model

The generator matrix of the Markov chain under (s, S) policy is given by:

$$Q = \begin{bmatrix} A_{00} & A_{01} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \dots \\ A_{10} & A_{11} & A_{12} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \dots \\ \mathbf{0} & A_{21} & F_1 & F_0 & \mathbf{0} & \mathbf{0} & \dots & \dots \\ \mathbf{0} & \mathbf{0} & F_2 & F_1 & F_0 & \mathbf{0} & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & F_2 & F_1 & F_0 & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \dots & \dots \end{bmatrix}.$$

The entries in the block matrices of $\mathbb{Q}$ are defined as follows:
$A_{00} = \begin{bmatrix} A_{00}^{11} & \mathbf{0} \\ A_{00}^{21} & A_{00}^{22} \end{bmatrix}$, $A_{00}^{11} = I_{S_1} \otimes P_1$, $A_{00}^{22} = I_{S_1} \otimes P_2$,

$$P_1 = \begin{bmatrix} C_1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_2 \\ \mathbf{0} & C_1 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_2 \\ \mathbf{0} & \mathbf{0} & C_1 & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_1 & \mathbf{0} & \dots & \mathbf{0} & C_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_3 & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_3 \end{bmatrix}, P_2 = \begin{bmatrix} C_5 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_6 \\ \mathbf{0} & C_5 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_6 \\ \mathbf{0} & \mathbf{0} & C_5 & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_6 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_5 & \mathbf{0} & \dots & \mathbf{0} & C_6 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_7 & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_7 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_7 \end{bmatrix},$$

$C_1 = D_0 - \beta I_m$, $C_2 = \beta I_m$, $C_3 = D_0$, $C_4 = I_{S_2} \otimes \tau I_m$, $C_5 = D_0 - (\tau + \beta) I_m$, $C_6 = \beta I_m$,
$C_7 = D_0 - \tau I_m$,

$$A_{00}^{21} = \begin{bmatrix} C_4 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_4 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_4 & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_4 \end{bmatrix}, \quad A_{01} = \begin{bmatrix} \mathbf{0} & A_{01}^{12} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{01}^{24} & \mathbf{0} \end{bmatrix},$$

$$A_{01}^{12} = \begin{bmatrix} C_8 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_8 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_8 & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_8 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_8 \end{bmatrix}, \quad A_{01}^{24} = \begin{bmatrix} C_9 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_9 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_9 & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_9 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_9 \end{bmatrix},$$

$C_8 = I_{S_2} \otimes \gamma_1 \otimes D_1$, $C_9 = I_{S_2} \otimes D_1$,

$$A_{10} = \begin{bmatrix} A_{10}^{11} & \mathbf{0} \\ A_{10}^{21} & \mathbf{0} \\ \mathbf{0} & A_{10}^{32} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_{10}^{52} \end{bmatrix}, A_{10}^{11} = I_{S_1} \otimes C_{10}, A_{10}^{21} = \begin{bmatrix} \mathbf{0} & C_{11} \\ I_{S_1 - 1} \otimes C_{11} & \mathbf{0} \end{bmatrix}, A_{10}^{32} = \begin{bmatrix} \mathbf{0} & C_{12} \\ I_{S_1 - 1} \otimes C_{12} & \mathbf{0} \end{bmatrix},$$

$A_{10}^{52} = I_{S_1} \otimes C_{13}$, $C_{10} = \begin{bmatrix} U_2^0 \otimes I_m & \mathbf{0} \\ I_{S_2 - 1} \otimes q_2 U_2^0 \otimes I_m & \mathbf{0} \end{bmatrix}$, $C_{11} = I_{S_2} \otimes q_1 U_1^0 \otimes I_m$,

$C_{12} = I_{S_2} \otimes \theta_1 U_1^0 \otimes I_m$, $C_{13} = \begin{bmatrix} U_2^0 \otimes I_m & \mathbf{0} \\ I_{S_2 - 1} \otimes q_2 U_2^0 \otimes I_m & \mathbf{0} \end{bmatrix}$,

$$A_{11} = \begin{bmatrix} A_{11}^{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_{11}^{22} & A_{11}^{23} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_{11}^{33} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_{11}^{42} & \mathbf{0} & A_{11}^{44} & A_{11}^{45} \\ A_{11}^{51} & \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{11}^{55} \end{bmatrix}, A_{11}^{11} = I_{S_1} \otimes P_3, A_{11}^{22} = \begin{bmatrix} P_4 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & P_5 \\ P_5 & P_4 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & P_5 & P_4 & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & P_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & P_5 & P_4 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} C_{14} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{15} \\ C_{16} & C_{14} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{15} \\ \mathbf{0} & C_{16} & C_{14} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{15} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{14} & \mathbf{0} & \dots & \mathbf{0} & C_{15} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{16} & C_{17} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{17} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{16} & C_{17} \end{bmatrix}, C_{14} = U_2 \oplus D_0 - \beta I_{n_2 m}, C_{15} = \beta I_{n_2 m},$$

$$C_{16} = p_2 U_2^0 \gamma_2 \otimes I_m, C_{17} = U_2 \oplus D_0, P_4 = \begin{bmatrix} C_{18} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{19} \\ \mathbf{0} & C_{18} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{19} \\ \mathbf{0} & \mathbf{0} & C_{18} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{19} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{18} & \mathbf{0} & \dots & \mathbf{0} & C_{19} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{20} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{20} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{20} \end{bmatrix},$$

$C_{18} = U_1 \oplus D_0 - (\beta + \psi) I_{n_1 m}$, $C_{19} = \beta I_{n_1 m}$, $C_{20} = U_1 \oplus D_0$, $P_5 = I_{S_2} \otimes p_1 U_1^0 \gamma_1 \otimes I_m$,

$$A_{11}^{23} = \begin{bmatrix} C_{21} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{21} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{21} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{21} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{21} \end{bmatrix}, C_{21} = I_{S_2} \otimes e_{n_1} \otimes \gamma_1 \otimes \psi I_m,$$

$$A_{11}^{33} = I_{S_1} \otimes P_6, A_{11}^{42} = \begin{bmatrix} C_{25} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{25} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{25} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{25} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{25} \end{bmatrix}, A_{11}^{44} = I_{S_1} \otimes P_7,$$

$$P_6 = \begin{bmatrix} C_{22} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{23} \\ \mathbf{0} & C_{22} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{23} \\ \mathbf{0} & \mathbf{0} & C_{22} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{23} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{22} & \mathbf{0} & \dots & C_{23} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{24} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{24} \end{bmatrix}, P_7 = \begin{bmatrix} C_{26} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{27} \\ \mathbf{0} & C_{26} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{27} \\ \mathbf{0} & \mathbf{0} & C_{26} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{27} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{26} & \mathbf{0} & \dots & C_{27} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{28} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{28} \end{bmatrix},$$

$C_{22} = \theta U_1 \oplus D_0 - \beta I_{n_1 m}$, $C_{23} = \beta I_{n_1 m}$, $C_{24} = \theta U_1 \oplus D_0$, $C_{25} = I_{S_2} \otimes \gamma_1 \otimes \tau I_m$,
$C_{26} = D_0 - (\tau + \eta + \beta) I_m$, $C_{27} = \beta I_m$, $C_{28} = D_0 - (\tau + \eta) I_m$,

$$A_{11}^{45} = \begin{bmatrix} C_{29} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{29} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{29} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{29} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{29} \end{bmatrix}, A_{11}^{51} = \begin{bmatrix} C_{30} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{30} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{30} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{30} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{30} \end{bmatrix},$$

$C_{29} = I_{S_2} \otimes \gamma_2 \otimes \eta I_m$, $C_{30} = I_{S_2} \otimes I_{n_2} \otimes \tau I_m$, $A_{11}^{55} = I_{S_1} \otimes P_7$,

$$P_7 = \begin{bmatrix} C_{31} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{32} \\ C_{33} & C_{31} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{32} \\ \mathbf{0} & C_{33} & C_{31} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{32} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{31} & \mathbf{0} & \dots & \mathbf{0} & C_{32} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{33} & C_{34} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{34} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{33} & C_{34} \end{bmatrix},$$

where
$C_{31} = U_2 \oplus D_0 - (\tau + \beta) I_{n_2 m}$, $C_{32} = \beta I_{n_2 m}$, $C_{33} = p_2 U_2^0 \gamma_2 \otimes I_m$, $C_{34} = U_2 \oplus D_0 - \tau I_{n_2 m}$,

$$A_{12} = \begin{bmatrix} \mathbf{0} & A_{12}^{12} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ A_{12}^{21} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_{12}^{33} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{12}^{45} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{12}^{55} \end{bmatrix}, A_{12}^{12} = \begin{bmatrix} C_{35} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{35} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{35} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{35} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{35} \end{bmatrix},$$

$$A_{12}^{21} = \begin{bmatrix} C_{36} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{36} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{36} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{36} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{36} \end{bmatrix}, C_{35} = I_{S_2} \otimes \gamma_1 \otimes I_{n_2} \otimes D_1, C_{36} = I_{S_2} \otimes I_{n_1} \otimes D_1,$$

$$A_{12}^{33} = \begin{bmatrix} C_{37} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{37} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{37} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{37} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{37} \end{bmatrix}, A_{12}^{45} = \begin{bmatrix} C_{38} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{38} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{38} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{38} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{38} \end{bmatrix},$$

$$C_{37} = I_{S_2} \otimes I_{n_1} \otimes D_1, C_{38} = I_{S_2} \otimes D_1, A_{12}^{45} = \begin{bmatrix} C_{39} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{39} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{39} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{39} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{39} \end{bmatrix},$$

$$C_{39} = I_{S_2} \otimes I_{n_2} \otimes D_1, \, A_{21} = \begin{bmatrix} \mathbf{0} & A_{21}^{12} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ A_{21}^{21} & A_{21}^{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{21}^{34} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_{21}^{43} & \mathbf{0} & A_{21}^{45} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & A_{21}^{65} \end{bmatrix}, \, A_{21}^{12} = \begin{bmatrix} \mathbf{0} & C_{40} \\ I_{S_1-1} \otimes C_{40} & \mathbf{0} \end{bmatrix},$$

$$A_{21}^{21} = I_{S_1} \otimes C_{41}, \, A_{21}^{22} = \begin{bmatrix} \mathbf{0} & C_{42} \\ I_{S_1-1} \otimes C_{42} & \mathbf{0} \end{bmatrix}, \, C_{40} = I_{S_2} \otimes q_1 U_1^0 \gamma_1 \otimes I_m,$$

$$C_{41} = \begin{bmatrix} e_{n_1} \otimes U_2^0 \gamma_1 \otimes I_m & \mathbf{0} \\ I_{S_2-1} \otimes e_{n_1} \otimes q_2 U_2^0 \gamma_1 \otimes I_m & \mathbf{0} \end{bmatrix}, \, C_{42} = I_{S_2} \otimes e_{n_2} \otimes q_1 U_1^0 \gamma_2 \otimes I_m, \, A_{21}^{34} = \begin{bmatrix} \mathbf{0} & C_{43} \\ I_{S_1-1} \otimes C_{43} & \mathbf{0} \end{bmatrix},$$

$$A_{21}^{43} = I_{S_1} \otimes C_{44}, \, A_{21}^{45} = \begin{bmatrix} \mathbf{0} & C_{45} \\ I_{S_1-1} \otimes C_{45} & \mathbf{0} \end{bmatrix}, \, A_{21}^{65} = I_{S_1} \otimes C_{46}, \, C_{43} = I_{S_2} \otimes \theta U_1^0 \otimes I_m,$$

$$C_{44} = \begin{bmatrix} I_{n_1} \otimes U_2^0 \otimes I_m & \mathbf{0} \\ I_{S_2-1} \otimes I_{n_1} \otimes q_2 U_2^0 \otimes I_m & \mathbf{0} \end{bmatrix}, \, C_{45} = I_{S_2} \otimes \theta U_1^0 \otimes I_{n_2 m}, \, C_{46} = \begin{bmatrix} U_2^0 \gamma_2 \otimes I_m & \mathbf{0} \\ I_{S_2-1} \otimes q_2 U_2^0 \gamma_2 \otimes I_m & \mathbf{0} \end{bmatrix},$$

$$F_1 = \begin{bmatrix} F_1^{11} & F_1^{12} & F_1^{13} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & F_1^{22} & F_1^{23} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & F_1^{33} & F_1^{34} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & F_1^{44} & \mathbf{0} & \mathbf{0} \\ F_1^{51} & \mathbf{0} & \mathbf{0} & \mathbf{0} & F_1^{55} & F_1^{56} \\ \mathbf{0} & F_1^{62} & \mathbf{0} & \mathbf{0} & \mathbf{0} & F_1^{66} \end{bmatrix}, \, F_1^{11} = \begin{bmatrix} P_8 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & P_9 \\ P_9 & P_8 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & P_9 & P_8 & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & P_8 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & P_9 & P_8 \end{bmatrix},$$

$$P_8 = \begin{bmatrix} C_{47} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{48} \\ \mathbf{0} & C_{47} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{48} \\ \mathbf{0} & \mathbf{0} & C_{47} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{48} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{47} & \mathbf{0} & \dots & \mathbf{0} & C_{48} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{49} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{49} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{49} \end{bmatrix}, \, P_9 = I_{S_2} \otimes p_1 U_1^0 \gamma_1 \otimes I_m,$$

$$C_{47} = U_1 \oplus D_0 - (\eta + \beta + \psi) I_{n_1 m}, \, C_{48} = \beta I_{n_1 m}, \, C_{49} = U_1 \oplus D_0 - (\eta + \psi) I_{n_1 m},$$

$$F_1^{12} = \begin{bmatrix} C_{50} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{50} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{50} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{50} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{50} \end{bmatrix}, \, F_1^{13} = \begin{bmatrix} C_{51} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{51} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{51} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{51} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{51} \end{bmatrix},$$

$$C_{50} = I_{S_2} \otimes I_{n_1} \gamma_2 \otimes \eta I_m, \, C_{51} = I_{S_2} \otimes e_{n_1} \otimes \gamma_1 \otimes \psi I_m,$$

$$F_1^{22} = \begin{bmatrix} P_{10} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & P_{11} \\ P_{11} & P_{10} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & P_{11} & P_{10} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & P_{10} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & P_{11} & P_{10} \end{bmatrix}, \, F_1^{24} = \begin{bmatrix} C_{57} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{57} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{57} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{57} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{57} \end{bmatrix},$$

$$P_{10} = \begin{bmatrix} C_{52} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{53} \\ C_{54} & C_{52} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{53} \\ \mathbf{0} & C_{54} & C_{52} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{53} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{52} & \mathbf{0} & \dots & \mathbf{0} & C_{52} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{54} & C_{55} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{55} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{54} & C_{55} \end{bmatrix}, P_{11} = \begin{bmatrix} C_{56} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{56} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{56} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{56} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{56} \end{bmatrix},$$

where
$C_{52} = U_1 \oplus U_2 \oplus D_0 - (\psi + \beta) I_{n_1 n_2 m}$, $C_{53} = \beta I_{n_1 n_2 m}$, $C_{54} = I_{n_1} \otimes p_2 U_2^0 \gamma_2 \otimes I_m$,
$C_{55} = U_1 \oplus U_2 \oplus D_0 - \psi I_{n_1 n_2 m}$, $C_{56} = I_{S_2} \otimes p_1 U_1^0 \gamma_1 \otimes I_{n_2 m}$, $C_{57} = I_{S_2} \otimes e_{n_1} \otimes \gamma_1 \otimes \psi I_{n_2 m}$,
$F_1^{33} = I_{S_1} \otimes P_{12}$,

$$P_{12} = \begin{bmatrix} C_{58} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{59} \\ \mathbf{0} & C_{58} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{59} \\ \mathbf{0} & \mathbf{0} & C_{58} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{59} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{58} & \mathbf{0} & \dots & \mathbf{0} & C_{59} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{60} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{60} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{60} \end{bmatrix},$$

$C_{58} = \theta U_1 \oplus D_0 - (\eta + \beta) I_{n_1 m}$, $C_{59} = \beta I_{n_1 m}$, $C_{60} = \theta U_1 \oplus D_0 - \eta I_{n_1 m}$,

$$F_1^{34} = \begin{bmatrix} C_{61} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{61} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{61} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{61} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{61} \end{bmatrix}, F_1^{44} = \begin{bmatrix} C_{62} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{63} \\ C_{64} & C_{62} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{63} \\ \mathbf{0} & C_{64} & C_{62} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{63} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{62} & \mathbf{0} & \dots & \mathbf{0} & C_{63} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{64} & C_{65} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{65} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{64} & C_{65} \end{bmatrix},$$

where
$C_{61} = I_{S_2} \otimes I_{n_1} \otimes \gamma_2 \otimes \eta I_m$, $C_{62} = \theta U_1 \oplus U_2 \oplus D_0 - \beta I_{n_1 n_2 m}$, $C_{63} = \beta I_{n_1 n_2 m}$,
$C_{64} = I_{n_1} \otimes p_2 U_2^0 \gamma_2 \otimes I_m$, $C_{65} = \theta U_1 \oplus U_2 \oplus D_0$,

$$F_1^{51} = \begin{bmatrix} C_{66} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{66} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{66} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{66} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{66} \end{bmatrix}, F_1^{55} = I_{S_1} \otimes P_{13},$$

$$F_1^{56} = \begin{bmatrix} C_{70} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{70} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{70} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{70} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{70} \end{bmatrix}, P_{13} = \begin{bmatrix} C_{67} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{68} \\ \mathbf{0} & C_{67} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{68} \\ \mathbf{0} & \mathbf{0} & C_{67} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{68} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{67} & \mathbf{0} & \dots & \mathbf{0} & C_{68} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{69} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{69} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{69} \end{bmatrix},$$

where
$C_{66} = I_{S_2} \otimes \gamma_1 \otimes \tau I_m$, $C_{67} = D_0 - (\tau + \eta + \beta) I_m$, $C_{68} = \beta I_m$, $C_{69} = D_0 - (\eta + \tau) I_m$,

$$C_{70} = I_{S_2} \otimes \gamma_2 \otimes \eta I_m,\ F_1^{62} = \begin{bmatrix} C_{71} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{71} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{71} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{71} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{71} \end{bmatrix},\ C_{71} = I_{S_2} \otimes \gamma_1 \otimes \tau I_{n_2 m},$$

$$F_1^{66} = \begin{bmatrix} C_{72} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{73} \\ C_{74} & C_{72} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{73} \\ \mathbf{0} & C_{74} & C_{72} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{73} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{72} & \mathbf{0} & \dots & \mathbf{0} & C_{73} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{74} & C_{75} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{75} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & C_{74} & C_{75} \end{bmatrix},$$

where
$C_{72} = U_2 \oplus D_0 - (\tau + \beta) I_{n_2 m}$, $C_{73} = \beta I_{n_2 m}$, $C_{74} = p_2 U_2^0 \gamma_2 \otimes I_m$, $C_{75} = U_2 \oplus D_0 - \tau I_{n_2 m}$,

$$F_0 = \begin{bmatrix} F_0^{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & F_0^{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & F_0^{33} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & F_0^{44} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & F_0^{55} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & F_0^{66} \end{bmatrix},\ F_0^{11} = \begin{bmatrix} C_{76} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{76} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{76} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{76} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{76} \end{bmatrix},$$

$$F_0^{22} = \begin{bmatrix} C_{77} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{77} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{77} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{77} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{77} \end{bmatrix},\ C_{76} = I_{S_2} \otimes I_{n_1} \otimes D_1,\ C_{77} = I_{S_2} \otimes I_{n_1 n_2} \otimes D_1,$$

$$F_0^{33} = \begin{bmatrix} C_{78} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{78} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{78} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{78} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{78} \end{bmatrix},\ F_0^{44} = \begin{bmatrix} C_{79} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{79} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{79} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{79} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{79} \end{bmatrix},$$

where
$C_{78} = I_{S_2} \otimes I_{n_1} \otimes D_1$, $C_{79} = I_{S_2} \otimes I_{n_1 n_2} \otimes D_1$

$$F_0^{55} = \begin{bmatrix} C_{80} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{80} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{80} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{80} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{80} \end{bmatrix},\ F_0^{66} = \begin{bmatrix} C_{81} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{81} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{81} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & C_{81} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & C_{81} \end{bmatrix},$$

$C_{80} = I_{S_2} \otimes D_1$, $C_{81} = I_{S_2} \otimes I_{n_2} \otimes D_1$,

$$F_2 = \begin{bmatrix} F_2^{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & F_2^{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & F_2^{35} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & F_2^{44} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & F_2^{66} \end{bmatrix}, F_2^{11} = \begin{bmatrix} \mathbf{0} & C_{82} \\ I_{S_1-1} \otimes C_{82} & \mathbf{0} \end{bmatrix},$$

$$F_2^{22} = \begin{bmatrix} P_{14} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & P_{15} \\ P_{15} & P_{14} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & P_{15} & P_{14} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & P_{14} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & P_{15} & P_{14} \end{bmatrix},$$

where

$$C_{82} = I_{S_2} \otimes q_1 U_1^0 \gamma_1 \otimes I_m, P_{14} = \begin{bmatrix} I_{n_1} \otimes U_2^0 \gamma_2 \otimes I_m & \mathbf{0} \\ I_{S_2-1} \otimes I_{n_1} \otimes q_2 U_2^0 \gamma_2 \otimes I_m & \mathbf{0} \end{bmatrix}, P_{15} = I_{S_2} \otimes q_1 U_1^0 \gamma_1 \otimes I_{n_2 m},$$

$$F_2^{35} = \begin{bmatrix} \mathbf{0} & C_{83} \\ I_{S_1-1} \otimes C_{83} & \mathbf{0} \end{bmatrix}, F_2^{44} = \begin{bmatrix} \mathbf{0} & C_{84} \\ I_{S_1-1} \otimes C_{84} & \mathbf{0} \end{bmatrix}, F_2^{66} = I_{S_2} \otimes C_{85},$$

where

$$C_{83} = I_{S_2} \otimes \theta U_1^0 \otimes I_m, C_{84} = I_{S_2} \otimes \theta U_1^0 \otimes I_{n_2 m}, C_{85} = \begin{bmatrix} U_2^0 \gamma_2 \otimes I_m & \mathbf{0} \\ I_{S_2-1} \otimes q_2 U_2^0 \gamma_2 \otimes I_m & \mathbf{0} \end{bmatrix}.$$

3.2. Stability condition

To discuss the stability condition, we first consider the generator matrix $F = F_0 + F_1 + F_2$. The vector χ is the invariant vector of the matrix F. Then, relations $\chi F = 0$ and $\chi e = 1$ and The LIQBD fashion with infinitesimal generator Q is stable if and only if

$$\chi F_0 e < \chi F_2 e.$$

The stability obtained after some mathematical rearranging is shown below:

$$\begin{aligned} &\boldsymbol{\chi_0}[e_{S_1 S_2 n_1} \otimes D_1 e_m] + \boldsymbol{\chi_1}[e_{S_1 S_2 n_1 n_2} \otimes D_1 e_m] + \boldsymbol{\chi_2}[e_{S_1 S_2 n_1} \otimes D_1 e_m] + \boldsymbol{\chi_3}[e_{S_1 S_2 n_1 n_2} \otimes D_1 e_m] \\ &+ \boldsymbol{\chi_4}[e_{S_1 S_2} \otimes D_1 e_m] + \boldsymbol{\chi_5}[e_{S_1 S_2 n_2} \otimes D_1 e_m] < \boldsymbol{\chi_0}[e_{S_2} \otimes q_1 U_1^0 \otimes e_m] \\ &+ \boldsymbol{\chi_1}\Big(e_{S_1} \otimes [(e_{n_1} \otimes U_2^0 \otimes e_m + q_1 U_1^0 \otimes e_{n_2 m}) + e_{S_2-1} \otimes (e_{n_1} q_2 U_2^0 \otimes e_m + q_1 U_1^0 \otimes e_{n_2 m})]\Big) \\ &+ \boldsymbol{\chi_2}[e_{S_1 S_2} \otimes \theta U_1^0 \otimes e_{n_1 m}] + \boldsymbol{\chi_3}[e_{S_1 S_2} \otimes \theta U_1^0 \otimes e_{n_2 m}] \\ &+ \boldsymbol{\chi_5}\Big(e_{S_1} \otimes [U_2^0 \otimes e_m + e_{S_2-1} \otimes q_2 U_2^0 \otimes e_m]\Big). \end{aligned}$$

3.3. The steady state probability vector

Let X be the steady state probability vector of the infintesimal generator $\mathbb{Q}$ of the process {X(t): $t \geq 0$}. The subdivision of $X = (x_0, x_1, x_2, ...)$, where x_0 is of dimension $2(S_1 S_2 m)$, x_1 is of dimension $2(S_1 S_2 n_2 m) + 2(S_1 S_2 n_1 m) + S_1 S_2 m$ and $x_2, x_3, ...$ are of dimension $2(S_1 S_2 n_1 m) + 2(S_1 S_2 n_1 n_2 m) + S_1 S_2 m + S_1 S_2 n_2 m$. As X is a vector satisfies the relation

$$X\mathbb{Q} = 0 \text{ and } Xe = 1.$$

The probability vector X follows a matrix geometric structure under the steady state is

$$x_j = x_2 R^{j-1}, \ j \geq 3 \tag{2}$$

where R is the quadratic equation's lowest non-negative solution

$$R^2F_2 + RF_1 + F_0 = 0$$

and the vector $x_0, x_1 and x_2$ are obtained with the help of succeeding equations:

$$x_0A_{00} + x_1A_{10} = 0, \tag{3}$$

$$x_0A_{01} + x_1A_{11} + x_2A_{21} = 0, \tag{4}$$

$$x_1A_{12} + x_2[F_1 + RF_2] = 0, \tag{5}$$

subject to a condition normalization

$$x_0e_0 + x_1e_1 + x_2[I - R]^{-1}e_2 = 1. \tag{6}$$

Computing the rate matrix R is necessary before attempting to solve the set of equations mentioned above. However, [9] used Logarithmic reduction approach, an algorithm that makes it simple to produce R.

4. System characteristics

- Probability of the system is empty:

$$P_{empty} = x_0e_0.$$

- The probability of the server-1 is idle:

$$P_{idle} = \sum_{u_1=1}^{S_1}\sum_{u_2=1}^{S_2}\sum_{u_5=1}^{m} x_{000u_1u_2u_5} + \sum_{u_1=1}^{S_1}\sum_{u_2=1}^{S_2}\sum_{u_4=1}^{n_2}\sum_{u_5=1}^{m} x_{101u_1u_2u_4u_5}.$$

- The probability of the server-2 is on vacation:

$$\begin{aligned} P_{vac} = &\sum_{u_1=1}^{S_1}\sum_{u_2=1}^{S_2}\sum_{u_5=1}^{m} x_{000u_1u_2u_5} + \sum_{u_1=1}^{S_1}\sum_{u_2=1}^{S_2}\sum_{u_5=1}^{m} x_{030u_1u_2u_5} \\ &+ \sum_{u_1=1}^{S_1}\sum_{u_2=1}^{S_2}\sum_{u_3=1}^{n_1}\sum_{u_5=1}^{m} x_{110u_1u_2u_3u_5} + \sum_{u_1=1}^{S_1}\sum_{u_2=1}^{S_2}\sum_{u_3=1}^{n_1}\sum_{u_5=1}^{m} x_{120u_1u_2u_3u_5} \\ &+ \sum_{u_1=1}^{S_1}\sum_{u_2=1}^{S_2}\sum_{u_5=1}^{m} x_{130u_1u_2u_5} + \sum_{i=2}^{\infty}\sum_{u_1=1}^{S_1}\sum_{u_2=1}^{S_2}\sum_{u_3=1}^{n_1}\sum_{u_5=1}^{m} x_{i10u_1u_2u_3u_5} \\ &+ \sum_{i=2}^{\infty}\sum_{u_1=1}^{S_1}\sum_{u_2=1}^{S_2}\sum_{u_3=1}^{n_1}\sum_{u_5=1}^{m} x_{i20u_1u_2u_3u_5} + \sum_{i=2}^{\infty}\sum_{u_1=1}^{S_1}\sum_{u_2=1}^{S_2}\sum_{u_5=1}^{m} x_{i30u_1u_2u_5}. \end{aligned}$$

- The probability of the server-1 is offering service in normal mode:

$$P_{S_1B} = \sum_{i=1}^{\infty}\sum_{u_1=1}^{S_1}\sum_{u_2=1}^{S_2}\sum_{u_3=1}^{n_1}\sum_{u_5=1}^{m} x_{i10u_1u_2u_3u_5} + \sum_{i=2}^{\infty}\sum_{u_1=1}^{S_1}\sum_{u_2=1}^{S_2}\sum_{u_3=1}^{n_1}\sum_{u_4=1}^{n_2}\sum_{u_5=1}^{m} x_{i11u_1u_2u_3u_4u_5}.$$

- The probability of the server-1 is offering service in working breakdown:

$$P_{S_1WBD} = \sum_{i=1}^{\infty}\sum_{u_1=1}^{S_1}\sum_{u_2=1}^{S_2}\sum_{u_3=1}^{n_1}\sum_{u_5=1}^{m} x_{i20u_1u_2u_3u_5} + \sum_{i=2}^{\infty}\sum_{u_1=1}^{S_1}\sum_{u_2=1}^{S_2}\sum_{u_3=1}^{n_1}\sum_{u_5=1}^{m} x_{i21u_1u_2u_3u_4u_5}.$$

- The probability of the server-2 is offering service in normal mode:

$$P_{S_2B} = \sum_{u_1=1}^{S_1}\sum_{u_2=1}^{S_2}\sum_{u_4=1}^{n_2}\sum_{u_5=1}^{m} x_{101u_1u_2u_4u_5} + \sum_{i=1}^{\infty}\sum_{u_1=1}^{S_1}\sum_{u_2=1}^{S_2}\sum_{u_4=1}^{n_2}\sum_{u_5=1}^{m} x_{i31u_1u_2u_4u_5}$$
$$+ \sum_{i=2}^{\infty}\sum_{u_1=1}^{S_1}\sum_{u_2=1}^{S_2}\sum_{u_3=1}^{n_1}\sum_{u_4=1}^{n_2}\sum_{u_5=1}^{m} x_{i21u_1u_2u_3u_4u_5}.$$

- The probability of the server-1 is on Repair:

$$P_{S_1R} = \sum_{u_1=1}^{S_1}\sum_{u_2=1}^{S_2}\sum_{u_5=1}^{m} x_{030u_1u_2u_5} + \sum_{i=1}^{\infty}\sum_{u_1=1}^{S_1}\sum_{u_2=1}^{S_2}\sum_{u_5=1}^{m} x_{i30u_1u_2u_5}$$
$$+ \sum_{i=1}^{\infty}\sum_{u_1=1}^{S_1}\sum_{u_2=1}^{S_2}\sum_{u_4=1}^{n_2}\sum_{u_5=1}^{m} x_{i31u_1u_2u_4u_5}.$$

- Expected number of customers in the system:

$$E_{system} = \sum_{i=1}^{\infty} i x_i e = x_1 e_1 + x_2[2(I-R)^{-1} + R(I-R)^{-2}]e_2.$$

- Expected first inventory level:

$$E_{IL_1} = \sum_{i=1}^{\infty}\sum_{u_1=1}^{S_1}\sum_{u_2=1}^{S_2} u_1 x_i(u_1, u_2)$$

- Expected first inventory level:

$$E_{IL_2} = \sum_{i=1}^{\infty}\sum_{u_1=1}^{S_1}\sum_{u_2=1}^{S_2} u_2 x_i(u_1, u_2)$$

- Expected reorder rate with first commodity

$$E_{RR_1} = \sum_{i=1}^{\infty}\sum_{u_2=1}^{S_2}\sum_{u_3=1}^{n_1}\sum_{u_5=1}^{m} [x_{i101u_2u_3u_5}(U_1^0\gamma_1 \otimes I_m)e + x_{i201u_2u_3u_5}(\theta U_1^0 \otimes I_m)e]$$
$$+ \sum_{i=2}^{\infty}\sum_{u_2=1}^{S_2}\sum_{u_3=1}^{n_1}\sum_{u_4=1}^{n_2}\sum_{u_5=1}^{m} x_{i111u_2u_3u_4u_5}(U_1^0\gamma_1 \otimes I_{n_2m})e$$
$$+ \sum_{i=2}^{\infty}\sum_{u_2=1}^{S_2}\sum_{u_3=1}^{n_1}\sum_{u_4=1}^{n_2}\sum_{u_5=1}^{m} x_{i211u_2u_3u_4u_5}(U_1^0\gamma_1 \otimes I_{n_2m})e.$$

- Expected reorder rate with second commodity

$$E_{RR_2} = \sum_{u_1=1}^{S_1}\sum_{u_4=1}^{n_2}\sum_{u_5=1}^{m} x_{101u_1(s_2+1)u_5}(U_2^0 \otimes I_m)e$$
$$+ \sum_{i=2}^{\infty}\sum_{u_2=1}^{S_2}\sum_{u_3=1}^{n_1}\sum_{u_4=1}^{n_2}\sum_{u_5=1}^{m} [x_{i11u_1(s_2+1)u_3u_4u_5} + x_{i21u_1(s_2+1)u_3u_4u_5}](I_{n_1} \otimes U_2^0\gamma_2 \otimes I_m)e$$
$$+ \sum_{i=1}^{\infty}\sum_{u_2=1}^{S_2}\sum_{u_4=1}^{n_2}\sum_{u_5=1}^{m} x_{i31u_1(s_2+1)u_4u_5}(U_2^0\gamma_2 \otimes I_{n_2m})e.$$

5. Cost Analysis

The cost function for our model was created with the premise that each cost element (per unit of time) correlates to a distinct system measure.

- C_{I_1} - the first item in inventory with a cost per unit
- C_{I_2} - the second item in the inventory with a cost per unit
- C_H - storing a customer's cost in the system for each unit of time.
- C_{R_1} -setup costs for each order of the primary item
- C_{R_2} - Setup costs for each order of complimentary items

$$TC = C_{I_1} E_{IL_1} + C_{I_2} E_{IL_2} + C_H E_{system} + C_{R_1} E_{RR_1} + C_{R_2} E_{RR_2}$$

6. Numerical Implementation

To compute numerical outcomes, we have employed diverse MAP demonstrations for the incoming arrival in a manner that ensures their mean values are 1, as recommended by [5].

- **Erlang arrival** (ERA):

$$D_0 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix} D_1 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

- **Exponential arrival** (EXA):

$$D_0 = [-1] D_1 = [1]$$

- **Hyper exponential arrival** $(HEXA)$:

$$D_0 = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix} D_1 = \begin{bmatrix} 1.710 & 0.190 \\ 0.171 & 0.019 \end{bmatrix}$$

- **MAP-Negative Correlation arrival** $(MNCA)$:

$$D_0 = \begin{bmatrix} -1.00243 & 1.00243 & 0 \\ 0 & -1.00243 & 0 \\ 0 & 0 & -225.797 \end{bmatrix} \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.01002 & 0 & 0.99241 \\ 223.539 & 0 & 2.258 \end{bmatrix}$$

- **MAP-Positive Correlation arrival** $(MPCA)$:

$$D_0 = \begin{bmatrix} -1.00243 & 1.00243 & 0 \\ 0 & -1.00243 & 0 \\ 0 & 0 & -225.797 \end{bmatrix} \quad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.99241 & 0 & 0.01002 \\ 2.258 & 0 & 223.539 \end{bmatrix}$$

Consider the following PH-distributions for the service and repair progression:

- **Erlang service** (ERS):

$$\gamma = [1, 0] \quad U = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$$

- **Exponential service** (EXS):

$$\gamma = [1] \quad U = [-1]$$

- **Hyper exponential service** $(HEXS)$:

$$\gamma = [0.8, 0.2] \quad U = \begin{bmatrix} -2.8 & 0 \\ 0 & -0.28 \end{bmatrix}$$

6.1. Illustrative 1

We examine the consequence of the service rate of server-2 (μ_2) versus the expected system size (E_{system}) in the Table 1-3. We fix $\lambda = 1$, $\mu_1 = 12$, $\eta = 1$, $\beta = 1$, $\psi = 1$, $\tau = 2$, $S_1 = 4$, $S_2 = 6$, $s_2 = 3$, $p_1 = 0.5$, $q_1 = 1 - p_1$, $p_2 = 0.5$, $q_2 = 1 - p_2$, $\theta = 0.6$, such that the system remains stable. The observation from Table 1-3 as follows:

- While we maximize the service rate of the server-2 then the corresponding E_{system} decreases with the combination of arrival and service times.
- From the point of view of arrival times, E_{system} decreases highly for $HYPA$ and decreases slowly for $ERLA$ while an increase the server-2's service rate. However, consider the service times, E_{system} decreases highly in $ERLS$ and decreases slowly in $HYPS$ with the combination of $EXPA$, $MNCA$ and $MPCA$, but in the case of $ERLA$ and $HYPA$, Esystem decreases slowly in $EXPS$ and decreases fastly in $HYPS$.

6.2. Illustration 2

With the support of the Tables 4-6, we visualize the influence of the service rate (μ_2) of the server-2 upon the probability of server-2 is undergoing vacation (P_{vac}). Fix $\lambda = 1$, $\mu_1 = 12$, $\eta = 1$, $\beta = 1$, $\psi = 1$, $\tau = 2$, $S_1 = 4$, $S_2 = 6$, $s_2 = 3$, $p_1 = 0.5$, $q_1 = 1 - p_1$, $p_2 = 0.5$, $q_2 = 1 - p_2$, $\theta = 0.6$, so that the stability condition is satisfied.

- Observation of Tables 4-6 discloses the fact that P_{vac} maximizes while maximizing the service rate of the server for the distinct feasible ordering of service and arrival times.
- This is because an increase in service rate leads to a decrease in the duration of service time respectively. As a result, the server will be getting more chances to go on vacation. Besides, the increment rate is higher in the case of $HYPA$ and lower in the case of $ERLA$. In the same way, from the service time point of view, the increment rate is rapid for $HYPS$ and gradual for $ERLS$.

6.3. Illustrative 3

With the support of Tables 7-9, we visualize the impact of the repair rate of the server-1(τ) on the Probability of server-1 being busy(P_{S_1B}). Fix $\lambda = 1$, $\mu_1 = 12$, $\mu_2 = 10$, $\eta = 1$, $\beta = 1$, $\psi = 1$, $S_1 = 4$, $S_2 = 6$, $s_2 = 3$, $p_1 = 0.5$, $q_1 = 1 - p_1$, $p_2 = 0.5$, $q_2 = 1 - p_2$, $\theta = 0.6$, so that the stability condition is satisfied.

- From Table 7-9, we may view that as the repair rate of server-1(τ) increases, (P_{S_1B}) increases for all possible groupings of service and arrival times.
- An increase in repair rate implies that the server takes minimum time to complete the repair process.
- As a result, the server will get more time to serve the customer and so (P_{S_1B}) increases. Hence, the system can focus on decreasing the time taken to repair the failed server for its optimal utilization. Moreover, the speed of increment of (P_{S_1B}) is high for $ERLA$ and low for $HYPA$. In the same way, it is high for $ERLS$ and low for $HYPS$.

Table 1: *Server-2 service rate (μ_2) vs E_{system} - ERLS*

μ_2	$ERLA$	$EXPA$	$HYPA$	$MNCA$	$MPCA$
8.0	0.197384249	0.224815927	0.2713795337	0.290089474	4.308746051
8.5	0.196918323	0.224009689	0.269865612	0.288694105	4.167282560
9.0	0.196498144	0.223275819	0.268486466	0.287407096	4.034807543
9.5	0.196117354	0.222604963	0.267225347	0.286215133	3.910580674
10.0	0.195770689	0.221989289	0.266068055	0.285107103	3.793926042
10.5	0.195453765	0.221422194	0.265002498	0.284073649	3.684228530
11.0	0.195162900	0.220898080	0.264018333	0.283106828	3.580929431
11.5	0.194894989	0.220412169	0.263106672	0.282199845	3.483521781
12.0	0.194647394	0.219960366	0.262259844	0.281346850	3.391545721
12.5	0.194417864	0.219539139	0.261471205	0.280542773	3.304584052

Table 2: *Server-2 service rate (μ_2) vs E_{system} - EXPS*

μ_2	$ERLA$	$EXPA$	$HYPA$	$MNCA$	$MPCA$
8.0	0.202436650	0.229325675	0.276190307	0.288189790	4.284049580
8.5	0.201923153	0.228463897	0.274610523	0.286720757	4.143969323
9.0	0.201458764	0.227678126	0.273168421	0.285367819	4.012769799
9.5	0.201036863	0.226958737	0.27184722	0.284116698	3.889718360
10.0	0.200651935	0.226297636	0.270632665	0.282955451	3.774147068
10.5	0.200299357	0.225687981	0.269512585	0.281874005	3.665448646
11.0	0.199975227	0.225123948	0.268476541	0.280863797	3.563071842
11.5	0.199676236	0.224600554	0.267515540	0.279917499	3.466516647
12.0	0.199399561	0.224113511	0.266621796	0.279028795	3.375329599
12.5	0.199142780	0.223659113	0.265788546	0.278192214	3.289099329

Table 3: *Server-2 service rate (μ_2) vs E_{system} - HYPS*

μ_2	$ERLA$	$EXPA$	$HYPA$	$MNCA$	$MPCA$
8.0	0.227728762	0.247910753	0.291707349	0.281256741	4.144213513
8.5	0.226937445	0.246779024	0.289865528	0.279466679	4.010955831
9.0	0.226211855	0.245738665	0.288170867	0.277819490	3.886146349
9.5	0.225544345	0.244779141	0.286606470	0.276298631	3.769067923
10.0	0.224928388	0.243891462	0.285157942	0.274889987	3.659073727
10.5	0.224358381	0.243067914	0.283812936	0.273581450	3.555580430
11.0	0.223829490	0.242301839	0.282560799	0.272362579	3.458061772
11.5	0.223337523	0.241587463	0.281392290	0.271224328	3.366042632
12.0	0.222878828	0.240919754	0.280299352	0.270158822	3.279093637
12.5	0.222450210	0.240294308	0.279274929	0.269159176	3.196826301

Table 4: *Server-2 service rate (μ_2) vs P_{vac} - ERLS*

μ_2	$ERLA$	$EXPA$	$HYPA$	$MNCA$	$MPCA$
8.0	0.992696356	0.988326139	0.981073741	0.978176264	0.960252482
8.5	0.993059887	0.988853514	0.981898129	0.979032770	0.961666866
9.0	0.993389158	0.989334618	0.982654586	0.979817153	0.962989215
9.5	0.993688677	0.989775249	0.983350966	0.980538758	0.964227717
10.0	0.993962221	0.990180294	0.983993994	0.981205329	0.965389685
10.5	0.994212971	0.990553889	0.984589456	0.981823341	0.966481656
11.0	0.994443622	0.990899561	0.985142349	0.982398253	0.967509488
11.5	0.994656468	0.991220336	0.985657014	0.982934708	0.968478435
12.0	0.994853476	0.991518816	0.986137231	0.983436680	0.969393223
12.5	0.995036336	0.991797261	0.986586309	0.983907592	0.970258104

Table 5: *Server-2 service rate (μ_2) vs P_{vac} - EXPS*

μ_2	$ERLA$	$EXPA$	$HYPA$	$MNCA$	$MPCA$
8.0	0.991933009	0.987434189	0.980074405	0.976954919	0.960264988
8.5	0.992324808	0.987994543	0.980928958	0.977881661	0.961679429
9.0	0.992680715	0.988506494	0.981713965	0.978729839	0.963001556
9.5	0.993005319	0.988976019	0.982437430	0.979509541	0.964239645
10.0	0.993302482	0.989408162	0.983106196	0.980229156	0.965401072
10.5	0.993575474	0.989807206	0.983726147	0.980895720	0.966492420
11.0	0.993827078	0.990176811	0.984302365	0.981515186	0.967519581
11.5	0.994059676	0.990520121	0.984839265	0.982092622	0.968487832
12.0	0.994275317	0.990839849	0.985340695	0.982632376	0.969401917
12.5	0.994475769	0.991138351	0.985810023	0.983138200	0.970266099

Table 6: *Server-2 service rate (μ_2) vs P_{vac} - HYPS*

μ_2	$ERLA$	$EXPA$	$HYPA$	$MNCA$	$MPCA$
8.0	0.987301352	0.982924032	0.975917875	0.971500408	0.960007628
8.5	0.987838969	0.983624068	0.976870150	0.972668334	0.961435754
9.0	0.988334016	0.984269283	0.977749342	0.973743753	0.962769344
9.5	0.988791273	0.984865824	0.978563672	0.974737072	0.964017128
10.0	0.989214835	0.985418958	0.979320150	0.975657230	0.965186832
10.5	0.989608227	0.985933220	0.980024790	0.976511937	0.966285315
11.0	0.989974506	0.986412541	0.980682791	0.977307876	0.967318682
11.5	0.990316333	0.986860343	0.981298668	0.978050865	0.968292380
12.0	0.990636036	0.987279615	0.981876366	0.97874599	0.969211284
12.5	0.990935659	0.987672986	0.982419347	0.979397714	0.970079769

Table 7: *Server-1 repair rate (τ) vs P_{S_1B} - ERLS*

τ	$ERLA$	$EXPA$	$HYPA$	$MNCA$	$MPCA$
2.1	0.141008388	0.13790536	0.13246213	0.130940643	0.075845503
2.2	0.141253797	0.138188589	0.132887094	0.131195807	0.075919075
2.3	0.141473036	0.138443996	0.133273312	0.131426853	0.07598541
2.4	0.141669716	0.138675233	0.133625511	0.131636941	0.076045499
2.5	0.141846844	0.138885360	0.133947705	0.131828717	0.076100165
2.6	0.142006944	0.139076961	0.134243323	0.132004409	0.076150095
2.7	0.142152144	0.139252229	0.134515310	0.132165904	0.076195868
2.8	0.142284249	0.139413038	0.134766207	0.132314815	0.076237974
2.9	0.142404799	0.139560993	0.134998218	0.132452523	0.076276831
3.0	0.142515115	0.139697482	0.135213260	0.132580217	0.076312796

Table 8: *Server-1 repair rate (τ) vs P_{S_1B} - EXPS*

τ	$ERLA$	$EXPA$	$HYPA$	$MNCA$	$MPCA$
2.1	0.138021661	0.134843332	0.129303375	0.127808856	0.074550912
2.2	0.138263883	0.135118502	0.129710507	0.128054649	0.074624807
2.3	0.138480662	0.135366864	0.130080690	0.128277310	0.074691540
2.4	0.138675482	0.135591923	0.130418429	0.128479865	0.074752084
2.5	0.138851247	0.135796620	0.130727549	0.128664843	0.074807246
2.6	0.139010396	0.135983435	0.131011317	0.128834378	0.074857702
2.7	0.139154988	0.136154477	0.131272540	0.128990278	0.074904020
2.8	0.139286770	0.136311545	0.131513638	0.129134086	0.074946683
2.9	0.139407235	0.136456186	0.131736709	0.129267126	0.074986104
3.0	0.139517664	0.136589733	0.131943580	0.129390538	0.075022635

Table 9: *Server-1 repair rate (τ) vs P_{S_1B} - HYPS*

τ	$ERLA$	$EXPA$	$HYPA$	$MNCA$	$MPCA$
2.1	0.121762987	0.118860529	0.113611432	0.112325780	0.067357513
2.2	0.121967364	0.119083143	0.113924156	0.112523046	0.067428853
2.3	0.122151232	0.119284568	0.114208672	0.112702075	0.067493742
2.4	0.122317344	0.119467554	0.114468443	0.112865232	0.067553018
2.5	0.122468001	0.119634405	0.114706401	0.113014495	0.067607384
2.6	0.122605135	0.119787070	0.114925047	0.113151535	0.067657429
2.7	0.122730382	0.119927203	0.115126525	0.113277768	0.067703651
2.8	0.122845133	0.120056218	0.115312681	0.113394403	0.067746476
2.9	0.122950579	0.120175328	0.115485115	0.113502478	0.067786270
3.0	0.123047742	0.120285582	0.115645216	0.113602890	0.067823346

7. Conclusion

In this study, we explain inventory management at service facilities using two types of servers: reliable servers and unreliable servers. Under steady-state conditions, matrix analytic techniques are used to determine the number of customers in the system, the server status, and the inventory level. Measures of important system features are derived in the steady state. We determined the optimality of this model by numerical analysis. As a result, this approach is appropriate for situations involving working vacation allocation when the server is reliable and for service disruption, or emergency vacation where the other server is unreliable.

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