A TWO-PARAMETER ARADHANA DISTRIBUTION WITH APPLICATIONS TO RELIABILITY ENGINEERING

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Abstract

The search for a statistical distribution for modelling the reliability data from reliability engineering is challenging and the main cause is the stochastic nature of the data and the presence of skewness, kurtosis and over-dispersion. During recent decades several one and two-parameter statistical distributions have been proposed in statistics literature but all these distributions were unable to capture the nature of data due to the presence of skewness, kurtosis and over-dispersion in the data. In the present paper, two-parameter Aradhana distribution, which includes one parameter Aradhana distribution as a particular case, has been proposed. Using convex combination approach of deriving a new statistical distribution, a two- parameter Aradhana distribution has been proposed. Various interesting and useful statistical properties including survival function, hazard function, reverse hazard function, mean residual life function, stochastic ordering, deviation from mean and median, stress-strength reliability, Bonferroni and Lorenz curve and their indices have been discussed. The raw moments, central moments and descriptive measures based on moments of the proposed distribution have been obtained. The estimation of parameters using the maximum likelihood method has been explained. The simulation study has been presented to know the performance in terms of consistency of maximum likelihood estimators as the sample size increases and. The goodness of test of the proposed distributions has been tested using the values of Akaike Information criterion and Kolmogorov-Smirnov statistics. Finally, two examples of real lifetime datasets from reliability engineering have been presented to demonstrate its applications and the goodness of fit, and it shows a better fit over two-parameter generalized Aradhana distribution, quasi Aradhana distribution, new quasi Aradhana distribution, Power Aradhana distribution, weighted Aradhana distribution, gamma distribution and Weibull distribution. The flexibility, tractability and usefulness of the proposed distribution show that it is very much useful for modelling reliability data from reliability engineering. As this is a new distribution and it has wide applications, it will draw the attention of researchers in reliability engineering and biomedical sciences to search many more applications in the future.

Keywords: Aradhana distribution, reliability properties, maximum likelihood estimation, applications.

I. INTRODUCTION

Shanker [1] proposed the Aradhana distribution, a one parameter lifetime distribution designed to characterize lifetime data originating from the fields of biomedical sciences and engineering. This distribution is characterized by its probability density function (pdf) and cumulative distribution function (cdf) as

$$f(x;\theta)=\frac{\theta^3}{\theta^2+2\theta+2}\left(1+x\right)^2 e^{-\theta x};x>0,\theta>0$$

$$F(x;\theta)=1-\left[1+\frac{\theta\, x(\theta\, x+2\theta+2)}{\theta^2+2\theta+2}\right]e^{-\theta x};x>0,\theta>0$$

It has been shown by Shanker [1] that in most of the real lifetime datasets it exhibited superior fit in comparison to exponential, Lindley [2], Shanker and Akash distributions introduced by Shanker [3,4]. The Aradhana distribution is a convex combination of exponential $\left(\theta\right)$, gamma $\left(2,\theta\right)$ and gamma $\left(3,\theta\right)$ distributions with their proportions $\frac{\theta^2}{\theta^2+2\theta+2},\frac{2\theta}{\theta^2+2\theta+2}$ and $\frac{2}{\theta^2+2\theta+2}$ respectively.

The mean and variance of Aradhana distribution are

$$E(X)=\frac{\theta^2+4\theta+6}{\theta\left(\theta^2+2\theta+2\right)}\text{ and } Var(X)=\frac{\theta^4+8\theta^3+24\theta^2+24\theta+12}{\theta^2\left(\theta^2+2\theta+2\right)^2}.$$

Important statistical properties of Aradhana distribution including shapes for varying values of parameter, moments related measures, hazard function, mean residual life function, stochastic ordering, mean deviations, distribution of order Statistics, Bonferroni and Lorenz curves, Renyi entropy measure and stress-strength reliability have been discussed and also studied estimation of parameter and applications of Aradhana distribution for modelling lifetime data by Shanker [1].

Shanker et al [5] have introduced a quasi Aradhana distribution (QAD) with its pdf and cdf as

$$f(x,\theta,\alpha)=\frac{\theta}{\alpha^2+2\alpha+2}\left(\alpha+\theta x\right)^2 e^{-\theta x}\ ;x>0,\theta>0,\alpha>0$$

$$F(x,\theta,\alpha)=1-\left[1+\frac{\theta x\left(\theta x+2\alpha+2\right)}{\alpha^2+2\alpha+2}\right]e^{-\theta x}\ ;x>0,\theta>0,\alpha>0.$$

The detailed studies on various statistical properties, estimation of parameters and applications of QAD are available in Shanker et al [5]. Anthony and Elangovan [6] discussed the length-biased version of QAD and study its properties and applications. Further, Anthony and Elangovan [7] proposed a new generalization of QAD by introducing an additional parameter in the QAD and study its statistical properties and applications.

Shanker et al [8] proposed a new quasi Aradhana distribution defined by its pdf and cdf

$$f(x;\theta,\alpha)=\frac{\theta^3}{\theta^4+2\theta^2\alpha+2\alpha^2}\left(\theta+\alpha x\right)^2 e^{-\theta x};x>0,\theta>0,\alpha>0$$

$$F(x;\theta,\alpha)=1-\left[1+\frac{\theta\,\alpha\, x(\theta\,\alpha\, x+2\theta^2+2\alpha)}{\theta^4+2\theta^2\alpha+2\alpha^2}\right]e^{-\theta x};x>0,\theta>0,\alpha>0.$$

The detailed studies on various statistical properties, estimation of parameters and applications of NQAD are available in Shanker et al [8].

In this paper an attempt has been made to suggest a two-parameter Aradhana distribution and study its statistical properties, estimation of parameters and applications. The whole paper is divided into eleven sections. Section one is introductory in nature. Section 2 deals with the derivation of pdf and cdf of a two-parameter Aradhana distribution and the behaviors of its pdf and cdf. Descriptive measures based on moments have been discussed in section three. Reliability properties of the distribution have been studied in section four. Deviation from mean and median, Bonferroni and Lorenz curves, stress-strength reliability have been discussed in sections five, six and seven, respectively. Maximum likelihood estimation and simulation study of the proposed distribution are given in sections eight and nine respectively. Finally, the applications and concluding remarks are presented in section ten and eleven, respectively.

II. A TWO-PARAMETER ARADHANA DISTRIBUTION

A two-parameter Aradhana distribution (ATPAD) can be defined by its pdf and cdf

$$f(x;\theta,\alpha)=\frac{\theta^3}{\theta^2\alpha^2+2\theta\alpha+2}(\alpha+x)^2 e^{-\theta x} \quad ; x>0,\ \theta>0, \alpha>0$$

$$F(x;\theta,\alpha)=1-\left[1+\frac{\theta x(\theta x+2\theta\alpha+2)}{\theta^2\alpha^2+2\theta\alpha+2}\right]e^{-\theta x} \quad ; x>0,\ \theta>0, \alpha>0$$

Aradhana distribution with parameter θ, gamma $(3,\theta)$ distribution and exponential distribution are special cases of ATPAD for $(\alpha=1)$, $(\alpha=0)$ and $\alpha\to\infty$. The behavior of the pdf and the cdf of ATPAD for different values of parameters are shown in figures 1 and 2 respectively.

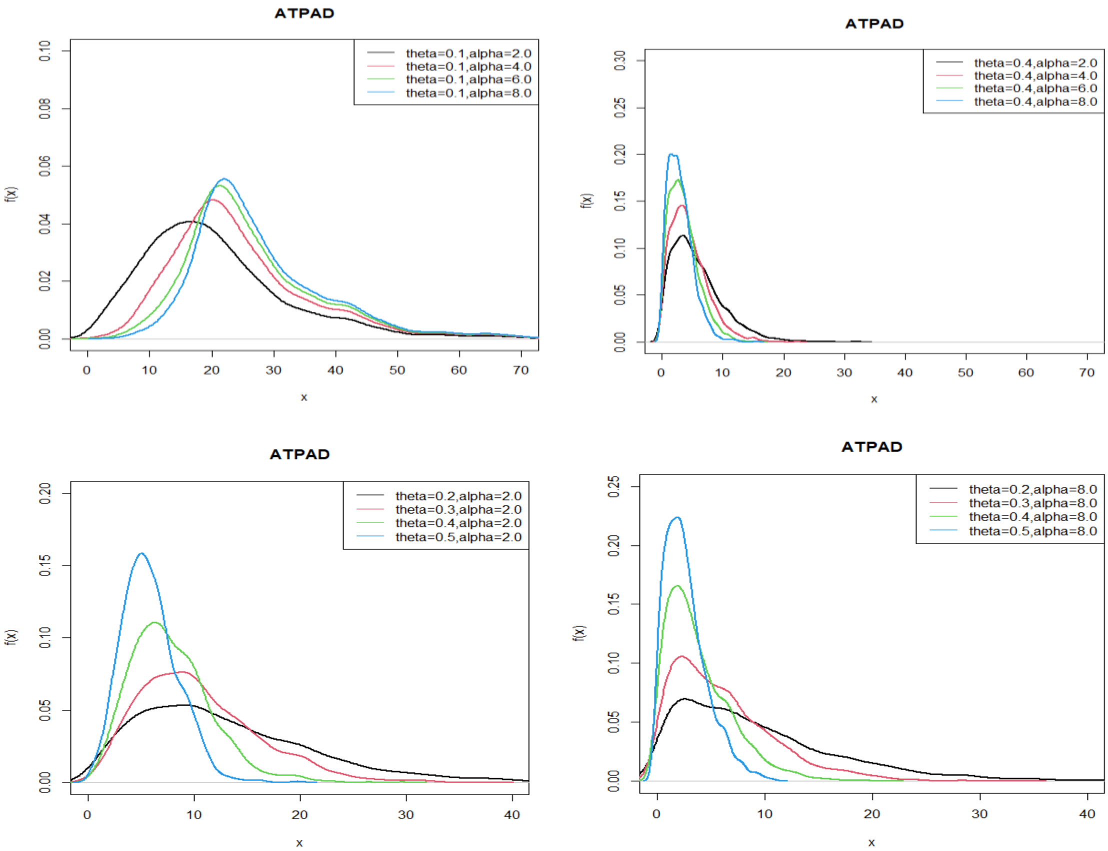

Figure 1: *pdf of ATPAD for different values of parameters*

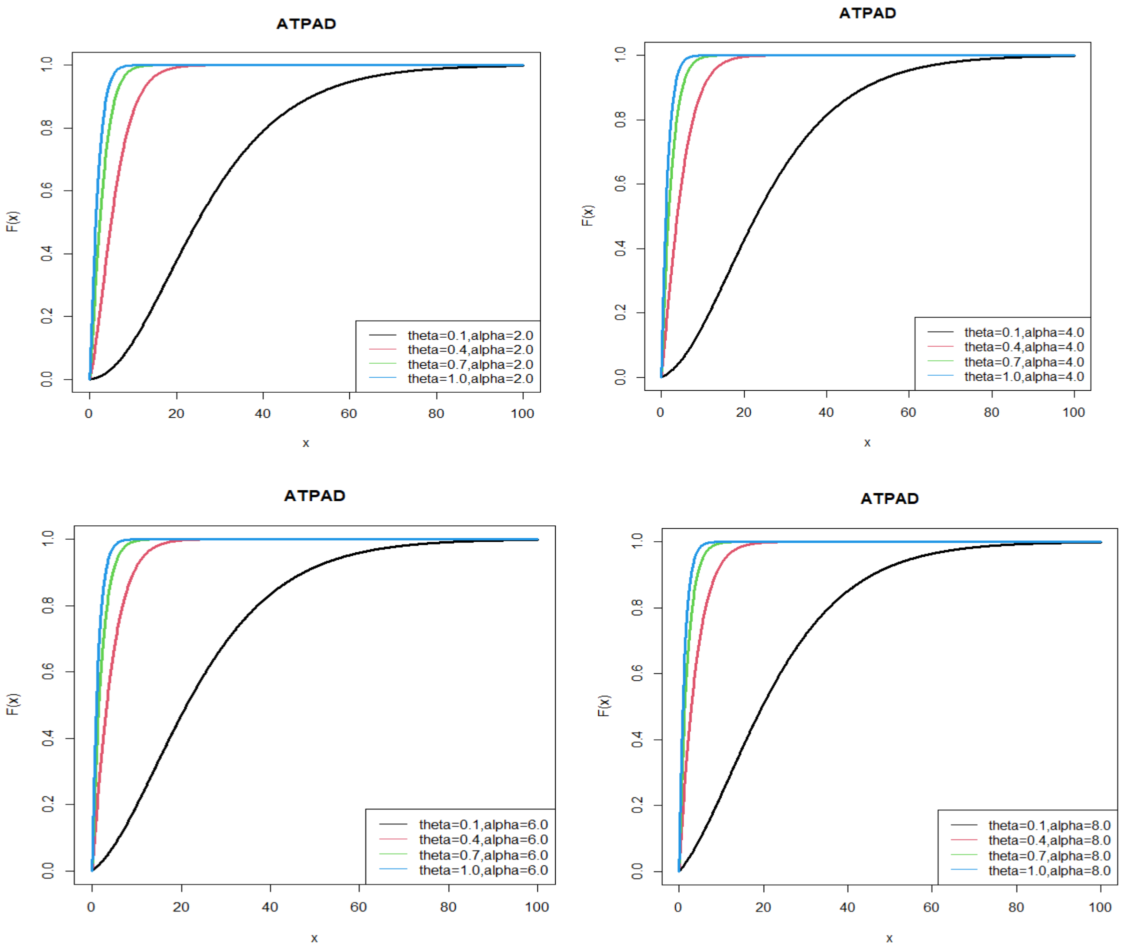

Figure 2: *cdf of ATPAD for different values of parameters*

III. DESCRIPTIVE MEASURES

The r th moment about origin of ATPAD can be obtained as

$$\mu_r' = E\left(X^r\right) = \frac{\theta^3}{\theta^2\alpha^2 + 2\theta\alpha + 2}\int_0^\infty x^r\left(\alpha^2 + 2\alpha x + x^2\right)e^{-\theta x}dx$$

$$= \frac{r!\left\{\theta^2\alpha^2 + 2(r+1)\theta\alpha + (r+1)(r+2)\right\}}{\theta^r\left(\theta^2\alpha^2 + 2\theta\alpha + 2\right)}; r = 1, 2, 3, \ldots$$

Substituting $r = 1, 2, 3, 4$ in the above expression, the first four moments about origin (raw moments) of ATPAD can be obtained as

$$\mu_1' = \frac{\theta^2\alpha^2 + 4\theta\alpha + 6}{\theta\left(\theta^2\alpha^2 + 2\theta\alpha + 2\right)}, \qquad \mu_2' = \frac{2\left(\theta^2\alpha^2 + 6\theta\alpha + 12\right)}{\theta^2\left(\theta^2\alpha^2 + 2\theta\alpha + 2\right)},$$

$$\mu_3' = \frac{6\left(\theta^2\alpha^2 + 8\theta\alpha + 20\right)}{\theta^3\left(\theta^2\alpha^2 + 2\theta\alpha + 2\right)}, \qquad \mu_4' = \frac{24\left(\theta^2\alpha^2 + 10\theta\alpha + 30\right)}{\theta^4\left(\theta^2\alpha^2 + 2\theta\alpha + 2\right)}$$

The central moments, using relationship between central moments and raw moments, can thus be obtained as

$$\mu_2 = \frac{\theta^4\alpha^4 + 8\theta^3\alpha^3 + 24\theta^2\alpha^2 + 24\theta\alpha + 12}{\theta^2\left(\theta^2\alpha^2 + 2\theta\alpha + 2\right)^2}$$

$$\mu_3 = \frac{2\left(\theta^6\alpha^6 + 12\theta^5\alpha^5 + 54\theta^4\alpha^4 + 100\theta^3\alpha^3 + 108\theta^2\alpha^2 + 72\theta\alpha + 24\right)}{\theta^3\left(\theta^2\alpha^2 + 2\theta\alpha + 2\right)^3}$$

$$\mu_4 = \frac{3\left(3\theta^8\alpha^8 + 48\theta^7\alpha^7 + 304\theta^6\alpha^6 + 944\theta^5\alpha^5 + 1816\theta^4\alpha^4 + 2304\theta^3\alpha^3 + 1920\theta^2\alpha^2 + 960\theta\alpha + 240\right)}{\theta^4\left(\theta^2\alpha^2 + 2\theta\alpha + 2\right)^4}$$

Thus, the coefficient of variation (C.V), coefficient of skewness $\left(\sqrt{\beta_1}\right)$, coefficient of kurtosis $\left(\beta_2\right)$, and index of dispersion $\left(\gamma\right)$ of ATPAD are obtained as

$$C.V. = \frac{\sigma}{\mu_1'} = \frac{\sqrt{\theta^4\alpha^4 + 8\theta^3\alpha^3 + 24\theta^2\alpha^2 + 24\theta\alpha + 12}}{\theta^2\alpha^2 + 4\theta\alpha + 6}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{2\left(\theta^6\alpha^6 + 12\theta^5\alpha^5 + 54\theta^4\alpha^4 + 100\theta^3\alpha^3 + 108\theta^2\alpha^2 + 72\theta\alpha + 24\right)}{\left(\theta^4\alpha^4 + 8\theta^3\alpha^3 + 24\theta^2\alpha^2 + 24\theta\alpha + 12\right)^{3/2}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{33\left(3\theta^8\alpha^8 + 48\theta^7\alpha^7 + 304\theta^6\alpha^6 + 944\theta^5\alpha^5 + 1816\theta^4\alpha^4 + 2304\theta^3\alpha^3 + 1920\theta^2\alpha^2 + 960\theta\alpha + 240\right)}{\left(\theta^4\alpha^4 + 8\theta^3\alpha^3 + 24\theta^2\alpha^2 + 24\theta\alpha + 12\right)^2}$$

$$\gamma = \frac{\sigma^2}{\mu_1'} = \frac{\theta^4\alpha^4 + 8\theta^3\alpha^3 + 24\theta^2\alpha^2 + 24\theta\alpha + 12}{\theta\left(\theta^2\alpha^2 + 2\theta\alpha + 2\right)\left(\theta^2\alpha^2 + 4\theta\alpha + 6\right)}$$

The graphical relationship between mean and variance of ATPAD to see the over-dispersion, equi-dispersion and under-dispersion are shown in the following figure 3.

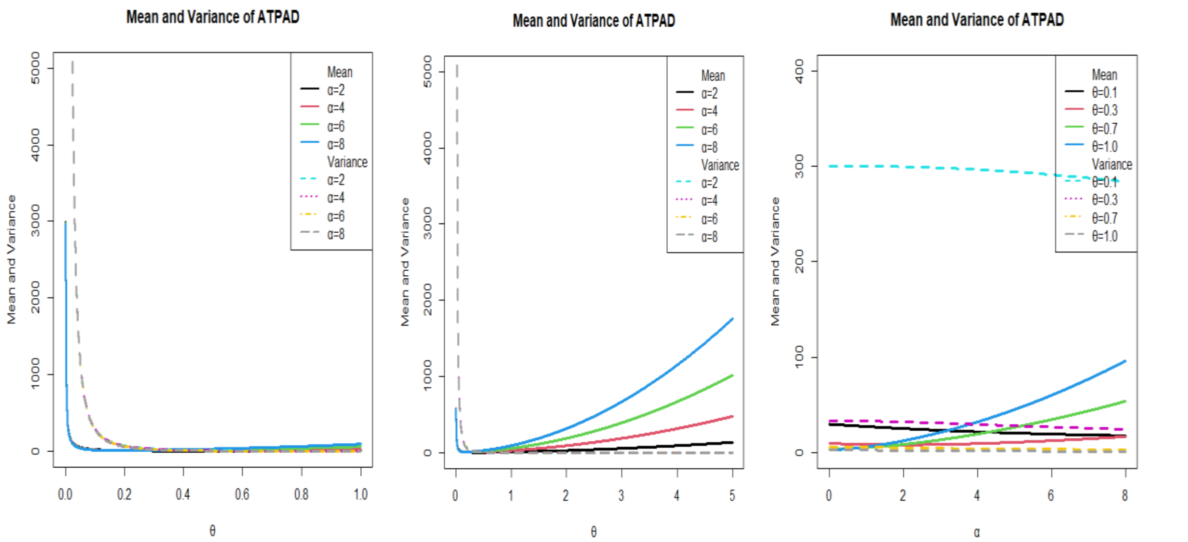

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Figure 3: *Mean and variance of ATPAD*

Behavior of coefficient of variation, skewness, kurtosis and index of dispersion of ATPAD shown in figure 4.

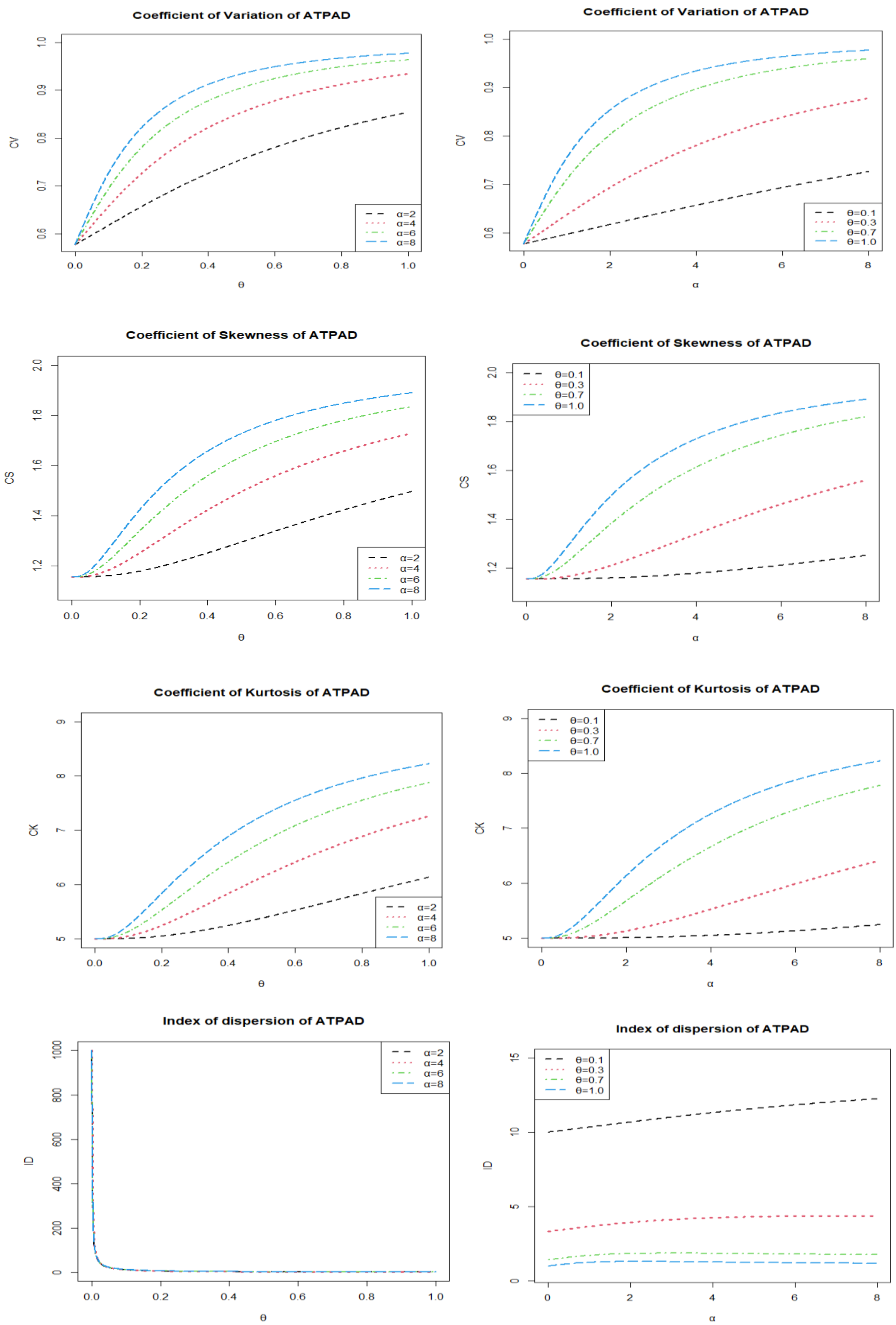

Figure 4: *Behaviors of coefficient of variation, skewness, kurtosis and index of dispersion of ATPAD*

IV. SOME RELIABILITY PROPERTIES

I. Survival Function

The survival function of ATPAD can be obtained as

$$S(x;\theta,\alpha)=1-F(x;\theta,\alpha)=\frac{\left[\theta x\left(\theta x+2\theta\alpha+2\right)+\left(\theta^2\alpha^2+2\theta\alpha+2\right)\right]e^{-\theta x}}{\theta^2\alpha^2+2\theta\alpha+2};x>0,\ \theta>0,\alpha>0$$

II. Hazard Function and Mean Residual Life Function

The hazard function of ATPAD can be obtained as

$$h(x;\theta,\alpha)=\frac{f(x;\theta,\alpha)}{S(x;\theta,\alpha)}=\frac{\theta^3(\alpha+x)^2}{\theta x\left(\theta x+2\theta\alpha+2\right)+\left(\theta^2\alpha^2+2\theta\alpha+2\right)}$$

The mean residual life function of ATPAD can be obtained as

$$m(x,\theta,\alpha)=\frac{1}{1-F(x;\theta,\alpha)}\int_0^x\left[1-F\left(t;\theta,\alpha\right)\right]dx$$

$$=\frac{\theta^2x^2+2\left(\theta\alpha+2\right)\theta x+\left(\theta^2\alpha^2+4\theta\alpha+6\right)}{\theta\left[\theta x\left(\theta x+2\theta\alpha+2\right)+\left(\theta^2\alpha^2+2\theta\alpha+2\right)\right]}$$

The graphical representation of hazard function and mean residual life function are presented in the figure 5 and 6 respectively. From the figure 5 it is cleared that all values of the parameters θ and α hazard function is monotonically increasing. From the figure 6 it is cleared that for all values of the parameters θ and α mean residual life function is monotonically decreasing.

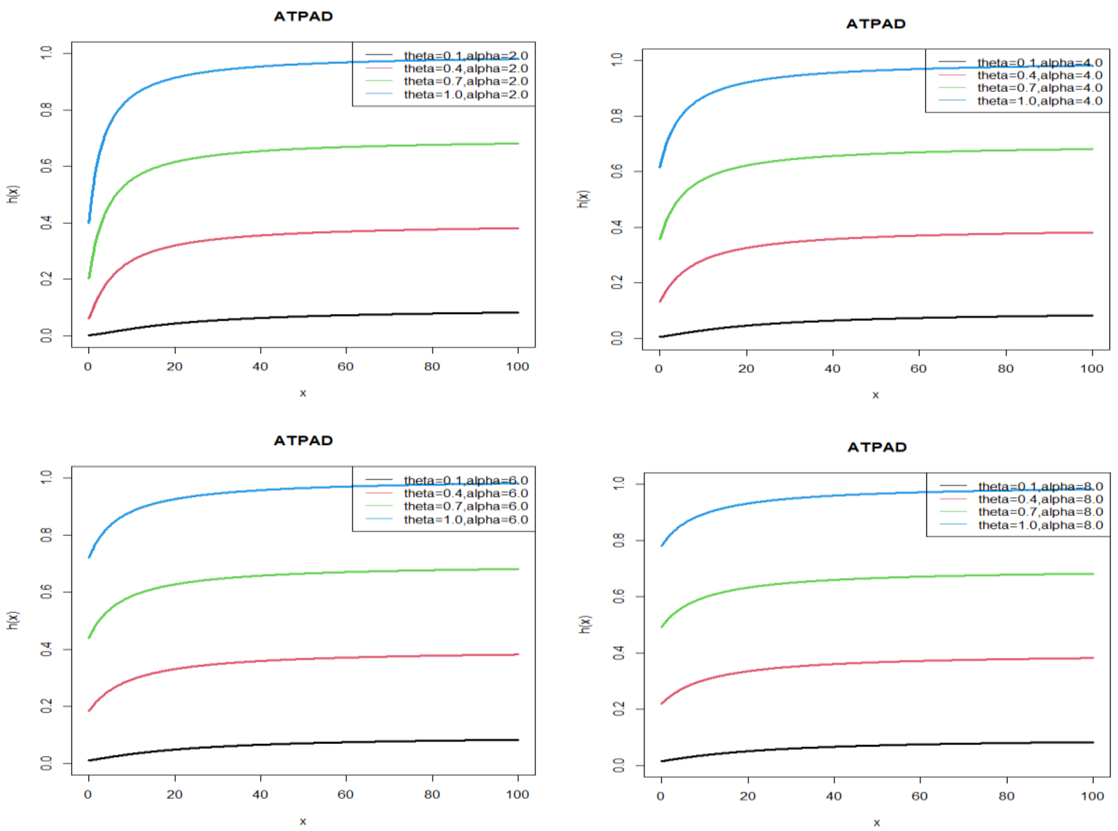

Figure 5: *Hazard function of ATPAD for various values of the parameters*

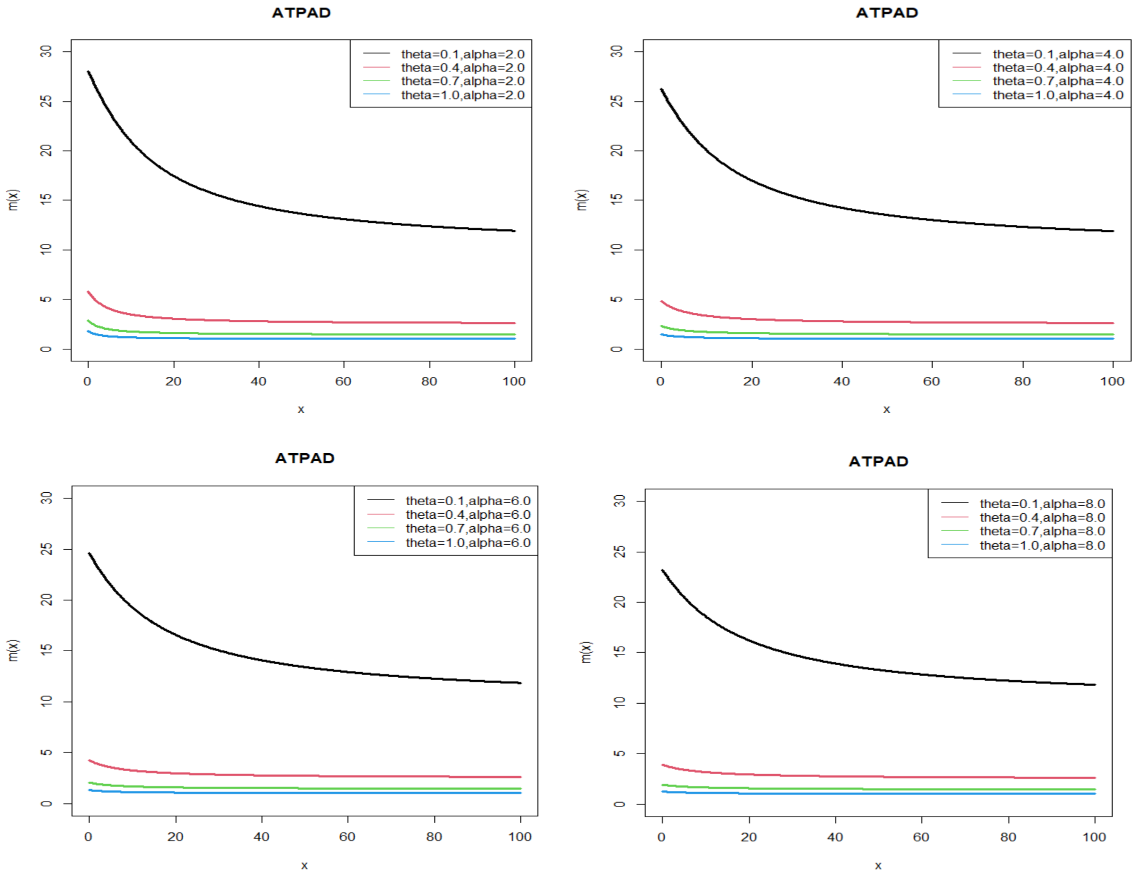

Figure 6: *Mean residual life function of ATPAD for various values of the parameters*

III. Reverse Hazard Function

Reverse hazard function of ATPAD can be obtained as

$$r_x(x;\theta,\alpha)=\frac{f(x;\theta,\alpha)}{F(x;\theta,\alpha)}=\frac{\theta^3(\alpha+x)^2 e^{-\theta x}}{(\theta^2\alpha^2+2\theta\alpha+2)-\left[\theta x(\theta x+2\theta\alpha+2)+(\theta^2\alpha^2+2\theta\alpha+2)\right]e^{-\theta x}}$$

IV. Stochastic Ordering

In probability theory and statistics, a stochastic order quantifies the concept of one random variable being bigger than another. A random variable X is said to be smaller than a random variable Y in the:

i. Stochastic order $(X \le_{st} Y)$ if $F_X(x) \ge F_Y(y)$ for all x
ii. Hazard rate order $(X \le_{hr} Y)$ if $h_X(x) \ge h_Y(y)$ for all x
iii. Mean residual life order $(X \le_{mrl} Y)$ if $m_X(x) \ge m_Y(y)$ for all x
iv. Likelihood ratio order $(X \le_{lr} Y)$ if $\dfrac{f_X(x)}{f_Y(y)}$ decrease in x

The following results due to Shaked and Shantikumar [9] are well known for establishing stochastic ordering of distributions

$$X <_{lr} Y \Rightarrow X <_{hr} Y \Rightarrow X <_{mrl} Y$$
$$\Downarrow$$
$$X <_{st} Y$$

Theorem: Let $X \sim$ ATPAD (θ_1, α_1) and $Y \sim$ ATPAD(θ_2, α_2). , If $\alpha_1 \ge \alpha_2$ and $\theta_1 > \theta_2$ or $\theta_1 = \theta_2$ and $\alpha_1 > \alpha_2$ then $X <_{lr} Y$ hence $X <_{hr} Y$, $X <_{mrl} Y$ and $X <_{st} Y$.

Proof: We have

$$\frac{f_X(x)}{f_Y(x)} = \frac{\theta_1^3\left(\alpha_2{}^2\theta_2^2 + 2\alpha_2\theta_2 + 2\right)}{\theta_2^3\left(\alpha_1{}^2\theta_1^2 + 2\alpha_1\theta_1 + 2\right)} \frac{\left(\alpha_1 + x\right)^2}{\left(\alpha_2 + x\right)^2} e^{-(\theta_1 - \theta_2)x}$$

Now

$$\log \frac{f_X(x)}{f_Y(x)} = \log\left[\frac{\theta_1^3\left(\alpha_2{}^2\theta_2^2 + 2\alpha_2\theta_2 + 2\right)}{\theta_2^3\left(\alpha_1{}^2\theta_1^2 + 2\alpha_1\theta_1 + 2\right)}\right] + 2\log\left(\frac{\alpha_1 + x}{\alpha_2 + x}\right) - \left(\theta_1 - \theta_2\right)x$$

Therefore

$$\frac{d}{dx}\left(\log \frac{f_X(x)}{f_Y(x)}\right) = \frac{2\left(\alpha_2 - \alpha_1\right)}{\left(\alpha_1 + x\right)\left(\alpha_2 + x\right)} - \left(\theta_1 - \theta_2\right)$$

Thus, If $\alpha_1 \ge \alpha_2$ and $\theta_1 > \theta_2$ or $\theta_1 = \theta_2$ and $\alpha_1 > \alpha_2$, $\frac{d}{dx}\left(\log \frac{f_X(x)}{f_Y(x)}\right) < 0$. This means $X <_{lr} Y$ hence $X <_{hr} Y$, $X <_{mrl} Y$ and $X <_{st} Y$.

V. DEVIATION FROM MEAN AND MEDIAN

The amount of scatter in a population is an evidently measured to some extent by the totality of deviations from the mean and median. These are known as the mean deviation about the mean and mean deviation about median and are defined by

$$\delta_1(x) = 2\mu F(\mu) - 2\int_0^{\mu} x\, f(x;\theta,\alpha)dx \quad \text{and} \quad \delta_2(x) = -\mu + 2\int_M^{\infty} x\, f(x;\theta,\alpha)dx$$

Thus $\delta_1(x)$ and $\delta_2(x)$ of ATPAD are obtained as

$$\delta_1(x) = \frac{2\left[\theta^2\mu^2 + 2\theta^2\alpha\mu + 4\theta\mu + \left(\theta^2\alpha^2 + 4\theta\alpha + 6\right)\right]e^{-\theta\mu}}{\theta\left(\theta^2\alpha^2 + 2\theta\alpha + 2\right)}$$

$$\delta_2(x) = \frac{2\left\{\theta^3 M^3 + \left(2\theta\alpha + 3\right)\theta^2 M^2 + \left(\theta^2\alpha^2 + 4\theta\alpha + 6\right)\theta M + \left(\theta^2\alpha^2 + 4\theta\alpha + 6\right)\right\}e^{-\theta M}}{\theta\left(\theta^2\alpha^2 + 2\theta\alpha + 2\right)} - \mu$$

VI. BONFERRONI AND LORENZ CURVES

The Bonferroni and Lorenz curves [10] and Bonferroni and Gini indices have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. The Bonferroni and Lorenz curves are defined as

$$\text{B(p)} = \frac{1}{p\mu}\int_0^q xf(x)dx = \frac{1}{p\mu}\left[\int_0^{\infty} xf(x)dx - \int_q^{\infty} xf(x)dx\right] = \frac{1}{p\mu}\left[\mu - \int_q^{\infty} xf(x)dx\right]$$

and

$$\mathrm{L(p)} = \frac{1}{\mu}\int_0^q x f(x)dx = \frac{1}{\mu}\left[\int_0^\infty x f(x)dx - \int_q^\infty x f(x)dx\right] = \frac{1}{\mu}\left[\mu - \int_q^\infty x f(x)dx\right]$$

respectively or equivalently.
The Bonferroni and Gini indices are obtained as

$$B = 1 - \int_0^1 B(p)dp \text{ and } G = 1 - 2\int_0^1 \mathrm{L}(p)dp \text{ ,respectively.}$$

Using the pdf of ATPAD, we get

$$B(p) = \frac{1}{p}\left[1 - \frac{\left\{\theta^3 q^3 + (2\theta\alpha+3)\theta^2 q^2 + \left(\theta^2\alpha^2 + 4\theta\alpha + 6\right)\theta q + \left(\theta^2\alpha^2 + 4\theta\alpha + 6\right)\right\} e^{-\theta q}}{\left(\theta^2\alpha^2 + 4\theta\alpha + 6\right)}\right]$$

$$L(p) = 1 - \frac{\left\{\theta^3 q^3 + (2\theta\alpha+3)\theta^2 q^2 + \left(\theta^2\alpha^2 + 4\theta\alpha + 6\right)\theta q + \left(\theta^2\alpha^2 + 4\theta\alpha + 6\right)\right\} e^{-\theta q}}{\left(\theta^2\alpha^2 + 4\theta\alpha + 6\right)}$$

Finally, after little algebraic simplification, the Bonferroni and Gini indices of ATPAD are obtained as

$$B = \frac{\left\{\theta^3 q^3 + (2\theta\alpha+3)\theta^2 q^2 + \left(\theta^2\alpha^2 + 4\theta\alpha + 6\right)\theta q + \left(\theta^2\alpha^2 + 4\theta\alpha + 6\right)\right\} e^{-\theta q}}{\left(\theta^2\alpha^2 + 4\theta\alpha + 6\right)}$$

$$G = \frac{2\left\{\theta^3 q^3 + (2\theta\alpha+3)\theta^2 q^2 + \left(\theta^2\alpha^2 + 4\theta\alpha + 6\right)\theta q + \left(\theta^2\alpha^2 + 4\theta\alpha + 6\right)\right\} e^{-\theta q}}{\left(\theta^2\alpha^2 + 4\theta\alpha + 6\right)} - 1$$

VII. STRESS-STRENGTH RELIABILITY

The stress-strength reliability of a component illustrates the life of the component which has random strength X that is subjected to a random stress Y. When the stress of the component Y applied to it exceeds the strength of the component X, the component fails instantly, and the component will function satisfactorily until $X > Y$. Therefore, $R = P(Y < X)$ is a measure of the component reliability and is known as stress-strength reliability in statistical literature. It has extensive applications in almost all areas of knowledge especially in engineering such as structures, deterioration of rocket motors, static fatigue of ceramic components, aging of concrete pressure vessels, etc.

$$R = P(\mathrm{Y} < \mathrm{X}) = \int_0^\infty P\left(Y < X \mid X = x\right) f_X(x)dx$$

$$= 1 - \frac{\theta_1^3\left[\begin{array}{l} 24\theta_2^2 + 48\left\{\theta_2^2\alpha_1 + \theta_2\left(\theta_2\alpha_2 + 1\right)\right\}\left(\theta_1 + \theta_2\right) \\ +2\left\{\theta_2^2\alpha_1 + 4\theta_2\left(\theta_2\alpha_2 + 1\right)\alpha_1 + \left(\theta_2^2\alpha_2^2 + 2\theta_2\alpha_2 + 2\right)\right\}\left(\theta_1 + \theta_2\right)^2 \\ +2\left\{\theta_2\left(\theta_2\alpha_2 + 1\right)\alpha_1^2 + \alpha_1\left(\theta_2^2\alpha_2^2 + 2\theta_2\alpha_2 + 2\right)\right\}\left(\theta_1 + \theta_2\right)^{32} \\ +\alpha_1^2\left(\theta_2^2\alpha_2^2 + 2\theta_2\alpha_2 + 2\right)\left(\theta_1 + \theta_2\right)^4 \end{array}\right]}{\left(\theta_1^2\alpha_1^2 + 2\theta_1\alpha_1 + 2\right)\left(\theta_2^2\alpha_2^2 + 2\theta_2\alpha_2 + 2\right)\left(\theta_1 + \theta_2\right)^5}$$

VIII. ESTIMATION AND INFERENCE

Let $(x_1, x_2, ..., x_n)$ be a random sample from ATPAD (θ, α), the likelihood function L and the log-likelihood function, $\log L$ are given by

$$L = \left(\frac{\theta^3}{\theta^2\alpha^2 + 2\theta\alpha + 2} \right)^2 \prod_{i=1}^{n} (\alpha + x_i)^2 e^{-n\theta\bar{x}}$$

$$\log L = 3n\log\theta - n\log\left(\theta^2\alpha^2 + 2\theta\alpha + 2\right) + 2\sum_{i=1}^{n}\log(\alpha + x_i) - n\theta\bar{x}$$

The maximum likelihood estimates (MLEs) $\hat{\theta}$ and $\hat{\alpha}$ of θ and α are then the solutions of the following non-linear equations

$$\frac{\partial \log L}{\partial \theta} = \frac{3n}{\theta} - \frac{2n\alpha(\theta\alpha + 1)}{\theta^2\alpha^2 + 2\theta\alpha + 2} - n\bar{x} = 0$$

$$\frac{\partial \log L}{\partial \alpha} = -\frac{2n\theta(\theta\alpha + 1)}{\theta^2\alpha^2 + 2\theta\alpha + 2} + 2\sum_{i=1}^{n}\frac{1}{\alpha + x_i} = 0$$

These two natural log likelihood equations do not seem to be solved directly. However, the Fisher's scoring method can be applied to solve these equations. We have

$$\frac{\partial^2 \log L}{\partial \theta^2} = -\frac{3n}{\theta^2} + \frac{2n\alpha^3\theta(\theta\alpha + 2)}{\left(\theta^2\alpha^2 + 2\theta\alpha + 2\right)^2}$$

$$\frac{\partial^2 \log L}{\partial \alpha^2} = \frac{2n\alpha\theta^3(\theta\alpha + 2)}{\left(\theta^2\alpha^2 + 2\theta\alpha + 2\right)^2}$$

$$\frac{\partial^2 \log L}{\partial \theta \partial \alpha} = \frac{-2n\left(\theta^2\alpha^2 + 4\theta\alpha + 2\right)}{\left(\theta^2\alpha^2 + 2\theta\alpha + 2\right)^2} = \frac{\partial^2 \log L}{\partial \alpha \partial \theta}$$

The following equations can be solved for MLEs $\hat{\theta}$ and $\hat{\alpha}$ of θ and α of ATPAD

$$\begin{pmatrix} \frac{\partial^2 \log L}{\partial \theta^2} & \frac{\partial^2 \log L}{\partial \theta \partial \alpha} \\ \frac{\partial^2 \log L}{\partial \alpha \partial \theta} & \frac{\partial^2 \log L}{\partial \alpha^2} \end{pmatrix}_{\substack{\hat{\theta}-\theta_0 \\ \hat{\alpha}-\alpha_0}} \begin{pmatrix} \hat{\theta} - \theta_0 \\ \hat{\alpha} - \alpha_0 \end{pmatrix} = \begin{pmatrix} \frac{\partial \log L}{\partial \theta} \\ \frac{\partial \log L}{\partial \alpha} \end{pmatrix}_{\substack{\hat{\theta}-\theta_0 \\ \hat{\alpha}-\alpha_0}}$$

IX. THE SIMULATION STUDY

In this section, we carried out simulation study to examine the performance of maximum likelihood estimators of the ATPAD. We examined the mean estimates, biases (B), mean square errors (MSEs) and variances of the MLEs. The mean, bias, MSE and variance are computed using the formulae

$$Mean = \frac{1}{n}\sum_{i=1}^{n}\hat{H}_i,\ B = \frac{1}{n}\sum_{i=1}^{n}\left(\hat{H}_i - H\right),\ MSE = \frac{1}{n}\sum_{i=1}^{n}\left(\hat{H}_i - H\right)^2,\ Variance = MSE - B^2$$

where $H = \theta, \alpha$ and $\hat{H} = \hat{\theta}_i, \hat{\alpha}_i$.

The simulation results for different parameter values of ATPAD are presented in tables 1 and 2 respectively. The steps for simulation study are as follows:

a. Data is generated using the acceptance-rejection method of simulation. The acceptance-rejection method is a commonly used approach in simulation studies to generate random samples from a target distribution when inverse transform method of simulation is not feasible or efficient. Acceptance rejection method for generating random samples from the

ATPAD consists of following steps.

i. Generate a random variable Y distributed as exponential (θ)

ii. Generate U distributed as Uniform $(0,1)$

iii. If $U \le \frac{f(y)}{M\, g(y)}$, then set $X = Y$ ("accept the sample"); otherwise ("reject the sample") and if reject then repeat the process: step (i-iii) until getting the required samples. Where M is a constant.

b. The sample sizes are taken as $n = 50, 100, 150, 200$

c. The parameter values are set as values $\theta = 0.2$, $\alpha = 1.8$ and $\theta = 0.2$, $\alpha = 4.0$

d. Each sample size is replicated 10000 times

The results obtained in Tables 1 and 2 show that as the sample size increases, biases, MSEs and variances of the MLEs of the parameters become smaller respectively. This result is in line with the first-order asymptotic theory.

Table-1: *The Mean values, Biases, MSEs and Variances of ATPAD for parameter* $\theta = 0.2$, $\alpha = 1.8$

Parameters	Sample Size	Mean	Bias	MSE	Variance
$\hat{\theta}$	20	0.20148	0.00148	0.000004212	0.000002011
	40	0.20094	0.00094	0.000002244	0.000001354
	50	0.20083	0.00083	0.000002477	0.000002009
	100	0.20081	0.00081	0.000002422	0.000001785
	150	0.20079	0.00079	0.000002254	0.000001591
	200	0.20068	0.00068	0.000002251	0.000001557
$\hat{\alpha}$	20	1.72273	-0.07726	0.127852500	0.121882500
	40	1.76513	-0.03486	0.062430860	0.061215480
	50	1.77232	-0.02767	0.051223530	0.050457760
	100	1.78933	-0.01066	0.026087550	0.025973800
	150	1.79298	-0.00701	0.017392750	0.017343480
	200	1.78781	-0.00121	0.015850450	0.025702030

Table-2: *The Mean values, Biases, MSEs and Variances of ATPAD for parameter* $\theta = 0.2$, $\alpha = 4.0$

Parameters	Sample Size	Mean	Bias	MSE	Variance
$\hat{\theta}$	20	0.20138	0.001380	0.00000342	0.000001517
	40	0.20110	0.001107	0.00000242	0.000001195
	50	0.20107	0.001077	0.00000224	0.000001082
	100	0.20107	0.001074	0.00000223	0.000001084
	150	0.20096	0.000963	0.00000206	0.000001139
	200	0.20074	0.000742	0.00000173	0.000001799
$\hat{\alpha}$	20	4.01832	0.018320	0.00659461	0.00625895
	40	4.00915	0.009156	0.00329730	0.00321346
	50	4.00726	0.007266	0.00263803	0.00258523
	100	4.00484	0.004849	0.00175868	0.00173517
	150	4.00364	0.003640	0.00141901	0.00140576
	200	4.00363	0.003630	0.00131901	0.00130576

X. APPLICATIONS

The following real lifetime datasets have been considered for testing the goodness of fit of ATPAD over the other two-parameter lifetime distributions. The goodness of fit based on K-S statistic, fitted plots of considered distributions for the datasets, p-p plots of considered distributions for the datasets and total time in test (TTT) plots for the datasets and the ATPAD confirm that among all considered distributions, ATPAD provides much closure fit.

Data set-1: This censored tri-modal data contains 30 items that is tested when test is stopped after20-th failure. The following data discussed by Murthy et al [11] and the values are:

0.0014, 0.0623, 1.3826, 2.0130, 2.5274, 2.8221, 3.1544, 4.9835, 5.5462, 5.8196, 5.8714, 7.4710, 7.5080, 7.6667, 8.6122, 9.0442, 9.1153, 9.6477, 10.1547, 10.7582.

Description of the data set-1

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
0.0014	2.7484	5.8455	5.7081	8.7202	10.7582

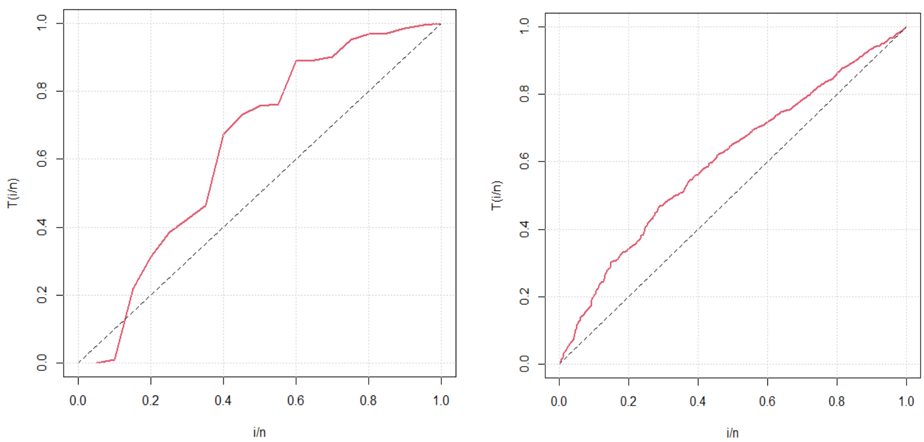

Figure 7: *TTT- plot of the observed dataset 1 and simulated data of ATPAD respectively*

Data set-2: The following skewed to right, a complete set of data, discussed by Murthy et al [11], and reports the failure time of 20 electric bulbs and the observations are:

1.32, 12.37, 6.56, 5.05, 11.58, 10.56, 21.82, 3.60, 1.33, 12.62, 5.36, 7.71, 3.53, 19.61, 36.63, 0.39, 21.35, 7.22, 12.42, 8.92.

Description of the data set-2

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
0.390	4.688	8.315	10.498	12.470	36.630

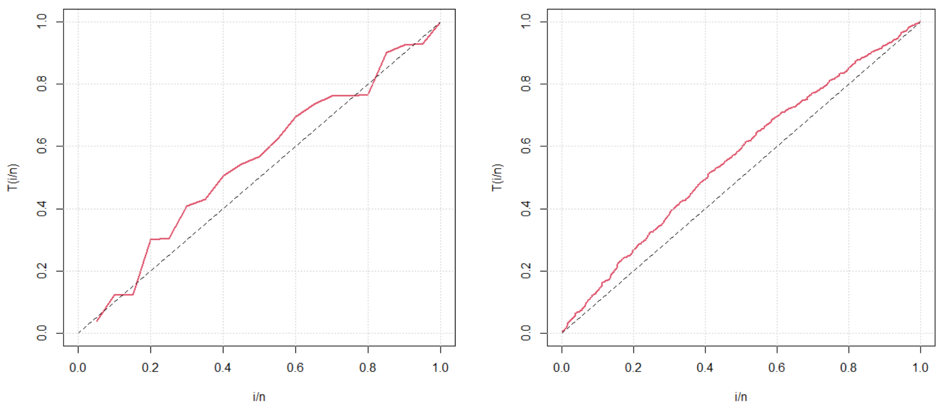

Figure 8: *TTT- plot of the observed dataset 2 and simulated data of ATPAD respectively*

In order to compare lifetime distributions, values of $-2\log L$, Akaike Information Criterion (AIC), Kolmogorov-Smirnov Statistics (K-S) and the corresponding probability value (p-value) for the above data set has been computed. The formulae for computing AIC and K-S are as follows:

$$AIC = -2logL + 2k\,, \quad D = \underset{x}{Sup}\,|\,F_n(x) - F_0(x)\,|$$

where, $k =$ number of parameters, $n =$ sample size, $F_n(x) =$ empirical cdf of considered distribution and $F_0(x) =$ cdf of considered distribution

The distribution corresponding to the lower values of $-2\log L$, AIC and K-S Statistics is the best fit distribution. The MLEs of parameters of the considered distributions along with their standard error, $-2\log L$, AIC, K-S and p-value of the considered distributions for datasets 1 and 2 are presented in tables 3 and 4 respectively.

Table 3: *ML estimates, Standard errors, $-2\log L$, AIC, K-S and p-value of the considered distributions for the dataset-1*

Distribution	ML estimates $\hat{\theta}$ $SE(\hat{\theta})$ $\hat{\alpha}$ $SE(\hat{\alpha})$	$-2\log L$	AIC	K-S	p-value
ATPAD	0.3896 (0.0806) 2.4576 (1.9436)	105.8912	109.8912	0.1837	0.5105
GAD	0.3896 (0.0806) 0.4068 (0.3217)	105.8912	109.8912	0.1949	0.4341
QAD	0.3896 (0.0806) 0.9577 (0.6199)	105.8912	109.8912	0.1971	0.3908
NQAD	0.3896 (0.0806) 0.1585 (0.1518)	105.8912	109.8912	0.1930	0.4696
PAD	0.5935 (0.1558) 0.8366 (0.1394)	106.0269	110.0269	0.1915	0.4687
WAD	0.4535 (0.1142) 0.0100 (0.5170)	107.4734	111.4734	0.1902	0.4346
GD	0.1513 (0.0552) 0.8637 (0.2373)	109.3792	113.3792	0.2530	0.1624
WD	0.1469 (0.0721) 1.0892 (0.2209)	109.5036	113.5036	0.9000	0.0000

Table 4: *ML estimates, Standard errors, $-2\log L$, AIC, K-S and p-value of the considered distributions for the dataset-2*

Distributions	ML estimates $\hat{\theta}$ $SE(\hat{\theta})$ $\hat{\alpha}$ $SE(\hat{\alpha})$	$-2\log L$	AIC	K-S	p-value
ATPAD	0.1866 (0.0565) 8.1128(9.4833)	132.9421	136.9421	0.1220	0.9276
GAD	0.1867 (0.0565) 0.1232 (0.1440)	132.9421	136.9421	0.1355	0.8570
QAD	0.1866 (0.0565) 1.5147 (1.3803)	132.9421	136.9421	0.1307	0.8972
NQAD	0.1869 (0.0558) 0.0231 (0.0329)	132.9421	136.9421	0.1459	0.8073
PAD	0.4421 (0.1225) 0.7755 (0.1097)	133.1672	137.1672	0.1343	0.8711
WAD	0.2625 (0.0710) 0.0100 (0.6242)	137.0825	141.0825	0.1425	0.8289
GD	0.1272(0.0438) 1.3361 (0.3811)	133.0916	137.0916	0.1340	0.8729
WD	0.1000 (0.0776) 1.0150 (0.2550)	134.0518	138.0518	0.9830	0.000

From Table-3 and 4 we observed that the ATPAD has the same $-2\log L$, AIC values but least K-S values as compared to GAD (Generalized Aradhana Distribution) of Daniel and Shanker [12] and , QAD (Quasi Aradhana Distribution), NQAD (New Quasi Aradhana distribution) and has the least $-2\log L$, AIC, K-S values as compared to PAD (Power Aradhana distribution) of Shanker and Shukla [13], WAD (Weighted Aradhana distribution) by Ganaie et al [14] and subsequently critical study done by Shanker et al [15], GD (gamma Distribution) and WD (Weibull distribution) by Weibull [16].

Hence, we may conclude that ATPAD provides the better fit than GAD, QAD, NQAD, PAD, WAD, GD and WD. Further, it is also clear from the fitted plot and P-P plot of two dataset of considered distributions in figure 9, 10 and 11, that ATPAD provides a much better fit over GAD, QAD, NQAD, PAD, WAD, GD and WD.

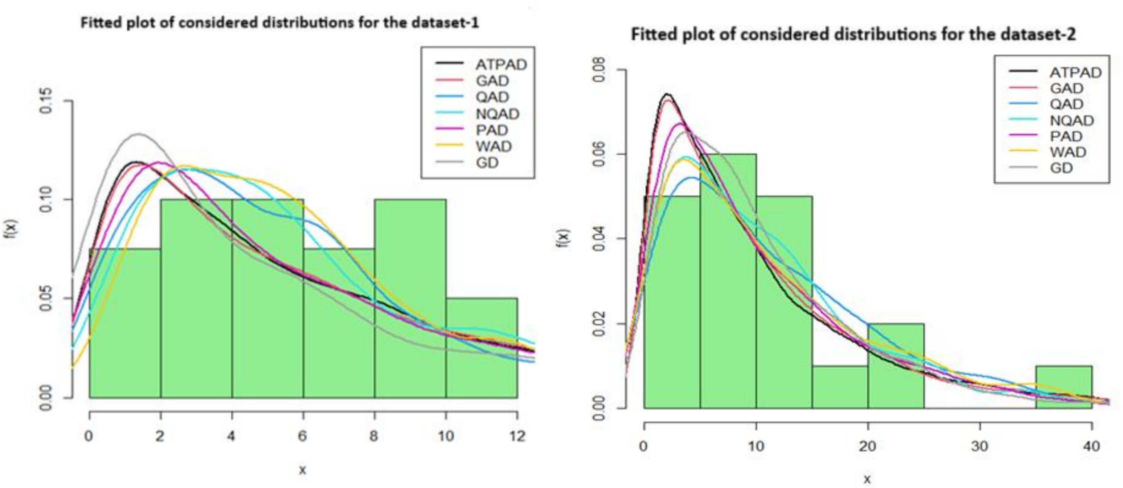

Figure 9: *Fitted plot of the considered distribution for the data set-1 and data set-2*

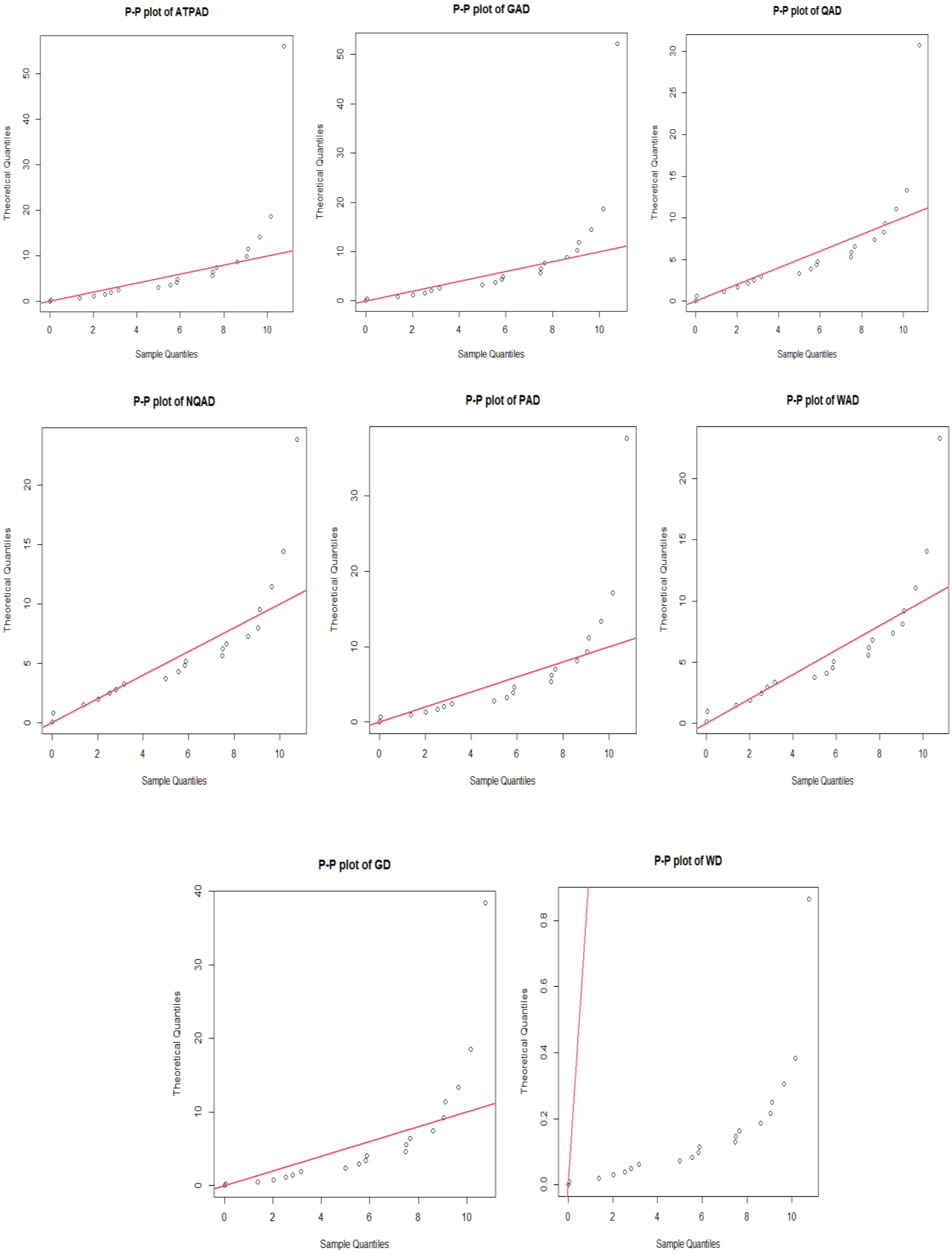

Figure 10: *P-P plot for considerd distributions of the data set-1*

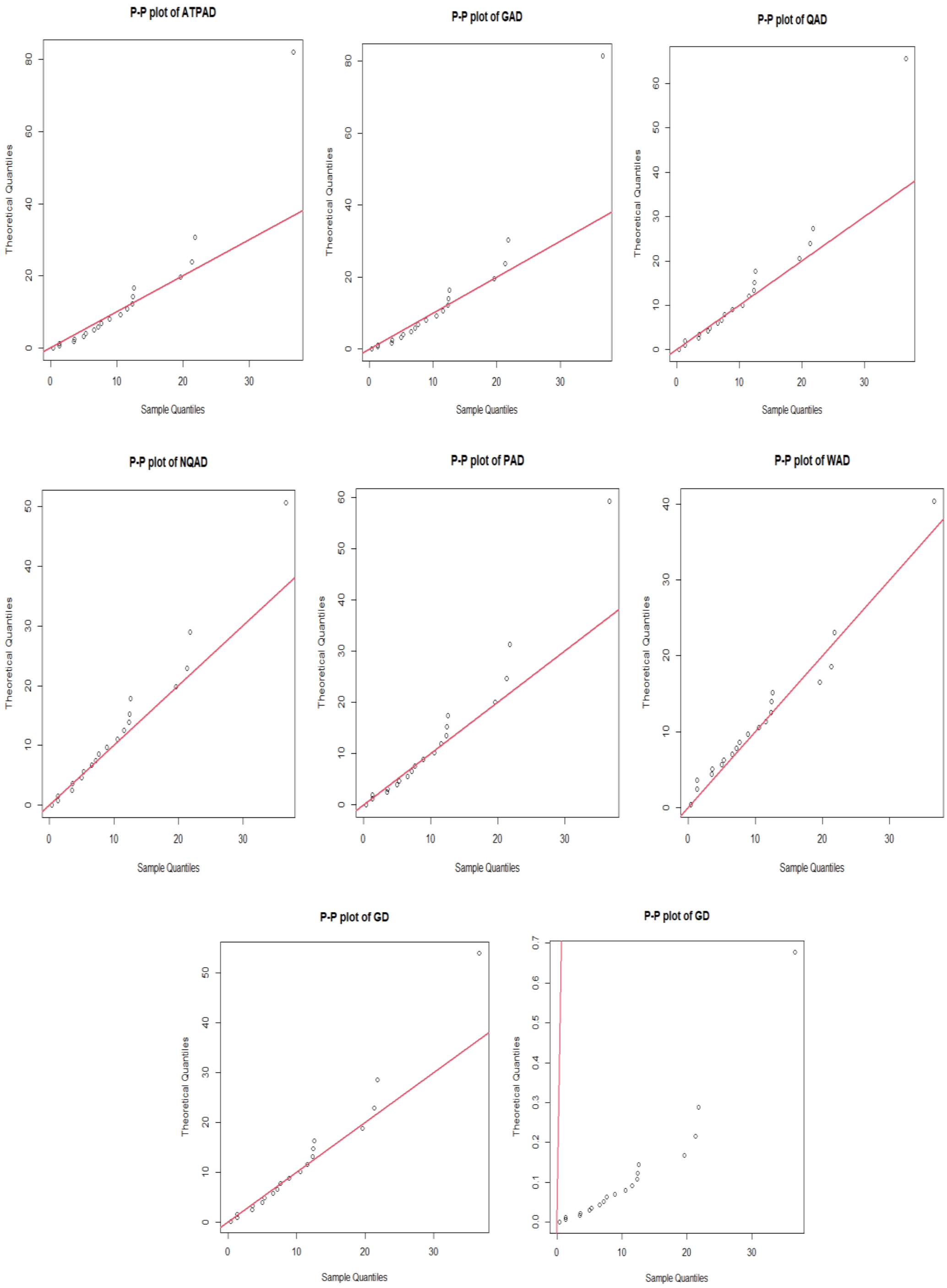

Figure 11: *P-P plot for considerd distributions of the data set-2*

XI. CONCLUSION AND FUTURE WORKS

In this paper, a two-parameter Aradhana distribution which includes Aradhana distribution, gamma distribution and exponential distribution are proposed. Its moments and statistical properties including survival function, hazard function, mean residual life function, reverse hazard function, stochastic ordering have been discussed. Deviations from mean and median, Bonferroni and Lorenz curve and their indices, stress strength reliability have also been discussed. The parameters of this distribution have been estimated using maximum likelihood estimation. To know the performance of maximum likelihood estimates of parameters, a simulation study has been presented. Finally, two examples of real lifetime datasets have been considered for applications and compared with GAD, QAD, NQAD, PAD, WAD, GD and WD. It has been found that ATPAD provide the best fit than the GAD, QAD, NQAD, PAD, WAD, GD and WD. As this is a new two-parameter lifetime distribution, it has the possibility of extension by adding more parameter in the distribution to see its performance over other lifetime distributions of same parameter. Further, Bayesian method of estimation and ranked set sampling method of estimation can also be considered in future to see the efficiency of these two methods of estimation over the classical maximum likelihood estimation.

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