BAYESIAN ESTIMATION OF PARAMETERS AND RELIABILITY CHARACTERISTICS IN THE INVERSE GOMPERTZ DISTRIBUTION

*1Taiwo. M. Adegoke, 2Latifat A. Abimbola, 3Oladapo M. Oladoja, 4Oyindamola. R. Oyebanjo & 5K.O. Obisesan

•

*1,2,3,4,5 Department of Mathematics and Statistics, First Technical University, Ibadan, Nigeria.
*1 taiwo.adegoke@tech-u.edu.ng, 2 latifat.abimbola@tech-u.edu.ng,
3 oladapo.oladoja@tech-u.edu.ng, 4 oyindamola.oyebanjo@tech-u.edu.ng
5 obisesan.olalekan@tech-u.edu.ng

Abstract

In this study, we derive Bayes' estimators for the unknown parameters of the Inverse Gompertz Distribution (IGD) using three alternative loss functions: the Squared Error Loss Function (SELF), the Entropy Loss Function (ELF), and the Linex Loss Function. Closed-form formulas for Bayes estimators are not possible when both parameters are unknown, hence Lindley's approximation (L-Approximation) is used for computation. We examine the performance of these estimators using their simulated hazards and assess their effectiveness in parameter estimation. It was discovered that as the sample size increases, parameter estimations became more precise and accurate across all functions. However, ELF consistently has lower MSE values than SELF and LINEX, indicating better parameter estimation. This pattern was also seen in the estimation of the hazard function, where ELF regularly beat SELF and LINEX, implying more efficient parameter estimation overall.

Keywords: Likelihood Function, Prior Distribution, Posterior Distribution, Bayes Estimates, Lindley Approximation

1. Introduction

Gompertz [1] proposed a probability distribution with two parameters, which is widely used in survival analysis to represent human mortality and behavioral sciences data. This distribution, a generalization of the exponential distribution, has many practical uses, particularly in medical and actuarial studies. It has considerable similarities to well-known distributions such as the Gumbel, Weibull, generalized logistic, exponential, and double exponential distributions [2].

However, the Gompertz distribution (GD) only shows an increasing failure rate, restricting its potential to represent occurrences across several fields. As a result, many authors have contributed to methodological studies and characterizations of this distribution to address real-world challenges in a variety of fields, including medical sciences, economics, behavioral sciences, engineering, biological studies, actuarial science, environmental studies, and lifetime analysis.

The Gompertz distribution and its variants have been the subject of extensive research. Read [3] offers a fundamental overview of the Gompertz distribution, including its features and applications in statistical fields. Makany [4] explores the theoretical foundations of Gompertz's curve and provides insights into its mathematical representation. Franses [5] discusses practical issues of fitting Gompertz curves to actual data. Wu and Lee [6] investigate combinations of Gompertz distributions, offering a framework for defining complicated systems. El-Gohary et al.

[7] introduce the generalized Gompertz distribution, which improves modeling flexibility. The beta-Gompertz distribution, proposed by Jafari et al. [8], enhances the flexibility of data capture. Khan et al. [9] introduce the transmuted Gompertz distribution, which can accommodate a wider range of data patterns. El-Bassiouny et al. [10, 11] study mixture models that combine the Gompertz distribution with other distributions to improve applicability in reliability and survival analysis. Rasool et al. [12] introduced the McDonald Gompertz distribution, which improves its ability to capture complicated data patterns. [13] introduced Topp-Leone Inverse Gompertz Distribution with different estimation procedures and application. Sanku et al. [14] assess and compare various estimating methodologies for the Gompertz distribution, assisting researchers and practitioners in selecting relevant methods.

2. Inverse Gompertz Distribution

The random variable X is said to have an Inverse Gaussian Distribution (IGD) with shape parameter λ and scale parameter γ, if its cumulative distribution function (CDF) is given by

$$F(x) = e^{-\frac{\lambda}{\gamma}\left(e^{\frac{\gamma}{x}}-1\right)}, \quad x > 0, \quad \lambda, \gamma > 0 \tag{1}$$

The probability density function (PDF) of the Inverse Gaussian Distribution (IGD) is expressed as

$$f(x) = \frac{\lambda}{x^2} e^{-\frac{\lambda}{\gamma}\left(e^{\frac{\gamma}{x}}-1\right)+\frac{\gamma}{x}} \tag{2}$$

Furthermore, the reliability function is provided as follows:

$$R(x) = 1 - e^{-\frac{\lambda}{\gamma}\left(e^{\frac{\gamma}{t}}-1\right)} \tag{3}$$

The quantile function for the IGD distribution can be expressed as

$$q = \frac{\gamma}{\ln(1 - \frac{\gamma}{\lambda}\ln q)}, \quad 0 < q < 1. \tag{4}$$

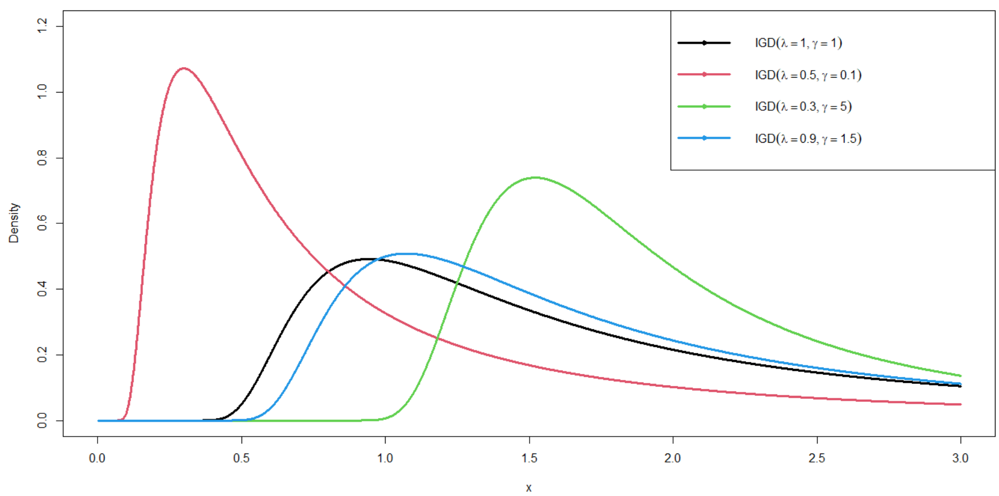

Figure 1: *PDF of the IGD*

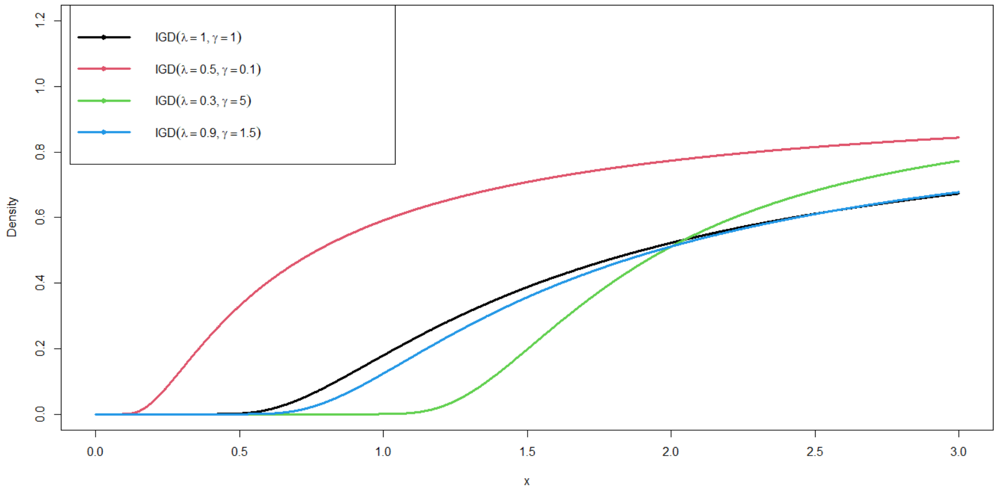

Figure 2: *CDF of the IGD*

3. Bayesian Estimation Techniques

Let $x = (x_1, x_2, \ldots x_n)$ be a random variable with parameters λ and γ having a size n. From the bayes' the posterior probability density function of the parameters λ and γ given x can be expressed as

$$\Pr(\lambda, \gamma, \eta | x) = \frac{\pi(\lambda, \gamma) l(\lambda, \gamma)}{\int\int\int \pi(\lambda, \gamma) l(\lambda, \gamma) \partial(\lambda, \gamma)} \tag{5}$$

where $l(\lambda, \gamma)$ is the likelihood and (λ, γ) is the prior probability distribution.

3.1. Likelihood Function

Given a series of observations $x = (x_1, x_2, \ldots x_n)$ with parameters λ and γ having a size n for IG distribution (2), the likelihood function can be expressed as

$$l = \frac{\lambda^n}{\sum x^2} e^{-\frac{\lambda}{\gamma} \sum \left(e^{\frac{\gamma}{x}} - 1\right) + \sum\left(\frac{\gamma}{x}\right)} \tag{6}$$

The log likelihood of IG distribution can be expressed as

$$L = \log l = n \log \lambda + \sum \left(\frac{\gamma}{x}\right) - 2 \sum \log(x) - \frac{\lambda}{\gamma} \sum \left(e^{\frac{\gamma}{x}} - 1\right) \tag{7}$$

The maximum likelihood estimator of the shape and scale parameters for the parameters λ and γ is obtained by differentiating the (7) on parameters λ and γ. The maximum likelihood differential equations are:

$$\frac{dL}{d\lambda} = \frac{1}{\lambda} - \sum_{i=1}^{n} \left(e^{\frac{\gamma}{x_i}} - 1\right) \frac{1}{\gamma} \tag{8}$$

$$\frac{dL}{d\gamma} = \sum_{i=1}^{n} \frac{1}{x_i} + \frac{\lambda \sum_{i=1}^{n} \left(\exp\left(\frac{\gamma}{x_i}\right) - 1\right)}{\gamma^2} - \frac{\lambda \sum_{i=1}^{n} \left(\frac{\exp\left(\frac{\gamma}{x_i}\right)}{x_i}\right)}{\gamma} \tag{9}$$

Analytical solutions to equations (8) and (9) are not viable. The estimated values for the parameters λ and γ can be derived numerically using an iterative approach known as the Newton-Raphson method [15, 17, 16]. The Fisher information matrix elements for parameters λ and γ can be represented as follows:

$$J_k = \begin{bmatrix} \frac{\partial^2 l(\lambda,\gamma)}{\partial\lambda^2} & \frac{\partial^2 l(\lambda,\gamma)}{\partial\lambda\partial\gamma} \\ \frac{\partial^2 l(\lambda,\gamma)}{\partial\lambda\partial\gamma} & \frac{\partial^2 l(\lambda,\gamma)}{\partial\gamma^2} \end{bmatrix} \tag{10}$$

The Jacobian matrix must be a non-singular symmetric matrix so its inverse must exist. So, using the Newton Raphson method we have

$$\begin{bmatrix} \lambda_{k+1} \\ \gamma_{k+1} \end{bmatrix} = \begin{bmatrix} \lambda_k \\ \gamma_k \end{bmatrix} - J_k^{-1} \begin{bmatrix} \frac{\partial l(\lambda,\gamma)}{\partial\lambda} \\ \frac{\partial l(\lambda,\gamma)}{\partial\gamma} \end{bmatrix} \tag{11}$$

with error term ϵ being the absolute differences between the new and the previous value of λ and γ in the iterative algorithm. That is

$$\epsilon \begin{bmatrix} \epsilon_{k+1}(\lambda) \\ \epsilon_{k+1}(\gamma) \end{bmatrix} = \left[\begin{bmatrix} \lambda_{k+1} \\ \gamma_{k+1} \end{bmatrix} - \begin{bmatrix} \lambda_k \\ \gamma_k \end{bmatrix} \right] \tag{12}$$

where λ_k and γ_k are the initial values of λ and γ respectively. .

where

$$L_{\lambda\lambda} = \frac{d^2L}{d\lambda^2} = -\frac{1}{\lambda^2} \tag{13}$$

$$L_{\gamma\gamma} = \frac{d^2L}{d\gamma^2} = -2 \cdot \frac{\lambda \sum_{i=1}^{n} \left(\exp\left(\frac{\gamma}{x_i}\right) - 1\right)}{\gamma^3} + \frac{2 \cdot \lambda \sum_{i=1}^{n} \left(\frac{\exp\left(\frac{\gamma}{x_i}\right)}{x_i}\right)}{\gamma^2} - \frac{\lambda \sum_{i=1}^{n} \left(\frac{\exp\left(\frac{\gamma}{x_i}\right)}{x_i^2}\right)}{\gamma} \tag{14}$$

$$L_{\lambda\gamma} = \frac{d^2L}{d\lambda d\gamma} = \frac{d^2L}{d\gamma d\lambda} = \frac{-n + \sum_{i=1}^{n} \exp\left(\frac{\gamma}{x_i}\right)}{\gamma^2} - \sum_{i=1}^{n} \left(\frac{\exp\left(\frac{\gamma}{x_i}\right)}{x_i}\right) \cdot \frac{1}{\gamma} \tag{15}$$

3.2. Prior Distribution

From (6), it can observed that there is no proper conjugate distribution for the parameters λ and γ. Therefore, we will consider the use of independent gamma prior distribution for the scale with parameters a_1 and b_1 and shape parameters a_2 and b_2. That is $\lambda \sim Gamma(a_1, b_1)$ and $\gamma \sim Gamma(a_2, b_2)$. The joint prior distribution can be expressed as

$$\pi(\lambda,\gamma) \propto \lambda^{a_1-1}\gamma^{a_2-1}e^{-b_1\lambda}e^{-b_2\gamma} \tag{16}$$

where a_1, a_2, b_1 and b_2 are hyper parameters.

3.3. Posterior Distribution

To obtain the posterior distribution for the IG distribution, we combine (6) and (16) and can be expressed as

$$P(\lambda,\gamma|X) = k^{-1}\lambda^{a_1+n-1}\gamma^{a_2-1}\sum x^{-2}e^{-\frac{\lambda}{\gamma}\Sigma\left(e^{\frac{\gamma}{x}}-1\right)+\Sigma\left(\frac{\gamma}{x}\right)-b_1\lambda-b_2\gamma} \tag{17}$$

where

$$k = \int_0^\infty \int_0^\infty \lambda^{a_1+n-1}\gamma^{a_2-1}\sum x^{-2}e^{-\frac{\lambda}{\gamma}\Sigma\left(e^{\frac{\gamma}{x}}-1\right)+\Sigma\left(\frac{\gamma}{x}\right)-b_1\lambda-b_2\gamma}\partial\lambda\partial\gamma$$

Analytical solutions for λ and γ from the posterior equation (17) are not viable due to its complicated nature, necessitating the use of numerical approaches such as Gibbs sampling, Metropolis-Hastings, EM algorithm, Lindley approximation, among others. In this study, we will use the Lindley approximation approach to obtain Bayesian estimates of λ and γ.

3.4. Loss Functions

The squared error is commonly employed as a loss function, however, its symmetric nature may not be acceptable in estimating issues with asymmetric losses. This disparity is especially pronounced in disciplines such as life testing and reliability estimation. In response, asymmetric loss functions, such as Varian's LINEX loss function [18], have gained popularity . [19] investigated the features of the LINEX loss function and discovered that the squared error loss is a specific instance of it. Another useful option is the entropy loss function.

In recent years, many authors have used Bayesian estimation for estimating the parameters of distributions. Examples include the works of Ahmed et al. [20], Basu & Ebrahimi [21], Nassar & Eissa [22], Pandey [23], Roio [24], Soliman et al. [31, 32, 33], Singh et al. [30, 25, 26], Adegoke et al [27], Ogunsanya et al. [28], Nzei et al. [29] , and others.

We achieve the appropriate Bayesian estimates by using predefined loss functions such squared error, LINEX, and entropy, which are defined as follows:

$$\begin{aligned} L_S(\hat{d}(\theta), d(\theta)) &= (\hat{d}(\theta) - d(\theta))^2, \\ L_L(\hat{d}(\theta), d(\theta)) &= e^{h(\hat{d}(\theta)-d(\theta))} - h(\hat{d}(\theta) - d(\theta)) - 1, \quad h \neq 0, \\ L_E(\hat{d}(\theta), d(\theta)) &\propto \left(\frac{\hat{d}(\theta)}{d(\theta)}\right)^w - w \log\left(\frac{\hat{d}(\theta)}{d(\theta)}\right) - 1, \quad w \neq 0, \end{aligned}$$

We get the desired Bayesian estimates.Here, $\hat{d}(\theta)$ is an estimate of $d(\theta)$. In the Bayesian paradigm, an optimal estimate for a certain loss function can be obtained by minimizing the average risk of $\hat{d}(\theta)$ relative to a weight function, also known as the prior distribution of θ. The Bayesian estimate, $\hat{d}_{BS}$, under the loss LS, corresponds to the posterior mean of $d(\theta)$. by applying specified loss functions: squared error, LINEX, and entropy, which are described as follows. The Bayesian estimate of $d(\theta)$ for the loss function LL is provided as:

$$\hat{d}_{BL} = -\frac{1}{h} \log\left(\mathbb{E}_\theta\left[e^{-h\theta}|x\right]\right)$$

the equivalent estimate for the loss function LE is as follows:

$$\hat{d}_{BE} = \left(\mathbb{E}_\theta(\theta^{-w}|x)\right)^{-\frac{1}{w}}$$

given that the corresponding expectations $\mathbb{E}_\theta(\cdot)$ exist. We use loss functions LS, LL, and LE to get Bayesian estimates of λ, γ, θ, the reliability function $R(t)$, and the hazard function $h(t)$.

Initially, we compute the Bayesian estimate for λ under the loss function L_S using the posterior distribution $P(\lambda, \gamma|x)$. This estimate is calculated as:

$$\hat{\lambda}_{BS} = k^{-1} \int_0^\infty \int_0^\infty \lambda^{a_1+n} \gamma^{a_2-1} \sum x^{-2} e^{-\frac{\lambda}{\gamma}\Sigma\left(e^{\frac{\gamma}{x}}-1\right)+\Sigma\left(\frac{\gamma}{x}\right)-b_1\lambda-b_2\gamma} \partial\lambda\partial\gamma \tag{18}$$

For the L_L loss function, the Bayesian estimate for λ is as follows:

$$\hat{\lambda}_{BL} = -\frac{1}{h} \log\left(\mathbb{E}\left[e^{-h\lambda}|x\right]\right) \qquad h \neq 0$$

where

$$\mathbb{E}_\lambda\left[e^{-h\lambda}|x\right] = k^{-1} \int_0^\infty \int_0^\infty \lambda^{a_1+n-1} \gamma^{a_2-1} \sum x^{-2} e^{-\frac{\lambda}{\gamma}\Sigma\left(e^{\frac{\gamma}{x}}-1\right)+\Sigma\left(\frac{\gamma}{x}\right)-b_1\lambda-b_2\gamma-h\lambda} \partial\lambda\partial\gamma \tag{19}$$

Finally, when considering the loss function LE, we determine that

$$\hat{\lambda}_{BE} = \left(\mathbb{E}(\lambda^{-w}|x)\right)^{-\frac{1}{w}}$$

where

$$\mathbb{E}_\lambda(\lambda^{-w}|x) = k^{-1}\int_0^\infty\int_0^\infty \lambda^{a_1+n-w-1}\gamma^{a_2-1}\sum x^{-2}e^{-\frac{\lambda}{\gamma}\Sigma\left(e^{\frac{\gamma}{x}}-1\right)+\Sigma\left(\frac{\gamma}{x}\right)-b_1\lambda-b_2\gamma}\partial\lambda\partial\gamma \quad (20)$$

Similarly, we proceed to derive Bayesian estimates for γ under the specified loss functions.

Assuming that λ and γ are unknown, we obtain equations for Bayesian estimates of the reliability function $R(t)$ in a similar manner. For the loss function L_S it is given as

$$\hat{R}(t) = k^{-1}\int_0^\infty\int_0^\infty \lambda^{a_1+n-1}\gamma^{a_2-1}\sum x^{-2}e^{-\frac{\lambda}{\gamma}\Sigma\left(e^{\frac{\gamma}{x}}-1\right)+\Sigma\left(\frac{\gamma}{x}\right)-b_1\lambda-b_2\gamma}\left(1-e^{-\frac{\lambda}{\gamma}\left(e^{\frac{\gamma}{t}}-1\right)}\right)\partial\lambda\partial\gamma \quad (21)$$

For the L_L loss function, we have

$$\hat{R}(t)_{BL} = -\frac{1}{h}\log\left(\mathbb{E}\left[e^{-hR(t)}|x\right]\right) \qquad h\neq 0$$

where

$$\mathbb{E}_\lambda\left[e^{-hR(t)}|x\right] = k^{-1}\int_0^\infty\int_0^\infty \lambda^{a_1+n-1}\gamma^{a_2-1}\sum x^{-2}e^{-\frac{\lambda}{\gamma}\Sigma\left(e^{\frac{\gamma}{x}}-1\right)+\Sigma\left(\frac{\gamma}{x}\right)-b_1\lambda-b_2\gamma}e^{-h\left(1-e^{-\frac{\lambda}{\gamma}\left(e^{\frac{\gamma}{t}}-1\right)}\right)}\partial\lambda\partial\gamma \quad (22)$$

Finally, for the loss function L_E, it is found that

$$\hat{\lambda}_{BE} = \left(\mathbb{E}(R(t)^{-w}|x)\right)^{-\frac{1}{w}}$$

$$\hat{R}(t)_{BE} = k^{-1}\int_0^\infty\int_0^\infty \lambda^{a_1+n-1}\gamma^{a_2-1}\sum x^{-2}e^{-\frac{\lambda}{\gamma}\Sigma\left(e^{\frac{\gamma}{x}}-1\right)+\Sigma\left(\frac{\gamma}{x}\right)-b_1\lambda-b_2\gamma}\left(1-e^{-\frac{\lambda}{\gamma}\left(e^{\frac{\gamma}{t}}-1\right)}\right)^{-w}\partial\lambda\partial\gamma \quad (23)$$

3.5. Lindley Approximation

In the preceding section, we derived Bayes estimators for λ, γ, and θ using various loss functions, such as squared error, linex, and entropy. It is worth noting that these estimators are expressed as ratios of two integrals, which resist simplification into closed forms. Nonetheless, using the methods developed by Lindley [34], these Bayes estimators can be estimated to a form devoid of integrals. In practice, this strategy produces simple Bayes estimators that are easy to implement. Consider the ratio of the integral $I(X)$,

$$I(x) = E[u(\lambda,\gamma|x)] = \frac{\int\int u(\lambda,\gamma)e^{L(\lambda,\gamma)+G(\lambda,\gamma)}d\lambda d\gamma}{\int\int e^{L(\lambda,\gamma)+\rho(\lambda,\gamma)}d\lambda d\gamma}, \quad (24)$$

where:

- $u(\lambda,\gamma)$ is a function of λ and γ only;
- $L(\lambda,\gamma)$ is the log of likelihood;
- $\rho(\lambda,\gamma)$ is the log of joint prior of λ and γ.

This can be evaluated as

$$\begin{aligned} I(x) =& u(\hat{\lambda},\hat{\gamma}) + \frac{1}{2}\left[(\hat{u}_{\gamma\gamma}+2\hat{u}_{\gamma}\hat{p}_{\gamma})\hat{\sigma}_{\gamma\gamma} + (\hat{u}_{\lambda\gamma}+2\hat{u}_{\lambda}\hat{p}_{\gamma})\hat{\sigma}_{\lambda\gamma}\right. \\ & \left. +(\hat{u}_{\gamma\lambda}+2\hat{u}_{\gamma}\hat{p}_{\lambda})\hat{\sigma}_{\gamma\lambda} + (\hat{u}_{\lambda\lambda}+2\hat{u}_{\lambda}\hat{p}_{\lambda})\hat{\sigma}_{\lambda\lambda}\right] \\ & + \frac{1}{2}\left[(\hat{u}_{\gamma}\hat{\sigma}_{\gamma\gamma}+\hat{u}_{\lambda}\hat{\sigma}_{\gamma\lambda})(L_{\gamma\gamma\gamma}\hat{\sigma}_{\gamma\gamma}+L_{\gamma\lambda\gamma}\hat{\sigma}_{\gamma\lambda}\right. \\ & +L_{\lambda\gamma\gamma}\hat{\sigma}_{\lambda\gamma}+L_{\lambda\lambda\gamma}\hat{\sigma}_{\lambda\lambda}) + (\hat{u}_{\gamma}\hat{\sigma}_{\lambda\gamma}+\hat{u}_{\lambda}\hat{\sigma}_{\lambda\lambda})(L_{\lambda\gamma\gamma}\hat{\sigma}_{\gamma\gamma} \\ & \left. +L_{\gamma\lambda\lambda}\hat{\sigma}_{\gamma\lambda}+L_{\lambda\gamma\lambda}\hat{\sigma}_{\lambda\gamma}+L_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda})\right] \end{aligned} \tag{25}$$

wher e:

- $\hat{\lambda}$ = MLE of λ;
- $\hat{\gamma}$ = MLE of γ;
- $\hat{u}_{\gamma} = \frac{\partial u(\hat{\lambda},\hat{\gamma})}{\partial\gamma}$, $\hat{u}_{\lambda} = \frac{\partial u(\hat{\lambda},\hat{\gamma})}{\partial\lambda}$, $\hat{u}_{\gamma\lambda} = \frac{\partial^2 u(\hat{\lambda},\hat{\gamma})}{\partial\gamma\partial\lambda}$, $\hat{u}_{\lambda\gamma} = \frac{\partial^2 u(\hat{\lambda},\hat{\gamma})}{\partial\lambda\partial\gamma}$;
- $\hat{u}_{\gamma\gamma} = \frac{\partial^2 u(\hat{\lambda},\hat{\gamma})}{\partial\gamma^2}$, $\hat{u}_{\lambda\lambda} = \frac{\partial^2 u(\hat{\lambda},\hat{\gamma})}{\partial\lambda^2}$;
- $\hat{L}_{\lambda\gamma\gamma} = \hat{L}_{\gamma\lambda\gamma} = \hat{L}_{\gamma\gamma\lambda} = \frac{\partial^3 L(\hat{\lambda},\hat{\gamma})}{\partial\gamma\partial\hat{\gamma}\partial\hat{\lambda}}$, $\hat{L}_{\gamma\gamma\gamma} = \frac{\partial^3 L(\hat{\lambda},\hat{\gamma})}{\partial\gamma\partial\hat{\gamma}\partial\hat{\gamma}}$, $\hat{L}_{\lambda\lambda\lambda} = \frac{\partial^3 L(\hat{\lambda},\hat{\gamma})}{\partial\lambda\partial\hat{\lambda}\partial\hat{\lambda}}$;
- $\hat{L}_{\gamma\lambda\lambda} = \hat{L}_{\lambda\lambda\gamma} = \hat{L}_{\lambda\gamma\lambda} = \frac{\partial^3 L(\hat{\lambda},\hat{\gamma})}{\partial\gamma\partial\hat{\lambda}\partial\hat{\lambda}}$;
- $\hat{p}_{\lambda} = \frac{\partial\pi(\hat{\lambda},\hat{\gamma})}{\partial\lambda}$, $\hat{p}_{\gamma} = \frac{\partial\pi(\hat{\lambda},\hat{\gamma})}{\partial\gamma}$.

wher e

$$L_{\lambda\lambda\gamma} = 0; \qquad L_{\lambda\lambda\lambda} = \frac{2}{\lambda^3} \tag{26}$$

$$L_{\gamma\gamma\lambda} = -2\cdot\frac{-n+\sum_{i=1}^{n}\exp\left(\frac{\gamma}{x_i}\right)}{\gamma^3} + \frac{2\cdot\sum_{i=1}^{n}\left(\frac{\exp\left(\frac{\gamma}{x_i}\right)}{x_i}\right)}{\gamma^2} - \frac{\sum_{i=1}^{n}\left(\frac{\exp\left(\frac{\gamma}{x_i}\right)}{x_i^2}\right)}{\gamma} \tag{27}$$

$$L_{\gamma\gamma\gamma} = 6\cdot\frac{\lambda\sum_{i=1}^{n}\left(\exp\left(\frac{\gamma}{x_i}\right)-1\right)}{\gamma^4} - 6\cdot\frac{\lambda\sum_{i=1}^{n}\left(\frac{\exp\left(\frac{\gamma}{x_i}\right)}{x_i}\right)}{\gamma^3} + 3\cdot\frac{\lambda\sum_{i=1}^{n}\left(\frac{\exp\left(\frac{\gamma}{x_i}\right)}{x_i^2}\right)}{\gamma^2} - \frac{\lambda\sum_{i=1}^{n}\left(\frac{\exp\left(\frac{\gamma}{x_i}\right)}{x_i^3}\right)}{\gamma} \tag{28}$$

$$\begin{aligned} \log\,\pi(\lambda,\gamma) &= (a_1-1)*\log(\lambda) + (a_2-1)*\log(\gamma) - b_1\lambda - b_2\gamma \\ \rho_{\lambda} &= \frac{a_1-1}{\lambda} - b_1; \qquad \rho_{\gamma} = \frac{a_2-1}{\gamma} - b_2 \end{aligned}$$

3.5.1 Bayes estimates of the parameters of IGD and its reliability

To obtain the bayes estimate under SELF for $\hat{\lambda}$, $u(\hat{\lambda},\hat{\gamma}) = \hat{\lambda}$, $u_{\lambda\lambda} = u_{\lambda\gamma} = u_{\gamma\gamma} = u_{\gamma\lambda} = u_{\gamma} = 0$ and $u_{\lambda} = 1$. Substituting these values into (25), we have

$$\hat{\lambda}_{BS} = \hat{\lambda} + \hat{p}_{\lambda}\hat{\sigma}_{\lambda\lambda} + \frac{1}{2}L_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda} \tag{29}$$

also to obtain the bayes estimate under SELF for $\hat{\gamma}$, $u(\hat{\lambda},\hat{\gamma}) = \hat{\gamma}$, $u_{\lambda\lambda} = u_{\lambda\gamma} = u_{\gamma\gamma} = u_{\gamma\lambda} = u_{\lambda} = 0$ and $u_{\gamma} = 1$. Substituting these values into (25), we have

$$\hat{\gamma}_{BS} = \hat{\gamma} + \hat{p}_{\gamma}\hat{\sigma}_{\gamma\gamma} + \frac{1}{2}\hat{\sigma}_{\gamma\gamma}L_{\gamma\gamma\gamma}\hat{\sigma}_{\gamma\gamma} \tag{30}$$

To obtain the bayes estimate of $\hat{\lambda}$ under the ELF, $u(\hat{\lambda},\hat{\gamma}) = \lambda^{-w}$, then $u_\lambda = -w\hat{\lambda}^{-w-1}$, $u_{\lambda\lambda} = w(w+1)\hat{\lambda}^{-w-2}$ and $u_{\lambda\gamma} = u_{\gamma\gamma} = u_{\gamma\lambda} = u_\gamma = 0$. Substituting these values into (25), we have

$$\hat{\lambda}_{BE} = \hat{\lambda}^{-w} + \frac{1}{2}\left[\hat{\sigma}_{\lambda\lambda}\left(\hat{u}_{\lambda\lambda} + 2\hat{u}_\lambda\rho_\lambda\right)\right] + \frac{1}{2}\left[\left(\hat{u}_\lambda\hat{\sigma}_{\lambda\lambda}\left(L_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}\right)\right]\right. \tag{31}$$

also to obtain the bayes estimate of $\hat{\gamma}$ under the ELF, $u(\hat{\lambda},\hat{\gamma}) = \gamma^{-w}$, then $u_\gamma = -w\hat{\lambda}^{-w-1}$, $u_{\gamma\gamma} = w(w+1)\hat{\lambda}^{-w-2}$ and $u_{\lambda\gamma} = u_{\lambda\lambda} = u_{\gamma\lambda} = u_\gamma = 0$. Substituting these values into (25), we have

$$\hat{\gamma}_{BE} = \hat{\gamma}^{-w} + \frac{1}{2}\left[\hat{\sigma}_{\gamma\gamma}\left(\hat{u}_{\gamma\gamma} + 2\hat{u}_\gamma\rho_\gamma\right)\right] + \frac{1}{2}\left[\left(\hat{u}_\gamma\hat{\sigma}_{\gamma\gamma}\left(L_{\gamma\gamma\gamma}\hat{\sigma}_{\gamma\gamma}\right)\right]\right. \tag{32}$$

To obtain the bayes estimate of λ under the LLF, $u(\hat{\lambda},\hat{\gamma}) = e^{-h\lambda}$, then $u_\lambda = -he^{-h\lambda}$, $u_{\lambda\lambda} = h^2e^{-h\lambda}$ and $u_{\lambda\gamma} = u_{\gamma\gamma} = u_{\gamma\lambda} = u_\gamma = 0$. Substituting these values into (25), we have

$$\hat{\lambda}_{BL} = e^{-h\lambda} + \frac{1}{2}\left[\hat{\sigma}_{\lambda\lambda}\left(\hat{u}_{\lambda\lambda} + 2\hat{u}_\lambda\rho_\lambda\right)\right] + \frac{1}{2}\left[\left(\hat{u}_\lambda\hat{\sigma}_{\lambda\lambda}\left(L_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}\right)\right]\right. \tag{33}$$

also to obtain the bayes estimate of γ under the LLF, $u(\hat{\lambda},\hat{\gamma}) = e^{-h\gamma}$, then $u_\gamma = -he^{-h\lambda}$, $u_{\gamma\gamma} = h^2e^{-h\lambda}$. and $u_{\lambda\gamma} = u_{\lambda\lambda} = u_{\gamma\lambda} = u_\gamma = 0$. Substituting these values into (25), we have

$$\hat{\gamma}_{BL} = e^{-h\gamma} + \frac{1}{2}\left[\hat{\sigma}_{\gamma\gamma}\left(\hat{u}_{\gamma\gamma} + 2\hat{u}_\gamma\rho_\gamma\right)\right] + \frac{1}{2}\left[\left(\hat{u}_\gamma\hat{\sigma}_{\gamma\gamma}\left(L_{\gamma\gamma\gamma}\hat{\sigma}_{\gamma\gamma}\right)\right]\right. \tag{34}$$

Under the SELF the bayes estimates for the reliability of IGD can be obtained by equating

$$\begin{aligned}
u &= 1 - e^{-\frac{\lambda}{\gamma}\left(e^{\frac{\gamma}{t}}-1\right)}; \qquad u_\lambda = \frac{(e^{\frac{\gamma}{t}}-1)\cdot e^{-\frac{\lambda}{\gamma}(e^{\frac{\gamma}{t}}-1)}}{\gamma} \\
u_\gamma &= -\left(\frac{\lambda(e^{\frac{\gamma}{t}}-1)}{\gamma^2} - \frac{\lambda e^{\frac{\gamma}{t}}}{\gamma t}\right)\cdot e^{-\frac{\lambda}{\gamma}(e^{\frac{\gamma}{t}}-1)}; \qquad u_{\lambda\lambda} = -\frac{(e^{\frac{\gamma}{t}}-1)^2\cdot e^{-\frac{\lambda}{\gamma}(e^{\frac{\gamma}{t}}-1)}}{\gamma^2} \\
u_{\gamma\gamma} &= -\left(-2\frac{\lambda(e^{\frac{\gamma}{t}}-1)}{\gamma^3} + 2\frac{\lambda e^{\frac{\gamma}{t}}}{\gamma^2 t} - \frac{\lambda e^{\frac{\gamma}{t}}}{\gamma t^2}\right)e^{-\frac{\lambda}{\gamma}(e^{\frac{\gamma}{t}}-1)} - \left(\frac{\lambda(e^{\frac{\gamma}{t}}-1)}{\gamma^2} - \frac{\lambda e^{\frac{\gamma}{t}}}{\gamma t}\right)^2 e^{-\frac{\lambda}{\gamma}(e^{\frac{\gamma}{t}}-1)} \\
u_{\lambda\gamma} &= u_{\gamma\lambda} = -\frac{(e^{\frac{\gamma}{t}}-1)\cdot e^{-\frac{\lambda}{\gamma}(e^{\frac{\gamma}{t}}-1)}}{\gamma^2} + \frac{e^{\frac{\gamma}{t}}\cdot e^{-\frac{\lambda}{\gamma}(e^{\frac{\gamma}{t}}-1)}}{\gamma t} + \frac{(e^{\frac{\gamma}{t}}-1)\cdot\left(\frac{\lambda(e^{\frac{\gamma}{t}}-1)}{\gamma^2} - \frac{\lambda e^{\frac{\gamma}{t}}}{\gamma t}\right)\cdot e^{-\frac{\lambda}{\gamma}(e^{\frac{\gamma}{t}}-1)}}{\gamma}
\end{aligned}$$

and substituting the values into (25). We have

$$\begin{aligned}
I(x) =& u(\hat{\lambda},\hat{\gamma}) + \frac{1}{2}\left[(\hat{u}_{\gamma\gamma} + 2\hat{u}_\gamma\hat{p}_\gamma)\hat{\sigma}_{\gamma\gamma} + (\hat{u}_{\lambda\lambda} + 2\hat{u}_\lambda\hat{p}_\lambda)\hat{\sigma}_{\lambda\lambda}\right] \\
&+ \frac{1}{2}\left[(\hat{u}_\gamma\hat{\sigma}_{\gamma\gamma})(L_{\gamma\gamma\gamma}\hat{\sigma}_{\gamma\gamma} + L_{\lambda\lambda\gamma}\hat{\sigma}_{\lambda\lambda}) + (\hat{u}_\lambda\hat{\sigma}_{\lambda\lambda})(L_{\lambda\gamma\gamma}\hat{\sigma}_{\gamma\gamma}\right. \\
&\left.+ L_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda})\right].
\end{aligned} \tag{35}$$

Similarly , we can evaluate the Bayes estimators for the reliability function using the ELF and LLF.

3.6. Simulation Study

In this part, we undertake a simulation research to estimate the parameters and reliability of the Inverse Gamma (IG) distribution across several λ and γ combinations: (0.9, 0.6), (1.0, 1.0), (1.0, 0.7), and (1.2, 0.8). The population parameter is created with R programming version 4.3.1. Sampling distributions are calculated for various sample sizes $n = [30, 50, 100, 500]$ using $R = 1000$ replications.Tables 1 and 2 show the calculated estimates and mean square errors (MSE) in brackets.

Table 1: *Bayes estimates for different parameter values under the SELF, ELF and LINEX*

		SELF		ELF		LINEX	
		$\hat{\lambda}_{BS}$	$\hat{\gamma}_{BS}$	$\hat{\lambda}_{BE}$	$\hat{\gamma}_{BE}$	$\hat{\lambda}_{BL}$	$\hat{\gamma}_{BL}$
n =30	$\lambda = 0.9$	0.9300	0.7191	0.8560	0.8274	0.6464	0.6600
	$\gamma = 0.6$	(0.0077)	(0.1106)	(0.0027)	(0.0884)	(0.0646)	(0.0182)
n =50	h = 0.6	0.9524	0.6090	0.8626	0.7691	0.6465	0.6988
	w= -0.5	(0.0061)	(0.0430)	(0.0018)	(0.0475)	(0.0647)	(0.0172)
n = 100	$a_1 = 1$	0.967	0.6103	0.8689	0.7744	0.6414	0.6964
	$a_2 = 1$	(0.0059)	(0.0273)	(0.0011)	(0.0417)	(0.0670)	(0.0139)
n = 500	$b_1 = 1$	0.9859	0.6026	0.8759	0.775	0.6373	0.6971
	$b_2 = 0.5$	(0.0076	(0.0050)	(0.0005)	(0.0327)	(0.0690)	(0.0103)
n =30	$\lambda = 1.0$	1.0589	1.2097	1.2692	0.9871	1.1225	1.1300
	$\gamma = 1.0$	(0.2014)	(0.3130)	(0.0623)	(0.0775)	(0.1425)	(0.0203)
n =50	h = -0.1	1.0731	1.0216	1.2397	1.0376	1.1253	1.1081
	w = 0.5	(0.1880)	(0.1199)	(0.0728)	(0.0482)	(0.1404)	(0.0131)
n = 100	$a_1 = 0.5$	1.0956	1.0206	1.2359	1.0171	1.1277	1.1078
	$a_2 = 0.5$	(0.1654)	(0.0760)	(0.0723)	(0.0226)	(0.1385)	(0.0125)
n = 500	$b_1 = 0.5$	1.1195	1.0051	1.2265	1.0021	1.1307	1.1058
	$b_2 = 0.5$	(0.1448)	(0.01408)	(0.0752)	(0.0033)	(0.1363)	(0.0113)
n =30	$\lambda = 1.0$	1.4021	0.8344	1.2489	0.8529	0.8726	0.9204
	$\gamma = 0.7$	.2226)	(0.1419)	(0.0903)	(0.1103)	(0.0166)	(0.04964)
n =50	h = 0.1	1.4760	0.7099	1.2998	0.7538	0.8666	0.9317
	w = -0.8	(0.2700)	(0.0553)	(0.1089)	(0.0441	(0.0180)	(0.0541)
n = 100	$a_1 = 1$	1.4773	0.7113	1.3025	0.7575	0.8662	0.9315
	$a_2 = 1$	0.2530)	(0.0350)	(0.1025)	(0.0288)	(0.018)	0.0538)
n = 500	$b_1 = 0.5$	1.4968	0.7030	1.3174	0.7536	0.8644	0.9321
	$b_2 = 0.5$	(0.2519)	(0.0065)	(0.1029)	(0.0076)	(0.0184)	(0.0539)
n =30	$\lambda = 1.2$	0.9083	0.9659	0.8734	0.9612	1.2290	1.217
	$\gamma = 0.8$	(0.1015)	(0.2014)	(0.1193)	(0.1689)	(0.0016)	(0.1844)
n =50	h = -0.2	0.8943	0.8157	0.8592	0.8283	1.2290	1.1788
	w = -0.9	(0.1192)	(0.0772)	(0.1357)	(0.0662)	(0.0019)	(0.1478)
n = 100	$a_1 = 1$	0.9272	0.8157)	0.8867	0.8299	1.2373	1.1782
	$a_2 = 1$	(0.0853	(0.0488	(0.1067	(0.0418)	(0.0018)	(0.1458)
n = 500	$b_1 = 1$	0.9533	0.8039	0.9077	0.8211	1.245	1.1746
	$b_2 = 1$	(0.0618)	.0090)	(0.0863)	(0.0080)	(0.0020)	(0.1408)

Table 2: *Bayes estimates for the hazard function under the SELF, ELF and LINEX*

		$\hat{R}(t)_{BS}$	$\hat{R}(t)_{BE}$	$\hat{R}(t)_{BL}$
n =30	$\lambda = 0.9$	0.5728	0.7343	0.7212
	$\gamma = 0.6$	(0.1114)	(0.0288)	(0.0326)
n =50	h = 0.6	0.5499	0.7208	0.7304
	w= -0.5	(0.1256)	(0.0328)	(0.0291)
n = 100	$a_1 = 1$	0.5517	0.7214	0.7301
	$a_2 = 1$	(0.1230)	(0.0323)	(0.0290)
n = 500	$b_1 = 1$	0.5501	0.7206	0.7309
	$b_2 = 0.5$	(0.1227)	(0.0322)	(0.0286)
	t =1			
n =30	$\lambda = 1.0$	0.3971	1.6789	1.0409
	$\gamma = 1.0$	(1.218)	(0.0375)	(0.2107)
n =50	h = -0.1	0.36002	1.9986	1.0371
	w = 0.5	(0.2917)	(1.2129)	(0.0188)
n = 100	$a_1 = 0.5$	0.36199	1.9976	1.0373
	$a_2 = 0.5$	(0.2895)	(1.2085)	(0.0188)
n = 500	$b_1 = 0.5$	0.3650	1.9876	1.0376
	$b_2 = 0.5$	(0.2862)	(1.1837)	(0.0189)
	t = 2			
n =30	$\lambda = 1$	0.4004	0.4639	0.9609
	$\gamma = 0.7$	(0.3605)	(0.2881)	(0.0015)
n =50	h = 0.1	0.4051	0.4674	0.9605
	w = -0.8	(0.3542)	(0.2838)	(0.0015)
n = 100	$a_1 = 1$	0.4077	0.4696	0.9603
	$a_2 = 1$	(0.3511)	(0.2813)	(0.0015)
n = 500	$b_1 = 0.5$	0.4124	0.4738	0.9598
	$b_2 = 0.5$	(0.3452)	(0.2768)	(0.0016)
	t =3			
n =30	$\lambda = 1.2$	0.1776	0.2037	0.9656
	$\gamma = 0.8$	(1.046)	(0.9935)	(0.0549)
n =50	h = -0.2	0.1695	0.1950	0.9672
	w = -0.9	(1.063)	(1.011)	(0.0542)
n = 100	$a_1 = 1$	0.1741	0.1995	0.9664
	$a_2 = 1$	(1.0529)	(1.0015)	(0.0545)
n = 500	$b_1 = 1$	0.1771	0.2021	0.9659
	$b_2 = 1$	(1.0464)	(0.9957)	(0.0548)
	t = 5			

Table 1 shows Bayesian estimates for various parameter values using three loss functions: SELF, ELF, and LINEX, with varied sample sizes. Each cell includes the estimated value of parameters ($\hat{\lambda}$ and $\hat{\gamma}$) with their standard errors in parentheses. Generally, as the sample size grows, the estimates get more precise, as evidenced by decreasing standard errors. The three loss functions act differently depending on the parameter values. However, it is clear that the ELF loss function consistently produces estimates with fewer standard errors than SELF and LINEX, implying greater performance in parameter estimation. This trend persists across a wide range of sample sizes and parameter values, demonstrating the efficiency of the ELF loss function in Bayesian estimation.

Table 2 shows Bayesian estimates of the hazard function for three different loss functions: SELF, ELF, and LINEX, across a range of sample sizes and parameter values. Increasing sample sizes often results in lower mean squared error (MSE) across all three functions, indicating better parameter estimate accuracy. However, performance differences exist amongst the loss algorithms at different parameter settings. with example, with $\lambda = 0.9$, $\gamma = 0.6$, $h = 0.6$, $w = -0.5$, $a_1 = 1$, $a_2 = 1$, $b_1 = 1$, and $b_2 = 0.5$, the ELF loss function consistently produces the lowest MSE compared to SELF and LINEX. This pattern holds true across other parameter settings, implying that the ELF function outperforms the MSE.

3.7. Real life Application

In this section, we look at the dataset published by Balakrishnan et al. [35], which includes 134entries representing scores on the General Rating of Affective Symptoms for Preschoolers (GRASP) scale. Using Bayesian approaches, we obtain the parameter estimates and reliability ratings for the Inverse Gamma (IG) distribution over a variety of loss functions.

Table 3: *Bayes estimate for the parameter of IGD under different loss functions when $a_1 = 1, a_2 = 1, b_1 = 0.5$ and $b_2 = 0.5$*

	SELF	ELF w = -0.7	ELF w = 1.2	LINEX h = -0.5	LINEX h = 0.5
$\hat{\lambda}$	0.2959	0.2962	0.29226	0.3520	0.3812
$\hat{\gamma}$	153.1028	152.3344	161.8959	156.055	156.0564

Table 4: *Bayes estimate for the reliability function under different loss functions for different parameter values*

		$a_1 = a_2 = 1$, $b_1 = b_2 = 0.5$	$a_1 = a_2 = 1$, $b_1 = b_2 = 1$
SELF	t = 1	0.3820	0.3804
	t = 5	0.3805	0.3789
ELF	w = -1.5 t = 1	0.3821	0.3811
	w = 1.5 t = 5	0.3830	0.3799
LINEX	h = 1 t =1	0.3678	0.3651
	h = -1 t =5	0.3679	0.3645

Table 3 shows the the Bayes estimates for the parameters of IG distribution under different loss functions. Also, Table 4 display the reliability estimates under different loss functions and parameter values.

4. Conclusion

Table 1 compares Bayesian parameter estimation for three different loss functions: SELF, ELF, and LINEX. Overall, as sample size grows, parameter estimates become more precise and accurate across all loss functions. However, the ELF loss function consistently produces lower mean squared error (MSE) values than SELF and LINEX, indicating more effective parameter estimation. This shows that the ELF loss function may perform better in terms of balancing precision and accuracy, making it an attractive option for Bayesian parameter estimation applications. Table 2 shows Bayesian estimates for the hazard function using three alternative loss functions: SELF, ELF, and LINEX. It demonstrates how the performance of these estimators fluctuates with sample size and parameter values. In general, as sample size increases, mean squared error (MSE) decreases across all three loss functions, indicating that parameter estimations are more accurate and precise. The ELF loss function regularly produces lower MSE values than SELF and LINEX, indicating more efficient parameter estimation.

References

[1] Gompertz, B. (1819). On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies. *Philosophical Transactions of the Royal Society of London*, 109, 513–585.

[2] Willekens, F. (1999). Continuous-time demographic models for the population of the United States: United States working paper. *United Nations Population Division*.

[3] Read, C. B. (1975). Gompertz distribution. *Encyclopedia of Statistical Sciences*. Wiley, New York.

[4] Makany, R. A. (2012). Theoretical basis of Gompertz's curve. *Biometrical Journal*, 33, 121–128.

[5] Franses, P. H. (1998). Fitting a Gompertz curve. *Journal of the Operational Research Society*, 45, 109–113.

[6] Wu, J. W. and Lee, W. C. (2003). Characterization of the mixtures of Gompertz distributions by conditional expectation of order statistics. *Biometrical Journal*, 41, 371–381.

[7] El-Gohary, A., Ahmad, A., and Adel Naif, A. (2010). The generalized Gompertz distribution. *Applied Mathematics and Modeling*, 37, 13–24.

[8] Jafari, A., Saeid, T., and Morad, A. (2016). The beta-Gompertz distribution. *Revista Colombiana de Estad?stica*, 37(1), 139–156.

[9] Khan, M. S., Robert, K., and Irene, L. H. (2018). Transmuted Gompertz distribution: properties and estimation. *Pakistan Journal of Statistics*, 32(3), 161–182.

[10] El-Bassiouny, A. H., Medhat, E. L.-D., Abdelfattah, M., and Eliwa, M. S. (2018). Mixture of exponentiated generalized Weibull–Gompertz distribution and its applications in reliability. *Journal of Statistical Applications in Probability*, 5(3), 1–14.

[11] El-Bassiouny, A. H., Medhat, E. L.-D., Abdelfattah, M., and Eliwa, M. S. (2019). Exponentiated generalized Weibull–Gompertz distribution with application in survival analysis. *Journal of Statistical Applications in Probability*, 6(1), 7–16.

[12] Rasool, R., Saeid, T., and Jafari, A. (2017). The McDonald Gompertz distribution: properties and applications. *Communications in Statistics - Simulation and Computation*, 47(5), 3341–3355.

[13] Adegoke, T. M., Oladoja, O. M., Bashir, S. O., Mustapha, A. A., Aderupatan, D. E., & Nzei, L. C. (2023). Topp-Leone inverse Gompertz distribution: Properties and different estimation techniques and applications. *Pakistan Journal of Statistics*, 39(4),433-456. https://www.pakjs.com/wp-content/uploads/2023/08/39402.pdf

[14] Sanku, D., Tanmay, K., and Yogesh, M. (2018). Evaluation and comparison of estimators in the Gompertz distribution. *Annals of Data Science*, 5(2), 235–258.

[15] Obisesan, K.O., Adegoke, T.M., Adekanmbi, D.B., and Lawal, M. (2015). Numerical approximation to intractable likelihood functions. *Perspectives and Developments in Mathematics*, 301-324.

[16] Bakari, H.R., Adegoke, T.M., and Yahya, A.M. (2016). Application of Newton Raphson method to non-linear models. *International Journal of Mathematics and Statistics Studies*, 4(4), 21-31.

[17] Yahaya, A.M., Dibal, N.P., Bakari, H.R., and Adegoke, T.M. (2016). Obtaining parameter estimate from the truncated Poisson probability distribution. *North Asian International Research Journal of Sciences, Engineering & I.T.*, 2(9), 3-10.

[18] Varian, H. R. (1975). A Bayesian approach to real estate assessment. In *Studies in Bayesian Econometrics and Statistics in Honor of Leonard J. Savage* (pp. 195–208).

[19] Zellner, A. (1986). Bayesian estimation and prediction using asymmetric loss functions. *Journal of the American Statistical Association*, 81.

[20] Ahmed, A., Al-Kutubi, H.,& Ibrahim, N. (2010). Comparison of the Bayesian and maximum likelihood estimation for Weibull distribution. *Journal of Mathematics and Statistics*, 6, 100–104.

[21] Basu, A.,& Ebrahimi, N. (1991). Bayesian approach to life testing and reliability estimation using asymmetric loss function. *Journal of Statistical Planning and Inference*, 29, 21–31.

[22] Nassar, M.,& Eissa, F. H. (2005). Bayesian estimation for the exponentiated Weibull model. *Communications in Statistics-Theory and Methods*, 33, 2343–2362.

[23] Pandey, B. (1997). Testimator of the scale parameter of the exponential distribution using linex loss function. *Communications in Statistics-Theory and Methods*, 26, 2191–2202.

[24] Roio, J. (1987). On the admissibility of c [xbar]+ d with respect to the linex loss function. *Communications in Statistics-Theory and Methods*, 16, 3745–3748.

[25] Singh, S. K., Singh, U.,& Sharma, V. K. (2013a). Bayesian estimation and prediction for flexible Weibull model under type-ii censoring scheme. *Journal of Probability and Statistics*, 2013.

[26] Singh, S. K., Singh, U.,& Sharma, V. K. (2013b). Bayesian estimation and prediction for the generalized Lindley distribution under asymmetric loss function. *Hacettepe Journal of Mathematics and Statistics*.

[27] Adegoke, T. M., Obisesan, K. O., Oladoja, O. M., & Adegoke, G. K. (2023). Bayesian and classical estimations of transmuted inverse Gompertz distribution. *Reliability: Theory & Applications*, 2(73), 24-38. DOI: 10.24412/1932-2321-2023-273-207-222

[28] Ogunsanya, A. S., Yahya, W. B., Adegoke, T. M., Iluno, C., Aderele, O. R., & Ekum, M. I. (2021). A new three-parameter Weibull inverse Rayleigh distribution: Theoretical development and applications. *Mathematics and Statistics*, 9(3), 249-272. DOI: 10.13189/ms.2021.090306

[29] Nzei, C. Lawrence, Adegoke, M. Taiwo, Ekhosuehi, N., & Mbegbu, I. Julian. (2024). Bayesian estimation of Topp-Leone Lindley (TLL) distribution parameters under different loss functions using Lindley approximation. *Reliability: Theory & Applications*, 1(77), 50-64. DOI: 10.24412/1932-2321-2024-177-50-64

[30] Singh, U., Gupta, P. K.,& Upadhyay, S. (2002). Estimation of exponentiated Weibull shape parameters under linex loss function. *Communications in Statistics-Simulation and Computation*, 31, 523–537.

[31] Soliman, A. et al. (2002). Reliability estimation in a generalized life-model with application to the Burr-XII. *IEEE Transactions on Reliability*, 51, 337–343.

[32] Soliman, A. et al. (2005). Estimation of parameters of life from progressively censored data using Burr-XII model. *IEEE Transactions on Reliability*, 54, 34–42.

[33] Soliman, A. A., Ellah, A. A.,& Sultan, K. (2006). Comparison of estimates using record statistics from Weibull model: Bayesian and non-Bayesian approaches. *Computational Statistics& Data Analysis*, 51, 2065–2077.

[34] Lindley, D. V. (1980). Approximate Bayesian methods. *Trabajos de Estad?stica y de Investigaci?n Operativa*, 31, 223–245.

[35] Balakrishnan, N., Victor, L., & Antonio, S. (2010). A mixture model based on Birnhaum-Saunders distributions: A study conducted by authors regarding the scores of the GRASP (general rating of affective symptoms for preschoolers), in a city located at the south part of Chile.