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WEIGHTED R-NORM ENTROPY FOR LIFETIME DISTRIBUTIONS: PROPERTIES AND APPLICATION

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Abstract

In the field of information theory, different uncertainty measures have been introduced by various researchers. These measures are widely used in reliability and survival studies. In this article, we introduce two new weighted uncertainty measures which are known as weighted R-Norm entropy (WRNE) and weighted R-Norm residual entropy (WRNRE). WRNE and WRNRE are "length-biased" shift-dependent uncertainty measures in which higher weight is assigned to large values of the observed random variable. Several important properties of these measures are studied. Some significant characterization results and the relationships of WRNRE with other reliability measures are presented. We also show that the survival function is uniquely determined by the WRNRE. Finally, based on a real life data set of bladder cancer patients, we illustrate the importance of WRNE and WRNRE.

Keywords: Weighted entropy, weighted R-Norm entropy, hazard rate function, mean residual life function and characterization results.

1. Introduction

A very important concept that has attracted the attention of researchers in the field of information theory is the measurement of uncertainty of probability distributions. The fundamental uncertainty measure (UM) which is well known by means of applications not only in the field of information theory but also in different other research fields is the Shannon's entropy [1]. Let Y be an absolutely continuous non-negative r.v with p.d.f $f(y)$, then the Shannon's entropy (SE) is defined as

$$H_Y(f) = -\int_0^\infty f(y)\log f(y)dy = -E[\log f(y)] \,. \tag{1}$$

Throughout this article, the notations r.v and p.d.f represent an absolutely continuous non-negative random variable and a probability density function respectively.

For a lifetime component that has survived up to an age t_0, the SE is not a useful technique for measuring the uncertainty about its residual life. So, the concept of residual entropy was proposed by Ebrahimi [2] and is defined as

$$H_Y(f;t_0) = -\int_{t_0}^\infty \frac{f(y)}{\bar{F}(t_0)}\log\frac{f(y)}{\bar{F}(t_0)}dy, \tag{2}$$

where, $\bar{F}(t_0) = 1 - F(t_0)$ is the survival function (s.f) of the r.v Y.

The above UM's have been widely used in different research fields, but these UM's consider that a lifetime system or component serves the same in its whole life from the aspects of some given qualitative characteristic set by the experimenter. Due to this drawback, these UM's provide the same importance to the occurrence of every event of a probabilistic experiment and therefore these measures take the designation of shift-independent UM's. But in our real life, there exist several situations where the shift-dependent UM's are desirable. So, with the contribution of Belis and Guiasu [3], the first shift-dependent UM, simply known as weighted entropy was introduced and is defined as

$$H_Y^w(f) = -\int_0^\infty w(y) f(y) \log f(y) dy$$
$$= -\int_0^\infty y f(y) \log f(y) dy, \tag{3}$$

where, the factor y in the integrand of (3) represents the weight which linearly emphasizes the occurrence of the event $\{Y = y\}$ and therefore yields a shift-dependent UM.

Similarly, Di Crescenzo and Longobardi [4] have extended the UM (3) to its dynamic (residual) version and therefore proposed the concept of weighted residual entropy as follows

$$H_Y^w(f; t_0) = -\int_{t_0}^\infty y \frac{f(y)}{\bar{F}(t_0)} \log \frac{f(y)}{\bar{F}(t_0)} dy. \tag{4}$$

It is clear from the available literature that the classical SE has been generalized in different ways by introducing some additional parameters to it. A well-known generalization that plays a very important role in different sciences is the concept of R-Norm entropy introduced by Boekee and Lubee [5]. For more work and applications of R-Norm entropy, one can see Kumar and Choudhary [6] and Kumar et al. [7]. The continuous version of R-Norm entropy (RNE) was given by Nanda and Das [8] and is given by

$$H_{(Y,R)}(f) = \frac{R}{R-1}\left[1 - \left\{\int_0^\infty f^R(y) dy\right\}^{\frac{1}{R}}\right], \ R > 0 (\neq 1). \tag{5}$$

Similarly, analogous to (2), Nanda and Das [8] have extended the R-Norm entropy to its dynamic (residual) version, known as R-Norm residual entropy for the residual lifetime $Y - t_0 | Y > t_0$ and is defined as

$$H_{(Y,R)}(f; t_0) = \frac{R}{R-1}\left[1 - \left\{\int_{t_0}^\infty \left(\frac{f(y)}{\bar{F}(t_0)}\right)^R dy\right\}^{\frac{1}{R}}\right], \ R > 0 (\neq 1). \tag{6}$$

From the recent literature, it is seen that the measurement of uncertainty (entropy) of probability distributions is widely being used in the research work of various researchers with respect to different sciences. After the existence of fundamental UM's, the various researchers have introduced their weighted versions (1.e weighted entropies) for measuring the uncertainty of such real life problems which are best fitted by weighted probability distributions. The researchers who have been attracted in the recent past by the concept of weighted entropy and therefore introduced some new flexible weighted UM's are: Bhat et al. [9], Bhat and Baig [10], Bhat et al. [11], Khammar and Jahanshahi [12], Kayal [13], Mirali and Baratpour [14], Nair et al. [15], Rajesh et al. [16], Nourbakhsh and Yari [17], Misagh et al. [18], Misagh and Yari [19] etc. Motivated with this research literature and the usefulness of R-Norm entropy and R-Norm residual entropy, here in this article, we introduce the concept of weighted R-Norm entropy and weighted R-Norm residual entropy. The article is continued as follows: In section 2, we consider the weighted R-Norm entropy (WRNE) in the form of its definition and several important properties. The section 3 studies the dynamic (residual) version of WRNE, known as weighted R-Norm residual entropy

(WRNRE) and also presents various significant characterization results of this UM. The various important properties of WRNRE and also its relationship with other well-known reliability measures are focused in section 4. The section 5 presents an application of the WRNE and WRNRE by using a real life data. Finally, in the last section, some concluding remarks are illustrated.

2. Weighted R-Norm Entropy (WRNE)

Analogous to (3), here in this section, we generate a new weighted UM which is actually the weighted version of R-Norm entropy (5) and is known as weighted R-Norm entropy (WRNE).

Definition 2.1 The WRNE for a r.v Y having p.d.f $f(y)$ denoted by $H^w_{(Y,R)}(f)$ is defined as

$$H^w_{(Y,R)}(f) = \frac{R}{R-1}\left[1 - \left(\int_0^\infty \left(yf(y)\right)^R dy\right)^{\frac{1}{R}}\right],\ R > 0(\neq 1), \tag{7}$$

where, the factor y in the integrand is defined in (3).

The following example makes it clear that two different probability distributions can have the same RNE's, but unequal WRNE's.

Example 2.1. Let Y and Z be two r.v's with pdf's

$$f_Y(t) = \begin{cases} \frac{t}{2}, & 0 < t < 2 \\ 0, & otherwise \end{cases} \qquad g_Z(t) = \begin{cases} 1 - \frac{t}{2}, & 0 < t < 2 \\ 0 & otherwise \end{cases}$$

By using (5), we obtain that

$$H_{(Y,R)}(f) = H_{(Z,R)}(g) = \frac{R}{R-1}\left[1 - \left(\frac{2}{R+1}\right)^{\frac{1}{R}}\right].$$

But, the WRNE's of Y and Z are not identical as follow

$$H^w_{(Y,R)}(f) = \frac{R}{R-1}\left[1 - \left(\frac{2^R}{2R+1}\right)^{\frac{1}{R}}\right] \text{ and } H^w_{(Z,R)}(g) = \frac{R}{R-1}\left[1 - \{2^{R+1}B(R+1, R+1)\}^{\frac{1}{R}}\right],$$

where,

$$B(\alpha,\beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},\ \alpha,\beta > 0.$$

Hence, even though $H_{(Y,R)}(f) = H_{(Z,R)}(g)$, but $H^w_{(Y,R)}(f) \neq H^w_{(Z,R)}(g), \forall\ R > 0(\neq 1)$.

Lemma 2.1. If $U = cY$, with $c > 0$, then

$$H^w_{(U,R)}(f) = \frac{R}{R-1}\left(1 - c^{\frac{1}{R}}\right) + c^{\frac{1}{R}} H^w_{(Y,R)}(f).$$

Example 2.2. Let $f(y)$ be the p.d.f of a r.v Y distributed as:

(a) Uniformly over $[m,n]$ with $f(y) = \frac{1}{n-m},\ m < y < n$, then

$$H_{(Y,R)}(f) = \frac{R}{R-1}\left[1 - (n-m)^{\frac{1-R}{R}}\right] \text{ and } H^w_{(Y,R)}(f) = \frac{R}{R-1}\left[1 - \left(\frac{n^{R+1} - m^{R+1}}{(n-m)^R}\right)^{\frac{1}{R}}\right].$$

(b) Exponentially with $f(y) = \eta e^{-\eta y},\ y > 0, \eta > 0$, then

$$H_{(Y,R)}(f)=\frac{R}{R-1}\left[1-\left(\frac{\eta^{R-1}}{R}\right)^{\frac{1}{R}}\right] \text{ and } H^{w}_{(Y,R)}(f)=\frac{R}{R-1}\left[1-\left(\frac{\Gamma(R+1)}{\eta R^{R+1}}\right)^{\frac{1}{R}}\right].$$

(c) Gamma with $f(y)=\frac{1}{\Gamma(\eta)}e^{-y}y^{\eta-1},\ 0<y<\infty,\eta>0$, then

$$H_{(Y,R)}(f)=\frac{R}{R-1}\left[1-\frac{1}{(\Gamma(\eta))^{R}}\left(\frac{\Gamma((\eta-1)R+1)}{R^{(\eta-1)R+1}}\right)^{\frac{1}{R}}\right] \text{ and } H^{w}_{(Y,R)}(f)=\frac{R}{R-1}\left[1-\frac{1}{(\Gamma(\eta))^{R}}\left(\frac{\Gamma(\eta R+1)}{R^{\eta R+1}}\right)^{\frac{1}{R}}\right].$$

(d) Weibull with $f(y)=\eta y^{\eta-1}e^{-y^{\eta}},\ y>0,\eta>0$, then

$$H_{(Y,R)}(f)=\frac{R}{R-1}\left[1-\left\{\frac{\eta^{R-1}\Gamma\left(\left(1-\frac{1}{\eta}\right)(R-1)+1\right)}{R^{(R-1)\left(1-\frac{1}{\eta}\right)+1}}\right\}^{\frac{1}{R}}\right] \text{ and } H^{w}_{(Y,R)}(f)=\frac{R}{R-1}\left[1-\left(\frac{\eta^{R-1}\Gamma\left(R+\frac{1}{\eta}\right)}{R^{R+\frac{1}{\eta}}}\right)^{\frac{1}{R}}\right].$$

(e) Rayleigh with $f(y)=\eta y e^{-\frac{\eta}{2}y^{2}},\ y\geq 0,\eta>0$, then

$$H_{(Y,R)}(f)=\frac{R}{R-1}\left[1-\left(\frac{(2\eta)^{\frac{R-1}{2}}\Gamma\left(\frac{R+1}{2}\right)}{(\sqrt{R})^{R+1}}\right)^{\frac{1}{R}}\right] \text{ and } H^{w}_{(Y,R)}(f)=\frac{R}{R-1}\left[1-\left\{\frac{\Gamma\left(R+\frac{1}{2}\right)\left(\frac{2^{2R-1}}{\eta}\right)^{\frac{1}{2}}}{R^{R+\frac{1}{2}}}\right\}^{\frac{1}{R}}\right].$$

Theorem 2.1. Let Y be a r.v having SE $H_Y(f)$, then

$$H^{w}_{(Y,R)}(f)\leq\frac{R}{R-1}\left[1-exp\left(\frac{(1-R)}{R}H_Y(f)+E(\log Y)\right)\right].$$

Proof. By applying the log-sum inequality, we have

$$\int_0^{\infty}f(y)\log\frac{f(y)}{(yf(y))^{R}}dy\geq\int_0^{\infty}f(y)dy\log\frac{\int_0^{\infty}f(y)dy}{\int_0^{\infty}(yf(y))^{R}dy}$$
$$=-\log\int_0^{\infty}(yf(y))^{R}dy.$$

Due to (7) and after simple simplification, we obtain the desired result.

3. Weighted R-Norm Residual Entropy (WRNRE)

This section presents the weighted R-Norm entropy for residual lifetimes by utilizing the equation (7) which is the weighted version of (6). Some important characterization results of this UM are also discussed.

Definition 3.1 For a r.v Y having p.d.f $f(y)$ and s.f $\bar{F}(t_0)$, the WRNRE of order R at time $t_0>0$ is defined as

$$H^{w}_{(Y,R)}(f;t_0)=\frac{R}{R-1}\left[1-\left\{\int_{t_0}^{\infty}\left(y\frac{f(y)}{\bar{F}(t_0)}\right)^{R}dy\right\}^{\frac{1}{R}}\right],\ R>0(\neq 1). \tag{8}$$

Here, we study the expressions of WRNRE of some well-known lifetime distributions.

Example 3.1. If a r.v Y has the p.d.f $f(y)$ and s.f $\bar{F}(t_0)$ as:

(a) $f(y)=\frac{1}{d-c},\ c<y<d$ and $\bar{F}(t_0)=\frac{d-t_0}{d-c}$, then

$$H_{(Y,R)}(y;t_0)=\frac{R}{R-1}\left[1-(d-t_0)^{\frac{1-R}{R}}\right] \quad \text{and} \quad H^{w}_{(Y,R)}(f;t_0)=\frac{R}{R-1}\left[1-\left\{\frac{d^{R+1}-t_0^{R+1}}{(d-t_0)^R(R+1)}\right\}^{\frac{1}{R}}\right].$$

(b) $f(y)=\eta e^{-\eta y},\ y>0,\eta>0$ and $\bar{F}(t_0)=e^{-\eta y}$, then

$$H_{(Y,R)}(f;t_0)=\frac{R}{R-1}\left(1-\frac{\eta^{\frac{R-1}{R}}}{R}\right) \quad \text{and} \quad H^{w}_{(Y,R)}(f;t_0)=\frac{R}{R-1}\left[1-\left(\frac{\Gamma(R+1,\eta R t_0)}{\eta e^{-\eta R t_0}R^{R+1}}\right)^{\frac{1}{R}}\right].$$

(c) $f(y)=\frac{\eta^{\mu}}{\Gamma(\mu)}e^{-\eta y}y^{\mu-1},\ 0<y<\infty,\eta,\mu>0$ and $\bar{F}(t_0)=\frac{\Gamma(\mu,\eta t_0)}{\Gamma(\mu)}$, then

$$H_{(Y,R)}(f;t_0)=\frac{R}{R-1}\left[1-\frac{\eta^{\mu}}{\Gamma(\mu,\eta t_0)}\left\{\frac{\Gamma(R(\mu-1)+1,\eta R t_0)}{(\eta R)^{R(\mu-1)+1}}\right\}^{\frac{1}{R}}\right]$$

and

$$H^{w}_{(Y,R)}(f;t_0)=\frac{R}{R-1}\left[1-\frac{\eta}{\Gamma(\mu,\eta t_0)}\left\{\frac{\Gamma(R\mu+1,\eta R t_0)}{(\eta R)^{R\mu+1}}\right\}^{\frac{1}{R}}\right].$$

(d) $f(y)=\eta y^{\eta-1}e^{-y^{\eta}},\ y>0,\eta>0$ and $\bar{F}(t_0)=e^{-y^{\eta}}$, then

$$H_{(Y,R)}(f;t_0)=\frac{R}{R-1}\left[1-\frac{\eta^{\frac{R-1}{R}}e^{t_0^{\eta}}}{R^{\frac{\eta(R+1)-R}{\eta R}}}\left\{\Gamma\left(\frac{R(\eta-1)+1}{\eta},Rt_0^{\eta}\right)\right\}^{\frac{1}{R}}\right]$$

and

$$H^{w}_{(Y,R)}(f;t_0)=\frac{R}{R-1}\left[1-\left\{\frac{\eta^{R-1}\Gamma\left(R+\frac{1}{\eta},Rt_0^{\eta}\right)}{R^{\frac{\eta R+1}{\eta}}e^{-Rt_0^{\eta}}}\right\}^{\frac{1}{R}}\right].$$

(e) $f(y)=\eta y e^{-\frac{\eta}{2}y^2},\ y\geq 0,\eta>0$ and $\bar{F}(t_0)=e^{-\frac{\eta}{2}y^2}$, then

$$H_{(Y,R)}(f;t_0)=\frac{R}{R-1}\left[1-e^{\frac{\eta}{2}t_0^2}\left\{\left(\frac{(2\eta)^{R-1}}{R^{R+1}}\right)^{\frac{1}{2}}\Gamma\left(\frac{R+1}{2},\frac{\eta R}{2}t_0^2\right)\right\}^{\frac{1}{R}}\right]$$

and

$$H^{w}_{(Y,R)}(f;t_0)=\frac{R}{R-1}\left[1-e^{\frac{\eta}{2}t_0^2}\left\{\Gamma\left(R+\frac{1}{2},\frac{\eta R}{2}t_0^2\right)\left(\frac{2^{2R-1}}{\eta R^{2R+1}}\right)^{\frac{1}{2R}}\right\}^{\frac{1}{R}}\right].$$

where, $\Gamma(\beta,\alpha z)=\alpha^{\beta}\int_z^{\infty}e^{-\alpha u}u^{\beta-1}du,\ \alpha,\beta>0$ is an upper incomplete gamma function.

Theorem 3.1 Let Y be a r.v having WRNRE and RNRE $H^{w}_{(Y,R)}(f;t_0)$ and $H_{(Y,R)}(f;t_0)$ respectively. Then for all $t_0>0$, wehave

$$H^{w}_{(Y,R)}(f;t_0)=\frac{R}{R-1}\left[1-\left\{t_0^R\left(1-\frac{(R-1)}{R}H_{(Y,R)}(f;t_0)\right)^R\right.\right.$$
$$\left.\left.+\int_{z=t_0}^{\infty}z^{R-1}\left(\frac{\bar{F}(z)}{\bar{F}(t_0)}\right)^R\left(R-(R-1)H_{(Y,R)}(f;z)\right)dz\right\}^{\frac{1}{R}}\right].$$

Proof.

$$\int_{t_0}^{\infty}\left(y\frac{f(y)}{\bar{F}(t_0)}\right)^R dy=\int_{t_0}^{\infty}\left(\int_0^y Rz^{R-1}dz\right)\left(\frac{f(y)}{\bar{F}(t_0)}\right)^R dy$$
$$=R\int_{t_0}^{\infty}\left[\int_0^{t_0} z^{R-1}dz+\int_{t_0}^{y} z^{R-1}dz\right]\left(\frac{f(y)}{\bar{F}(t_0)}\right)^R dy$$
$$=t_0^R\int_{t_0}^{\infty}\left(\frac{f(y)}{\bar{F}(t_0)}\right)^R dy+R\int_{z=t_0}^{\infty} z^{R-1}\left(\int_{y=z}^{\infty}\left(\frac{f(y)}{\bar{F}(t_0)}\right)^R dy\right)dz. \quad (9)$$

From (6), we have

$$\int_{t_0}^{\infty}\left(\frac{f(y)}{\bar{F}(t_0)}\right)^R dy=\left(1-\frac{(R-1)}{R}H_{(Y,R)}(f;t_0)\right)^R. \quad (10)$$

and

$$\int_{t_0}^{\infty} f^R(y)dy=\bar{F}^R(t_0)\left(1-\frac{(R-1)}{R}H_{(Y,R)}(f;t_0)\right)^R. \quad (11)$$

Using (9), (10) and (11) in (8), the required result will be obtained.

The following theorem shows that $H_{(Y,R)}^{w}(f;t_0)$ determines the s.f $\bar{F}(t_0)$ uniquely.

Theorem 3.2. Let Y be a r.v having p.d.f $f(y)$, s.f $\bar{F}(t_0)$ and WRNRE $H_{(Y,R)}^{w}(f;t_0)<\infty, \forall\ R>0(\neq 1)$ respectively. If $H_{(Y,R)}^{w}(f;t_0)$ is increasing in t_0, then $H_{(Y,R)}^{w}(f;t_0)$ uniquely determines the corresponding s.f $\bar{F}(t_0)$.

Proof. Rewriting (8) as

$$1-\frac{(R-1)}{R}H_{(Y,R)}^{w}(f;t_0)=\left(\int_{t_0}^{\infty}\left(y\frac{f(y)}{\bar{F}(t_0)}\right)^R dy\right)^{\frac{1}{R}}. \quad (12)$$

Differentiating (12) both sides w.r.t t_0, we have

$$\frac{(1-R)}{R}\frac{\partial}{\partial t_0}H_{(Y,R)}^{w}(f;t_0)=\frac{1}{R}\left(\int_{t_0}^{\infty}\left(y\frac{f(y)}{\bar{F}(t_0)}\right)^R dy\right)^{\frac{1-R}{R}}\left[Rh_F(t_0)\int_{t_0}^{\infty}\left(y\frac{f(y)}{\bar{F}(t_0)}\right)^R dy-t_0^R h_F^R(t_0)\right], \quad (13)$$

where, $h_F(t_0)=\frac{f(t_0)}{\bar{F}(t_0)}$ represents the hazard rate of Y. Using (12), we can rewrite (13) as

$$t_0^R\left(1-\frac{(R-1)}{R}H_{(Y,R)}^{w}(f;t_0)\right)^{1-R}h_F^R(t_0)-\left\{R-(R-1)H_{(Y,R)}^{w}(f;t_0)\right\}h_F(t_0)-(R-1)\frac{\partial}{\partial t_0}H_{(Y,R)}^{w}(f;t_0)=0\ . \quad (14)$$

For fixed $t_0>0$, $h_F(t_0)$ ia a solution of $\psi(x)=0$, where

$$\psi(x)=t_0^R\left(1-\frac{(R-1)}{R}H_{(Y,R)}^{w}(f;t_0)\right)^{1-R}x^R-R\left(1-\frac{(R-1)}{R}H_{(Y,R)}^{w}(f;t_0)\right)x-(R-1)\frac{\partial}{\partial t_0}H_{(Y,R)}^{w}(f;t_0)=0\ .$$

Differentiating $\psi(x)$ w.r.t x, we have

$$\frac{\partial}{\partial x}\psi(x)=Rt_0^R\left(1-\frac{(R-1)}{R}H_{(Y,R)}^{w}(f;t_0)\right)^{1-R}x^{R-1}-R\left(1-\frac{(R-1)}{R}H_{(Y,R)}^{w}(f;t_0)\right).$$

Also,

$$\frac{\partial^2}{\partial x^2}\psi(x)=R(R-1)t_0^R\left(1-\frac{(R-1)}{R}H_{(Y,R)}^{w}(f;t_0)\right)^{1-R}x^{R-2}\ .$$

Now, $\frac{\partial}{\partial X}\psi(x)=0$ gives

$$x = \left(\frac{R-(R-1)H^{w}_{(Y,R)}(f;t_0)}{Rt_0}\right)^{\frac{R}{R-1}} = x_0 \text{ (say).}$$

Case I. Let $R > 1$, then $\frac{\partial^2}{\partial x^2}\psi(x_0) > 0$. Thus, $\psi(x)$ attains minimum at x_0. Also, $\psi(0) < 0$ and $\psi(\infty) = \infty$. Further, we can also observe it that $\psi(x)$ first decreases for $0 < x < x_0$ and then increases for $x > x_0$. So, $x = h_F(t_0)$ is the unique solution to $\psi(x) = 0$.

Case II. Let $R < 1$, then $\frac{\partial^2}{\partial x^2}\psi(x_0) < 0$. Thus, $\psi(x)$ attains maximum at x_0. Also, $\psi(0) > 0$ and $\psi(\infty) = -\infty$. Further, we can easily see it that $\psi(x)$ first increases for $0 < x < x_0$ and then decreases for $x > x_0$. So, $x = h_F(t_0)$ is the unique solution to $\psi(x) = 0$. By combining both the cases, it is concluded that $H^{w}_{(Y,R)}(f;t_0)$ uniquely determines $h_F(t_0)$, which in turns determines $\bar{F}(t_0)$.

4. Properties and Inequalities of $H^{w}_{(Y,R)}(f;t_0)$

In this section, we study some interesting properties and inequalities of WRNRE.

Definition 4.1. A r.v Y is said to be smaller than in WRNRE of order R (denoted by$Y \overset{WRNRE}{\leq} Z$), if $H^{w}_{(Y,R)}(f;t_0) \leq H^{w}_{(Z,R)}(f;t_0), t_0 > 0$.

Definition 4.2. A r.v Y or a s.f $\bar{F}$ has increasing (decreasing) R-Norm entropy for residual life IWRNERL (DWRNERL), if $H^{w}_{(Y,R)}(f;t_0)$ is increasing (decreasing) in $t_0, t_0 > 0$.

Lemma 4.1. If $Z = \lambda Y$, with $\lambda > 0$ is a constant, then

$$H^{w}_{(Z,R)}(f;t_0) = \frac{R}{R--1}\left(1 - \lambda^{\frac{1}{R}}\right) + \lambda^{\frac{1}{R}} H^{w}_{(Y,R)}\left(f;\frac{t_0}{\lambda}\right).$$

Proof.

$$H^{w}_{(Z,R)}(f;t_0) = \frac{R}{R-1}\left[1 - \left\{\int_{t_0}^{\infty}\left(z\frac{f(z)}{Pr(Z>t_0)}\right)^{R} dz\right\}^{\frac{1}{R}}\right],$$

where, $f(z)$ is the p.d.f of Z.

Setting $Z = \lambda Y$, we obtain

$$H^{w}_{(Z,R)}(f;t_0) = \frac{R}{R-1}\left[1 - \left\{\int_{\frac{t_0}{\lambda}}^{\infty}\lambda\left(y\frac{f(y)}{\bar{F}\left(\frac{t_0}{\lambda}\right)}\right)^{R} dy\right\}^{\frac{1}{R}}\right].$$

By using (8), we obtain the required result.

Theorem 4.1. For two r.v's Y and Z, let us define $X_1 = \alpha_1 Y$ and $X_2 = \alpha_2 Z$, with $\alpha_1, \alpha_2 > 0$. Let $Y \overset{WRNRE}{\leq} Z$ and $\alpha_1 \leq \alpha_2$. Then $X_1 \overset{WRNRE}{\leq} X_2$, if $H^{w}_{(Y,R)}(f;t_0)$ or $H^{w}_{(Z,R)}(f;t_0)$ is decreasing in $t_0 > 0$.

Poof. Suppose $H^{w}_{(Y,R)}(f;t_0)$ is decreasing in t_0.

Now, $Y \overset{WRNRE}{\leq} Z$ implies

$$H^{w}_{(Y,R)}\left(f;\frac{t_0}{\alpha_2}\right) \leq H^{w}_{(Z,R)}\left(f;\frac{t_0}{\alpha_2}\right). \tag{15}$$

Further, since $\frac{t_0}{\alpha_1} \geq \frac{t_0}{\alpha_2}$, we have

$$H^w_{(Y,R)}\left(f; \frac{t_0}{\alpha_1}\right) \leq H^w_{(Y,R)}\left(f; \frac{t_0}{\alpha_2}\right). \tag{16}$$

From (15) and (16), we get

$$H^w_{(Y,R)}\left(f; \frac{t_0}{\alpha_1}\right) \leq H^w_{(Z,R)}\left(f; \frac{t_0}{\alpha_2}\right). \tag{17}$$

Using Lemma 4.1 in (17), we obtain $X_1 \overset{WRNRE}{\leq} X_2$.

Theorem 4.2. Let Y be a r.v with support $(0,m], m > 0$, p.d.f $f(y)$ and s.f $\bar{F}(t_0), t_0 > 0$, then for $R > 0 (\neq 1)$, the following inequality holds

$$H^w_{(Y,R)}(f; t_0) \geq \frac{R}{R-1}\left[1 - exp\left\{\frac{\int_{t_0}^{m}\left(y\frac{f(y)}{\bar{F}(t_0)}\right)^R \log\left(y\frac{f(y)}{\bar{F}(t_0)}\right)^R dy}{R\int_{t_0}^{m}\left(y\frac{f(y)}{\bar{F}(t_0)}\right)^R dy} + \log(m - t_0)\right\}\right].$$

Proof. Using log-sum inequality and (8), we have

$$\int_{t_0}^{m}\left(y\frac{f(y)}{\bar{F}(t_0)}\right)^R \log\left(y\frac{f(y)}{\bar{F}(t_0)}\right)^R dy \geq \int_{t_0}^{m}\left(y\frac{f(y)}{\bar{F}(t_0)}\right)^R dy \log\frac{\int_{t_0}^{m}(yf(y))^R dy}{\int_{t_0}^{m}(\bar{F}(t_0))^R dy}$$

$$= \int_{t_0}^{m}\left(y\frac{f(y)}{\bar{F}(t_0)}\right)^R dy\left[\log\left\{1 - \frac{(R-1)}{R}H^w_{(Y,R)}(f; t_0)\right\}^R - \log(m - t_0)\right].$$

After simple calculations, we can easily obtain the required result.

Theorem 4.3. If Y is IWRNERL (DWRNERL) and $R > 0(\neq 1)$, then

$$h_F(t_0) \leq (\geq)\left[\frac{R}{t_0^R}\left\{1 - \frac{(R-1)}{R}H^w_{(Y,R)}(f; t_0)\right\}^R\right]^{\frac{1}{R-1}}.$$

Proof. From (14), we have

$$(R-1)\frac{\partial}{\partial t_0}H^w_{(Y,R)}(f; t_0) = t_0^R\left(1 - \frac{(R-1)}{R}H^w_{(Y,R)}(f; t_0)\right)^{1-R} h_F^R(t_0) - R\left(1 - \frac{(R-1)}{R}H^w_{(Y,R)}(f; t_0)\right)h_F(t_0).$$

Since Y is IWRNERL (DWRNERL), therefore

$$h_F(t_0)\left[t_0^R\left\{1 - \frac{(R-1)}{R}H^w_{(Y,R)}(f; t_0)\right\}^{1-R} h_F^{R-1}(t_0) - R\left\{1 - \frac{(R-1)}{R}H^w_{(Y,R)}(f; t_0)\right\}\right] \geq (\leq) 0.$$

which leads to

$$h_F(t_0) \leq (\geq)\left[\frac{R}{t_0^R}\left\{1 - \frac{(R-1)}{R}H^w_{(Y,R)}(f; t_0)\right\}^R\right]^{\frac{1}{R-1}}.$$

Theorem 4.4. If $\bar{F}$ is IWRNERL (DWRNERL), then

$$H^w_{(Y,R)}(f; t_0) \geq (\leq)\frac{R}{R-1}\left[1 - t_0\left\{\frac{\left(1 + \frac{\partial}{\partial t_0}\delta_F(t_0)\right)^{R-1}}{\delta_F(t_0)}\right\}^{\frac{1}{R}}\right],$$

where $\delta_F(t_0)$ is the mean residual life function of Y.

Proof. From (14), we have

$$\frac{\partial}{\partial t_0} H^w_{(Y,R)}(f;t_0) = \frac{1}{R-1}\left[t_0^R\left\{1-\frac{(R-1)}{R}H^w_{(Y,R)}(f;t_0)\right\}^{1-R} h_F^R(t_0) - R\left\{1-\frac{(R-1)}{R}H^w_{(Y,R)}(f;t_0)\right\} h_F^R(t_0)\right].$$

Since, $\bar{F}$ is IWRNERL and $R>0(\neq 1)$, therefore, we have

$$t_0^R\left\{1-\frac{(R-1)}{R}H^w_{(Y,R)}(f;t_0)\right\}^{1-R} h_F^R(t_0) - R\left\{1-\frac{(R-1)}{R}H^w_{(Y,R)}(f;t_0)\right\} h_F(t_0).$$

which gives

$$H^w_{(Y,R)}(f;t_0) \geq \frac{R}{(R-1)}\left[1-t_0\left(\frac{h_F^{R-1}(t_0)}{R}\right)^{\frac{1}{R}}\right].$$

Using $h_F(t_0) = \frac{1+\frac{\partial}{\partial t_0}\delta_F(t_0)}{\delta_F(t_0)}$, we get

$$H^w_{(Y,R)}(f;t_0) \geq \frac{R}{R-1}\left[1-t_0\left\{\frac{1}{R}\left(\frac{1+\frac{\partial}{\partial t_0}\delta_F(t_0)}{\frac{\partial}{\partial t_0}\delta_F(t_0)}\right)^{R-1}\right\}^{\frac{1}{R}}\right].$$

The proof of DWRNERL is similar.

Theorem 4.5. Let Y be the lifetime of a system with p.d.f $f(y)$ and s.f $\bar{F}(t_o), t_0>0$, then for $R>0(\neq 1)$, we have

$$H^w_{(Y,R)}(f;t_0) \leq \frac{R}{R-1}\left[1-exp\left\{R\int_{t_0}^{\infty}\frac{f(y)}{\bar{F}(t_0)}\log y dy + (1-R)H_Y(f;t_0)\right\}\right]. \quad (18)$$

Proof. From log-sum inequality, we have

$$\int_{t_0}^{\infty} f(y)\log\frac{f(y)}{\left(y\frac{f(y)}{\bar{F}(t_0)}\right)^R}dy \geq \int_{t_0}^{\infty} f(y)dy\log\frac{\int_{t_0}^{\infty}f(y)dy}{\int_{t_0}^{\infty}\left(y\frac{f(y)}{\bar{F}(t_0)}\right)^R dy}$$

$$= \bar{F}(t_0)\left[\log\bar{F}(t_0) - R\log\left\{1-\frac{(R-1)}{R}H^w_{(Y,R)}(f;t_0)\right\}\right]. \quad (19)$$

where (19) is obtained from (8).

The L.H.S of (19) leads to

$$(1-R)\int_{t_0}^{\infty} f(y)\log f(y)dy - R\int_{t_0}^{\infty} f(y)\log y dy + R\bar{F}(t_0)\log\bar{F}(t_0). \quad (20)$$

Using (20) in (19), we obtain (18).

5. Application

In this section, we demonstrate a real life data set to analyze the performance of WRNE and WRNRE in practice. The data set represents the remission times (in months) of a random sample of 128 bladder cancer patients given in Lee and Wang [20] and is given as follows:

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

According to Afaq et al. [21] this data set is best fitted by length biased Lomax distribution (LBLD). So, for computing the uncertainty of this data set, the simple entropy techniques are not appropriate. Therefore, it is necessary to apply the weighted entropy techniques rather than the simple entropy. For the weighted entropy, here we must consider the parameters of the Lomax distribution (LD) not of the LBLD. The MLE's of the parameters of LD having p.d.f $f(y) = \frac{\alpha}{\beta}\left(1+\frac{y}{\beta}\right)^{-(\alpha+1)}, y > 0, \alpha, \beta > 0$ from this data set are obtained as: $\alpha = 8.43$ (shape parameter) and $\beta = 70.29$ (scale parameter) respectively. Now, for $\alpha = 8.43, \beta = 70.29, R = 2$ and $t_0 = 5$, the values of WRNE and WRNRE are obtained as: $H^{w}_{(Y,R)}(f) = 1.028$ and $H^{w}_{(Y,R)}(f;t_0) = 0.585$. Similarly, for the same values of α and β, if we take $R = 4$ and $t_0 = 10$, we can obtain $H^{w}_{(Y,R)}(f) = 1.111$ and $H^{w}_{(Y,R)}(f;t_0) = 1.015$ respectively.

6. Conclusion

In this article, we considered weighted R-Norm entropy of order R and also its dynamic (residual) version. These are shift-dependent uncertainty measures which assign the higher weight to the larger values of the observed random variable. We have also studied the various significant properties of these measures. Some of the important relationships of the proposed dynamic measure with hazard rate and mean residual life functions have been discussed. Finally, we have illustrated the importance of the proposed measures with the help of a real life data set.

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