BAYESIAN AND E-BAYESIAN ESTIMATION OF EXPONENTIATED INVERSE RAYLEIGH DISTRIBUTION USING CONJUGATE PRIOR

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Abstract

This study explores the application of Bayesian and E-Bayesian techniques to estimate the scale parameter of the Exponentiated Inverse Rayleigh distribution. Bayesian estimates for the parameter are derived using an informative Gamma prior and evaluated under three distinct loss functions: De-Groot, Squared Error, and Al-Bayyati loss functions. Various Properties of the E-Bayesian estimators under different loss functions have also been studied. To compare the effectiveness of E-Bayesian estimates against the Bayesian counterpart, a simulation study is conducted using MatLab. The various derived estimators were compared in terms of their Mean Squared Error. The results of a simulation study reveal that E-Bayesian estimates exhibit a smaller Mean Squared Error in comparison to Bayesian estimates, thereby demonstrating their enhanced efficiency. Among the E-Bayesian estimates, the third one stands out as the most effective. Moreover, the analysis highlights that the Squared Error loss function outperforms the Al-Bayyati and De-Groot loss functions, exhibiting a smaller MSE. Furthermore, the efficacy of these estimators is demonstrated through an analysis of a real-life dataset.

Keywords: Al-Bayyati loss function, De-Groot loss function, Exponentiated inverse Rayleigh distribution, Gamma prior, Squared error loss function.

1. Introduction

The Exponentiated Inverse Rayleigh distribution (EIRD) finds extensive utility in life testing and reliability studies, playing a crucial role in domains like electronic component longevity and wind speed analysis. Its significance also extends to physics and signal processing, facilitating investigations into radiations, sounds, and light phenomena. This versatility prompts statisticians to frequently employ the EIRD across diverse datasets.

Rehman and Dar [1] conducted a comprehensive examination of the Exponentiated Inverse Rayleigh distribution, delving into its mathematical properties and harnessing Bayesian estimation techniques for parameter estimation. The probability density function (PDF) and cumulative distribution function (CDF) of the EIRD, characterized by scale parameter θ and shape parameter α, are as follows:

$$f(x,\theta,\alpha)=\frac{2\alpha\theta e^{-\frac{\alpha\theta}{x^2}}}{x^3} \quad ; x>0, \alpha\,,>0 \tag{1}$$

$$F(x,\theta,\alpha)=e^{-\frac{\alpha\theta}{x^2}} \quad ; x>0, \alpha\,,\theta>0 \tag{2}$$

Numerous authors have explored the Inverse Rayleigh distribution (IRD) from various angles. Voda [2] delved into essential properties such as Maximum Likelihood Estimation (MLE), confidence intervals, and hypothesis tests. Siddiqui [3] focused on the diverse practical applications of the Inverse Rayleigh Distribution. Soliman et al. [4] utilized squared error and zero-one loss functions to devise Bayesian estimators for IRD, centered around lower record values. Reshi et al. [5] tackled parameter estimation for the Generalized Inverse Rayleigh distribution.

Dey [6] derived Bayes estimators for IRD parameters using distinct loss functions and a non-informative prior. Sindhua et al. [7] explored Bayesian estimators and associated risks for IRD parameters, emphasizing left-censored data and showcasing the efficacy of the gamma prior under Quasi-Quadratic loss functions. Okasha [8] explored E-Bayesian estimation for the Lomax distribution with type-II censored data.

This paper's objective is to conduct a statistical comparison between Bayesian estimators and Expected Bayesian estimators for the Exponentiated Inverse Rayleigh distribution's scale parameter. The analysis involves the utilization of gamma priors and different loss functions. The ensuing layout of the paper is outlined as follows: Section 2 outlines the derivation of the likelihood function, prior distribution, and posterior distribution. Section 3 presents Bayesian estimators for the EIRD scale parameter using Al-Bayyati, Squared Error, and De-Groot loss functions. In Section 4, E-Bayesian estimates are derived and their properties are examined. Section 5 is dedicated to a simulation study comparing Bayes and E-Bayes estimates. Real data analysis is tackled in Section 6, while Section 7 concludes by summarizing the findings.

2. Likelihood function, Prior and Posterior Distribution

2.1 Likelihood function

Let $\underline{x} = x_1, x_2, \dots, x_n$ be a random sample of size n drawn from EIRD. Then the likelihood function is given by

$$L(\underline{x}, \theta) = 2^n \alpha^n \theta^n \prod_{i=1}^{n} \frac{1}{x_i^3} e^{-\alpha\theta \sum_{i=1}^{n} x_i^{-2}} \tag{3}$$

In the context of Bayesian estimation, the selection of an appropriate prior holds paramount importance in parameter estimation. When a substantial understanding of the parameter(s) is available, the inclination is towards informative priors; however, when such knowledge is lacking, non-informative priors may be more appropriate. In this study, we opt for an informative prior, specifically the Gamma Prior, to derive the corresponding posterior distribution.

2.2 Prior distribution

The gamma distribution is employed as a conjugate prior distribution for the parameter θ. The subsequent Probability Density Function (PDF) is formulated using the shape parameter 'c' and the scale parameter 'r'.

$$h(\theta|c, r) = \frac{r^c \, \theta^{c-1} e^{-\theta r}}{\Gamma c} \qquad ; \; c, r > 0\,, \theta > 0 \tag{4}$$

2.3 Posterior distribution

The posterior distribution for the parameter θ using (3) and (4), is given as

$$g(\underline{x}, \theta) = \frac{L(\underline{x}, \theta) * h(\theta)}{\int_0^\infty L(\underline{x}, \theta) * h(\theta) d\theta}$$

$$= \frac{2^n \alpha^n \theta^n \prod_{i=1}^{n} \frac{1}{x_i^3} e^{-\alpha\theta \sum_{i=1}^{n} x_i^{-2}} * \frac{r^c \theta^{c-1} e^{-\theta r}}{\Gamma c}}{\int_0^\infty 2^n \alpha^n \theta^n \prod_{i=1}^{n} \frac{1}{x_i^3} e^{-\alpha\theta \sum_{i=1}^{n} x_i^{-2}} * \frac{r^c \theta^{c-1} e^{-\theta r}}{\Gamma c} d\theta}$$

As a result, the posterior distribution of is equal to

$$= \frac{\theta^{(n+c)-1} e^{-\theta\{\alpha \sum_{i=1}^{n} x_i^{-2} + r\}}}{\int_0^\infty \theta^{(n+c)-1} e^{-\theta\{\alpha \sum_{i=1}^{n} x_i^{-2} + r\}} d\theta}$$

$$g(\underline{x}, \theta) = \frac{S^{n+c} \theta^{(n+c)-1} e^{-S\theta}}{\Gamma(n+c)}, \quad \theta > 0 \quad where\ S = \left\{\alpha \sum_{i=1}^{n} x_i^{-2} + r\right\} \tag{5}$$

3. Bayesian Estimation

In this section, we find the Bayes estimate of scale parameter of EIRD under three different loss functions as:

3.1 Under the Al-Bayyati loss function

Al-Bayyati [9] proposed a loss function, defined as
$L(\hat{\theta}, \theta) = \theta^d (\hat{\theta} - \theta)^2$; where $\hat{\theta}$ is the estimate of θ.
By using the Al-Bayyati loss function, the bayes estimator is given as

$$E\{L(\hat{\theta}, \theta)\} = \int_0^\infty L(\hat{\theta}, \theta) * g(\underline{x}, \theta) d\theta$$

$$= \int_0^\infty \theta^d (\hat{\theta} - \theta)^2 * \frac{S^{n+c} \theta^{(n+c)-1} e^{-S\theta}}{\Gamma(n+c)} d\theta$$

$$= \int_0^\infty \theta^d (\hat{\theta}^2 + \theta^2 - 2\hat{\theta}\theta)^2 * \frac{S^{n+c} \theta^{(n+c)-1} e^{-S\theta}}{\Gamma(n+c)} d\theta$$

$$= \hat{\theta}^2 \int_0^\infty \frac{S^{n+c} \theta^{(n+c+d)-1} e^{-S\theta}}{\Gamma(n+c)} d\theta + \int_0^\infty \frac{S^{n+c} \theta^{(n+c+d+2)-1} e^{-S\theta}}{\Gamma(n+c)} d\theta - 2\hat{\theta} \int_0^\infty \frac{S^{n+c} \hat{\theta}^{(n+c+d+1)-1} e^{-S\theta}}{\Gamma(n+c)} d\theta$$

By solving the above integral, finally we get

$$E\{L(\hat{\theta}, \theta)\} = \hat{\theta}^2 * \frac{\Gamma(n+c+d)}{S^d \Gamma(n+c)} + \frac{\Gamma(n+c+d+2)}{S^{d+2} \Gamma(n+c)} - 2\hat{\theta} * \frac{\Gamma(n+c+d+1)}{S^{d+1} \Gamma(n+c)}$$

And consequently, the Bayes estimator is

$$\hat{\theta}_{BA} = \frac{(n+c+d)}{S} \tag{6}$$

3.2 Under the Squared error loss function

The Squared Error Loss Function [10] is defined as follows:
$L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$
By using the Squared error loss function, the bayes estimator is given as

$$E\{L(\hat{\theta}, \theta)\} = \int_0^\infty L(\hat{\theta}, \theta) * g(\underline{x}, \theta) d\theta$$

$$= \int_0^\infty (\hat{\theta} - \theta)^2 * \frac{S^{n+c} \theta^{(n+c)-1} e^{-S\theta}}{\Gamma(n+c)} d\theta$$

$$= \int_0^\infty \left(\hat{\theta}^2 + \theta^2 - 2\hat{\theta}\theta\right)^2 * \frac{S^{n+c}\theta^{(n+c)-1}e^{-S\theta}}{\Gamma(n+c)}\, d\theta$$

By solving the above integral, finally we get

$$E\{L(\hat{\theta}, \theta)\} = \hat{\theta}^2 + \frac{\Gamma(n+c+2)}{S^2\Gamma(n+c)} - 2\hat{\theta}\frac{(n+c)}{S}$$

And consequently, the Bayes estimator as

$$\hat{\theta}_{BS} = \frac{(n+c)}{S} \tag{7}$$

3. 3 Under the De-Groot loss function

The De-Groot loss function [11] is defined as follows:

$$L(\hat{\theta}, \theta) = \frac{(\hat{\theta} - \theta)^2}{\hat{\theta}^2}$$

By using the De-Groot loss function, the bayes estimator is given as

$$E\{L(\hat{\theta}, \theta)\} = \int_0^\infty L(\hat{\theta}, \theta) * g(\underline{x}, \theta) d\theta$$

$$= \int_0^\infty \frac{(\hat{\theta} - \theta)^2}{\hat{\theta}^2} * \frac{S^{n+c}\theta^{(n+c)-1}e^{-S\theta}}{\Gamma(n+c)}\, d\theta$$

$$= \frac{1}{\hat{\theta}^2}\int_0^\infty (\hat{\theta}^2 + \theta^2 - 2\hat{\theta}\theta)^2 * \frac{S^{n+c}\theta^{(n+c)-1}e^{-S\theta}}{\Gamma(n+c)}\, d\theta$$

By solving the above integral, finally we get

$$E\{L(\hat{\theta}, \theta)\} = 1 + \frac{1}{\hat{\theta}^2}\frac{\Gamma(n+c+2)}{S^2\Gamma(n+c)} - \frac{2}{\hat{\theta}}\frac{\Gamma(n+c+1)}{S\Gamma(n+c)}$$

And consequently, the Bayes estimator as

$$\hat{\theta}_{BD} = \frac{(n+c+1)}{S}. \tag{8}$$

4. E-Bayesian Estimation

According to Han [12], the prior parameters 'c' and 'r' should be chosen so that the prior given in (4) is a decreasing function of θ.

$$\frac{d}{d\theta}h(\theta|c,r) = \frac{r^c}{\Gamma c}\theta^{c-2}e^{-\theta r}\{(c-1) - r\theta\},$$

As a result, our prior distribution (4) becomes a decreasing function of θ, for 0<c<1 and r>0.
The E-Bayesian estimate of is calculated as follows:

$$\hat{\theta}_{EB} = \int_0^1\int_0^t \hat{\theta}_{BE} * \pi(\theta, c, r) * \mathrm{drdc}$$

The intervals of integration for the first and second integrals correspond to the domains of the hyperparameters 'c' and 'r,' respectively, ensuring that our prior density function exhibits a decreasing trend with respect to θ. The Bayesian estimate of θ, denoted as $\hat{\theta}_{EB}$, is calculated utilizing three distinct loss functions.

Subsequently, for the E-Bayesian estimates of θ, we deliberate on the choice of prior distributions for the hyperparameters 'c' and 'r.' These distributions serve primarily to explore the influence of different prior choices on the E-Bayesian estimations of θ. The hyperparameters 'c' and 'r' are governed by the following distributions

$$\pi_1(\theta, \mathrm{c}, \mathrm{r}) = \frac{2(\mathrm{t} - \mathrm{r})}{\mathrm{t}^2} \;;\quad 0 < c < 1;\quad 0 < r < t \tag{9}$$

$$\pi_2(\theta, \mathrm{c}, \mathrm{r}) = \frac{1}{\mathrm{t}} \qquad ; \quad 0 < c < 1; \quad 0 < r < t \tag{10}$$

$$\pi_3(\theta, \mathrm{c}, \mathrm{r}) = \frac{2\mathrm{r}}{\mathrm{t}^2} \qquad ; \quad 0 < c < 1; \quad 0 < r < t \tag{11}$$

Now follows the E- Bayesian estimates of the scale parameter of EIRD under proposed loss functions

4.1 E-Bayesian estimation of θ under the Al-Bayyati loss function

E-Bayesian estimate of the parameter θ based on $\pi_1(\theta, \mathrm{c}, \mathrm{r})$, is provided by

$$\hat{\theta}_{EBA_1} = \int_0^1 \int_0^t \hat{\theta}_{BA} * \pi_1(\theta, \mathrm{c}, \mathrm{r}) \mathrm{drdc}$$

$$= \int_0^1 \int_0^{\mathrm{t}} \left\{\frac{n + c + d}{S}\right\} * \left\{\frac{2(\mathrm{t} - \mathrm{r})}{\mathrm{t}^2}\right\} * \mathrm{drdc}$$

$$= \frac{2}{\mathrm{t}^2} \left\{ \int_0^1 (n + c + d) dc * \int_0^{\mathrm{t}} \left(\frac{\mathrm{t} - \mathrm{r}}{S}\right) * \mathrm{dr} \right\}$$

On solving the above equation, we get

$$\hat{\theta}_{EBA_1} = \frac{2n + 2d + 1}{\mathrm{t}^2} * \left\{ (\mathrm{t} + \mathrm{P}) * \log\left(\frac{\mathrm{t} + \mathrm{P}}{\mathrm{P}}\right) - t \right\} \tag{12}$$

E-Bayesian estimate of the parameter θ based on $\pi_2(\theta, \mathrm{c}, \mathrm{r})$, is provided by

$$\hat{\theta}_{EBA_2} = \int_0^1 \int_0^t \hat{\theta}_{BA} * \pi_2(\theta, \mathrm{c}, \mathrm{r}) \mathrm{drdc}$$

$$= \int_0^1 \int_0^{\mathrm{t}} \left\{\frac{n + c + d}{S}\right\} * \frac{1}{\mathrm{t}} * \mathrm{drdc}$$

where $P = (\alpha \sum_{i=1}^{n} x_i^{-2})$ and $\int_0^1 (n + c + d) dc = \frac{2n+2d+1}{2}$

Hence, on solving we get

$$\hat{\theta}_{EBA_2} = \frac{2n + 2d + 1}{2\mathrm{t}} * \left\{ \log\left(\frac{\mathrm{t} + \mathrm{P}}{\mathrm{P}}\right) \right\} \tag{13}$$

E-Bayesian estimate of the parameter θ based on $\pi_3(\theta, \mathrm{c}, \mathrm{r})$, and is given by

$$\hat{\theta}_{EBA_3} = \int_0^1 \int_0^t \hat{\theta}_{BA} * \pi_3(\theta, \mathrm{c}, \mathrm{r}) \mathrm{drdc}$$

$$= \int_0^1 \int_0^{\mathrm{t}} \left\{\frac{n + c + d}{S}\right\} * \frac{2\mathrm{r}}{\mathrm{t}^2} * \mathrm{drdc}$$

Hence on solving, we get

$$\hat{\theta}_{EBA_3} = \frac{(2n + 2d + 1)}{t^2} * \left\{ \mathrm{P} * \log\left(\frac{\mathrm{P}}{\mathrm{t} + \mathrm{P}}\right) + \mathrm{t} \right\} \tag{14}$$

4.2 E-Bayesian estimation of θ under the Squared error loss function

E-Bayesian estimate of the parameter θ under based on $\pi_1(\theta, \mathrm{c}, \mathrm{r})$, and is provided by

$$\theta_{EBS_1} = \int_0^1 \int_0^t \hat{\theta}_{BS} * \pi_1(\theta, \mathrm{c}, \mathrm{r}) \mathrm{drdc}$$

$$= \int_0^1 \int_0^t \left(\frac{n + c}{S}\right) * \left\{\frac{2(\mathrm{t} - \mathrm{r})}{\mathrm{t}^2}\right\} * \mathrm{drdc}$$

$$= \frac{2}{\mathrm{t}^2} \left\{ \int_0^1 (n + c) dc * \int_0^{\mathrm{t}} \left(\frac{\mathrm{t} - \mathrm{r}}{P + r}\right) * \mathrm{dr} \right\}$$

Where $S = \left\{\alpha \sum_{i=1}^{n} x_i^{-2} + r\right\}$ *and* $P = \alpha \sum_{i=1}^{n} x_i^{-2}$

On solving the above integrals, finally we get

$$\hat{\theta}_{EBS_1} = \frac{2n+1}{\mathrm{t}^2} * \left\{(\mathrm{t}+\mathrm{P}) * \log\left(\frac{\mathrm{t}+\mathrm{P}}{\mathrm{P}}\right) - t\right\} \tag{15}$$

E-Bayesian estimate of the parameter θ based on $\pi_2(\theta, \mathrm{c}, \mathrm{r})$, and is provided by

$$\hat{\theta}_{EBS_2} = \int_0^1 \int_0^t \hat{\theta}_{BS} * \pi_2(\theta, \mathrm{c}, \mathrm{r}) \mathrm{drdc}$$

$$= \int_0^1 \int_0^{\mathrm{t}} \left\{\frac{n+c}{S}\right\} * \frac{1}{\mathrm{t}} * \mathrm{drdc}$$

$$= \frac{1}{\mathrm{t}} \left\{ \int_0^1 (n+c) dc * \int_0^{\mathrm{t}} \left(\frac{1}{S}\right) * \mathrm{dr} \right\}$$

where $S = \left\{\alpha \sum_{i=1}^{n} x_i^{-2} + r\right\}$ *and* $P = \alpha \sum_{i=1}^{n} x_i^{-2}$

On solving the above integrals, finally we get

$$\hat{\theta}_{EBS_2} = \frac{2n+1}{2\mathrm{t}} * \left\{\log\left(\frac{\mathrm{t}+\mathrm{P}}{\mathrm{P}}\right)\right\} \tag{16}$$

E-Bayesian estimate of the parameter θ based on $\pi_3(\theta, \mathrm{c}, \mathrm{r})$, and is provided by

$$\hat{\theta}_{EBS_3} = \int_0^1 \int_0^t \hat{\theta}_{BS} * \pi_3(\theta, \mathrm{c}, \mathrm{r}) \mathrm{drdc}$$

$$= \int_0^1 \int_0^{\mathrm{t}} \left\{\frac{n+c}{S}\right\} * \frac{2\mathrm{r}}{\mathrm{t}^2} * \mathrm{drdc}$$

on solving the above intervals, finally we get

$$\hat{\theta}_{EBS_3} = \left\{\left(\frac{2n+1}{\mathrm{t}^2}\right) * \left\{\mathrm{P} * \log\left(\frac{\mathrm{p}}{\mathrm{t}+\mathrm{P}}\right) + \mathrm{t}\right\}\right\} \tag{17}$$

4.3 E-Bayesian estimation of θ under the De-Groot loss function

E-Bayesian estimate of the parameter θ based on $\pi_1(\theta, \mathrm{c}, \mathrm{r})$, and is provided by

$$\hat{\theta}_{EBD_1} = \int_0^1 \int_0^t \hat{\theta}_{BD} * \pi_1(\theta, \mathrm{c}, \mathrm{r}) \mathrm{drdc}$$

$$\hat{\theta}_{EBD_1} = \int_0^1 \int_0^t \left(\frac{n+c+1}{S}\right) * \left\{\frac{2(\mathrm{t}-\mathrm{r})}{\mathrm{t}^2}\right\} * \mathrm{drdc}$$

$$= \frac{2}{\mathrm{t}^2} \left\{ \int_0^1 (n+c+1) dc * \int_0^{\mathrm{t}} \left(\frac{\mathrm{t}-\mathrm{r}}{P+r}\right) * \mathrm{dr} \right\}$$

Where $S = \left\{\alpha \sum_{i=1}^{n} x_i^{-2} + r\right\}$ *and* $P = \alpha \sum_{i=1}^{n} x_i^{-2}$

On solving the above integrals, we get

$$\hat{\theta}_{EBD_1} = \frac{(2n+3)}{\mathrm{t}^2} * \left\{(\mathrm{t}+\mathrm{P}) * \log\left(\frac{\mathrm{t}+\mathrm{P}}{\mathrm{P}}\right) - t\right\} \tag{18}$$

E-Bayesian estimate of the parameter θ based on $\pi_2(\theta, \mathrm{c}, \mathrm{r})$, and is provided by

$$\hat{\theta}_{EBD_2} = \int_0^1 \int_0^t \hat{\theta}_{BD} * \pi_2(\theta, \mathrm{c}, \mathrm{r}) \mathrm{drdc}$$

$$= \int_0^1 \int_0^{\mathrm{t}} \left\{\frac{n+c+1}{S}\right\} * \frac{1}{\mathrm{t}} * \mathrm{drdc}$$

$$= \frac{1}{\mathrm{t}} \left\{ \int_0^1 (n+c) dc * \int_0^{\mathrm{t}} \left(\frac{1}{S}\right) * \mathrm{dr} \right\}$$

$$= \frac{1}{\mathrm{t}} \left\{ \int_0^1 (n+c+1) dc * \int_0^{\mathrm{t}} \left(\frac{1}{P+r}\right) * \mathrm{dr} \right\}$$

Where $S=\left\{\alpha\sum_{i=1}^{n}x_i^{-2}+r\right\}$ and $P=\alpha\sum_{i=1}^{n}x_i^{-2}$

$$\hat{\theta}_{EBD_2}=\frac{(2n+3)}{2t}*\left\{\log\left(\frac{t+P}{P}\right)\right\} \tag{19}$$

E-Bayesian estimate of the parameter θ based on $\pi_3(\theta,c,r)$, and is provided by

$$\begin{aligned}\hat{\theta}_{EBD_3}&=\int_0^1\int_0^t\hat{\theta}_{BD}*\pi_3(\theta,c,r)drdc\\&=\int_0^1\int_0^t\left\{\frac{n+c+1}{S}\right\}*\frac{2r}{t^2}*drdc\\&=\frac{2}{t^2}\left\{\int_0^1(n+c+1)dc*\int_0^t\left(\frac{r}{P+r}\right)*dr\right\}\\&=\frac{2}{t^2}\left\{\frac{2n+3}{2}*\left\{t-P*\log\left(\frac{t+P}{P}\right)\right\}\right\}\end{aligned}$$

On solving the above intervals, finally we get

$$\hat{\theta}_{EBD_3}=\left\{\frac{(2n+3)}{t^2}*\left\{P*\log\left(\frac{p}{t+P}\right)+t\right\}\right\} \tag{20}$$

4.4 Properties of E-Bayesian estimates under Different Loss Functions

In this section, we will discuss the relationship amongst the different E-Bayesian estimators obtained under the Al-Bayyati loss function i.e, $\hat{\theta}_{EBA_1},\hat{\theta}_{EBA_2},\hat{\theta}_{EBA_3}$ $(i=1,2,3)$

Theorem 4.1 E-Bayesian estimators obtained under the Al-Bayyati loss function will follow the following results:

(i) $\hat{\theta}_{EBA_3}<\hat{\theta}_{EBA_2}<\hat{\theta}_{EBA_1}$

(ii) $\lim_{P\to\infty}\left(\hat{\theta}_{EBA_1}\right)=\lim_{P\to\infty}\left(\hat{\theta}_{EBA_2}\right)=\lim_{P\to\infty}\left(\hat{\theta}_{EBA_3}\right)$

Proof (i) From (12) and (13), we get

$$\begin{aligned}\hat{\theta}_{EBA_1}-\hat{\theta}_{EBA_2}&=\frac{(2n+2d+1)}{t^2}*\left\{(t+P)*\log\left(\frac{t+P}{P}\right)-t\right\}-\frac{2n+2d+1}{2t}*\left\{\log\left(\frac{t+P}{P}\right)\right\}\\&=\frac{(2n+2d+1)}{t}\left\{\log\left(1+\frac{t}{P}\right)\left(\frac{P}{t}+\frac{1}{2}\right)-1\right\}\end{aligned} \tag{21}$$

From (13) and (14), we get

$$\begin{aligned}\hat{\theta}_{EBA_2}-\hat{\theta}_{EBA_3}&=\frac{(2n+2d+1)}{2t}*\left\{\log\left(\frac{t+P}{P}\right)\right\}-\frac{(2n+2d+1)}{t^2}*\left\{t-P*\log\left(\frac{t+P}{P}\right)\right\}\\&=\frac{(2n+2d+1)}{t}\left\{\log\left(1+\frac{t}{P}\right)\left(\frac{P}{t}+\frac{1}{2}\right)-1\right\}\end{aligned} \tag{22}$$

Since $\log\left(1+\frac{t}{P}\right)=\left\{\frac{t}{P}-\frac{t^2}{2P^2}+\frac{t^3}{3P^3}-\cdots\ldots\right\}$

$$\hat{\theta}_{EBA_1}-\hat{\theta}_{EBA_2}=\frac{(2n+2d+1)}{t}\left\{\frac{t^2}{12P^2}+\frac{t^3}{6P^3}-\cdots\ldots\right\}$$

$\hat{\theta}_{EBA_1}-\hat{\theta}_{EBA_2}>0$

hence

$$\hat{\theta}_{EBA_1}>\hat{\theta}_{EBA_2} \tag{23}$$

Similarly,

$$\hat{\theta}_{EBA_2}-\hat{\theta}_{EBA_3}=\frac{(2n+2d+1)}{t}\left\{\frac{t^2}{12P^2}+\frac{t^3}{6P^3}-\cdots\ldots\right\}$$

$\hat{\theta}_{EBA_2}-\hat{\theta}_{EBA_3}>0$, and

hence

$$\hat{\theta}_{EBA_2}>\hat{\theta}_{EBA_3} \tag{24}$$

Combining (23) and (24), we get

$\hat{\theta}_{EBA_3}<\hat{\theta}_{EBA_2}<\hat{\theta}_{EBA_1}$

Proof (ii): From (21) and (22), we get

$$\hat{\theta}_{EBA_1}-\hat{\theta}_{EBA_2}=\hat{\theta}_{EBA_2}-\hat{\theta}_{EBA_3}\frac{(2n+2d+1)}{t}\left\{\log\left(1+\frac{t}{P}\right)\left(\frac{P}{t}+\frac{1}{2}\right)-1\right\}$$

After taking the limit, we get

$$\lim_{P\to\infty}\left(\hat{\theta}_{EBA_1}-\hat{\theta}_{EBA_2}\right)=\lim_{P\to\infty}\frac{(2n+2d+1)}{t}\left\{\frac{t^2}{12P^2}+\frac{t^3}{6P^3}\dots\dots\right\}$$
$$\lim_{P\to\infty}\left(\hat{\theta}_{EBA_2}-\ \hat{\theta}_{EBA_3}\right)=\lim_{P\to\infty}\frac{(2n+2d+1)}{t}\left\{\frac{t^2}{12P^2}+\frac{t^3}{6P^3}\dots\dots\right\}$$

On solving the above , we have

$$\lim_{P\to\infty}\left(\hat{\theta}_{EBA_1}-\hat{\theta}_{EBA_2}\right)=\lim_{P\to\infty}\left(\hat{\theta}_{EBA_2}-\ \hat{\theta}_{EBA_3}\right)=0$$

Hence

$$\lim_{P\to\infty}\left(\hat{\theta}_{EBA_1}\right)=\lim_{P\to\infty}\left(\hat{\theta}_{EBA_2}\right)=\lim_{P\to\infty}\left(\hat{\theta}_{EBA_3}\right)$$

Now we will discuss the relationship amongst the different E-Bayesian estimators obtained under the Square Error loss function i.e, $\hat{\theta}_{EBS_1}, \hat{\theta}_{EBS_2}, \hat{\theta}_{EBS_3}$ $(i=1,2,3)$

Theorem 4.2 E-Bayesian estimators obtained under the Squared Error loss function will follow the following results:

(i) $\hat{\theta}_{EBS_3}<\hat{\theta}_{EBS_2}<\hat{\theta}_{EBS_1}$

(ii) $\lim_{P\to\infty}\left(\hat{\theta}_{EBS_1}\right)=\lim_{P\to\infty}\left(\hat{\theta}_{EBS_2}\ \right)=\lim_{P\to\infty}\left(\hat{\theta}_{EBS_3}\ \right)$

Proof (i): From (15) and (16), we get

$$\hat{\theta}_{EBS_1}-\hat{\theta}_{EBS_2}=\frac{2n+1}{t^2}*\left\{(t+P)*\log\left(\frac{t+P}{P}\right)-t\right\}-\frac{2n+1}{2t}*\left\{\log\left(\frac{t+P}{P}\right)\right\}$$
$$\hat{\theta}_{EBS_1}-\hat{\theta}_{EBS_2}=\frac{(2n+1)}{t}*\left\{\log\left(\frac{t+P}{P}\right)\left(\frac{P}{t}+\frac{1}{2}\right)-1\right\}\tag{25}$$
$$\hat{\theta}_{EBS_1}-\hat{\theta}_{EBS_2}>0$$
$$\hat{\theta}_{EBS_1}>\hat{\theta}_{EBS_2}\tag{26}$$

Similarly, from (16) and (17), we get

$$\hat{\theta}_{EBS_2}-\hat{\theta}_{EBS_3}=\frac{(2n+1)}{2t}*\left\{\log\left(\frac{t+P}{P}\right)\right\}-\frac{(2n+1)}{t^2}*\left\{t-P*\log\left(\frac{t+P}{P}\right)\right\}$$
$$\hat{\theta}_{EBS_2}-\hat{\theta}_{EBS_3}=\frac{(2n+1)}{t}\left\{\log\left(1+\frac{t}{P}\right)\left(\frac{P}{t}+\frac{1}{2}\right)-1\right\}\tag{27}$$
$$\hat{\theta}_{EBS_2}-\hat{\theta}_{EBS_3}>0$$
$$\hat{\theta}_{EBS_2}>\hat{\theta}_{EBS_3}\tag{28}$$

Combining (26) and (28), we get

$$\hat{\theta}_{EBS_3}<\hat{\theta}_{EBS_2}<\hat{\theta}_{EBS_1}$$

Proof (ii): From (25) and (27), we get

$$\hat{\theta}_{EBS_1}-\hat{\theta}_{EBS_2}=\frac{(2n+1)}{t}\left\{\log\left(1+\frac{t}{P}\right)\left(\frac{P}{t}+\frac{1}{2}\right)-1\right\}=\hat{\theta}_{EBS_2}-\hat{\theta}_{EBS_3}$$

After taking the limit, we get

$$\lim_{P\to\infty}\left(\hat{\theta}_{EBS_1}-\hat{\theta}_{EBS_2}\right)=\lim_{P\to\infty}\frac{(2n+1)}{t}\left\{\frac{t^2}{12P^2}+\frac{t^3}{6P^3}-\cdots\dots\right\}=0$$

and

$$\lim_{P\to\infty}\left(\hat{\theta}_{EBS_2}-\ \hat{\theta}_{EBS_3}\right)=\lim_{P\to\infty}\frac{(2n+1)}{t}\left\{\frac{t^2}{12P^2}+\frac{t^3}{6P^3}-\cdots\dots\right\}=0$$

On solving, we have

$$\lim_{P\to\infty}\left(\hat{\theta}_{EBS_1}-\hat{\theta}_{EBS_2}\right)=\lim_{P\to\infty}\left(\hat{\theta}_{EBS_2}-\ \hat{\theta}_{EBS_3}\right)=0$$

hence

$$\lim_{P\to\infty}\left(\hat{\theta}_{EBS_1}\right)=\lim_{P\to\infty}\left(\hat{\theta}_{EBS_2}\right)=\lim_{P\to\infty}\left(\hat{\theta}_{EBS_1}\right)$$

Now we will discuss the relationship amongst the different E-Bayesian estimators obtained under the De-Groot loss function i.e, $\hat{\theta}_{EBS_1}, \hat{\theta}_{EBS_2}, \hat{\theta}_{EBS_3}$ $(i=1,2,3)$

Theorem 4.3 E-Bayesian estimators obtained under the Square Error loss function will follow the following results:

(i)$\hat{\theta}_{EBD_3}<\hat{\theta}_{EBD_2}<\hat{\theta}_{EBD_1}$

(ii) $\lim_{P\to\infty}\left(\hat{\theta}_{EBD_1}\right)=\lim_{P\to\infty}\left(\hat{\theta}_{EBD_2}\ \right)=\lim_{P\to\infty}\left(\hat{\theta}_{EBD_3}\ \right)$

Proof (i): From (18) and (19), we get

$$\hat{\theta}_{EBD_1} - \hat{\theta}_{EBD_2} = \frac{2n+3}{t^2} * \left\{(t+P) * \log\left(\frac{t+P}{P}\right) - t\right\} - \frac{2n+3}{2t} * \left\{\log\left(\frac{t+P}{P}\right)\right\}$$

$$\hat{\theta}_{EBD_1} - \hat{\theta}_{EBD_2} = \frac{(2n+3)}{t} * \left\{\log\left(\frac{t+P}{P}\right)\left(\frac{P}{t} + \frac{1}{2}\right) - 1\right\} \quad (29)$$

$$\hat{\theta}_{EBD_1} - \hat{\theta}_{EBD_2} > 0$$

$$\hat{\theta}_{EBD_1} > \hat{\theta}_{EBD_2} \quad (30)$$

Similarly, from eq. (19) and (20), we get

$$\hat{\theta}_{EBD_2} - \hat{\theta}_{EBD_3} = \frac{(2n+3)}{2t} * \left\{\log\left(\frac{t+P}{P}\right)\right\} - \frac{(2n+3)}{t^2} * \left\{t - P * \log\left(\frac{t+P}{P}\right)\right\}$$

$$\hat{\theta}_{EBD_2} - \hat{\theta}_{EBD_3} = \frac{(2n+3)}{t}\left\{\log\left(1 + \frac{t}{P}\right)\left(\frac{P}{t} + \frac{1}{2}\right) - 1\right\} \quad (31)$$

$$\hat{\theta}_{EBD_2} - \hat{\theta}_{EBD_3} > 0$$

$$\hat{\theta}_{EBD_2} > \hat{\theta}_{EBD_3} \quad (32)$$

Combining (30) and (32), we get

$$\hat{\theta}_{EBD_3} < \hat{\theta}_{EBD_2} < \hat{\theta}_{EBD_1}$$

Proof (ii): From (29) and (31), we get

$$\hat{\theta}_{EBD_1} - \hat{\theta}_{EBD_2} = \frac{(2n+3)}{t}\left\{\log\left(1 + \frac{t}{P}\right)\left(\frac{P}{t} + \frac{1}{2}\right) - 1\right\} = \hat{\theta}_{EBD_2} - \hat{\theta}_{EBD_3}$$

After taking the limit , we get

$$\lim_{P\to\infty}\left(\hat{\theta}_{EBD_1} - \hat{\theta}_{EBD_2}\right) = \lim_{P\to\infty} \frac{(2n+3)}{t}\left\{\frac{t^2}{12P^2} + \frac{t^3}{6P^3} - \cdots \ldots\right\} = 0$$

And

$$\lim_{P\to\infty}\left(\hat{\theta}_{EBD_2} - \hat{\theta}_{EBD_3}\right) = \lim_{P\to\infty} \frac{(2n+3)}{t}\left\{\frac{t^2}{12P^2} + \frac{t^3}{6P^3} - \cdots \ldots\right\} = 0$$

On solving, we have

$$\lim_{P\to\infty}\left(\hat{\theta}_{EBD_1} - \hat{\theta}_{EBD_2}\right) = \lim_{P\to\infty}\left(\hat{\theta}_{EBD_2} - \hat{\theta}_{EBD_3}\right) = 0$$

$$\lim_{P\to\infty}\left(\hat{\theta}_{EBD_1}\right) = \lim_{P\to\infty}\left(\hat{\theta}_{EBD_2}\right) = \lim_{P\to\infty}\left(\hat{\theta}_{EBD_3}\right)$$

Hence the proof is complete.

The part (a) of theorem 4.(i), 4.(ii) and 4.(iii) shows that with different priors (9)-(11) of the parameters c and r, the associated E-Bayesian estimate $\hat{\theta}_{EBA_i}, \hat{\theta}_{EBD_i}, and\ \hat{\theta}_{EBS_i}$; (i=1,2,3) are different. The property (b) of the theorems shows that $\hat{\theta}_{EBA_i}, c$; (i=1,2,3) are asymptotically equivalent to each other as $\sum_{i=1}^{n} x_i^{-2} \to \infty$, that means $\hat{\theta}_{EBA_i}$; (i=1,2,3) are all close to each other when $\sum_{i=1}^{n} x_i^{-2}$ is sufficiently large and $\hat{\theta}_{EBD_i}, and\ \hat{\theta}_{EBS_i}$; $(i = 1,2,3)$ are also close to each other.

5. Simulation Study

In order to compare the performance of Bayesian and E-Bayesian techniques of estimation, a simulation study was conducted using MatLab. We chose a sample of size of n=20, 50, 70, 100, 120 to represent small, medium and large data set. The following steps were conducted:

1. The shape (α) and scale (θ) parameters has been fixed at 0.5 and 0.25 respectively.
2. For given value of t, we generate c and r from uniform and gamma distribution respectively.
3. For given value of α and θ, we generate a random sample of different sizes from Exponentiated inverse Rayleigh distribution (EIRD) using the quantile function.
4. The above steps are iterated 1000 times to find the MSE of Bayesian and E-Bayesian estimates of scale parameter using different loss functions.

5. The MSE for Bayesian and E-Bayesian estimates under different loss function are shown in table 1.
6. The MSE of θ for Bayesian and E-Bayesian estimation under different loss functions are also illustrated in Figure 1, 2 and 3.

Table 1: *Mean Squared Error (MSE) of* θ *under different loss functions for* $\alpha = 0.25, \theta = 0.5\ , t = 1.5, d = 3$.

n	$\hat{\theta}_{BA}$	$\hat{\theta}_{EBA_1}$	$\hat{\theta}_{EBA_2}$	$\hat{\theta}_{EBA_3}$	$\hat{\theta}_{BS}$	$\hat{\theta}_{EBS_1}$	$\hat{\theta}_{EBS_2}$	$\hat{\theta}_{EBS_3}$	$\hat{\theta}_{BD}$	$\hat{\theta}_{EBD_1}$	$\hat{\theta}_{EBD_2}$	$\hat{\theta}_{EBD_3}$
20	0.1357	0.1315	0.1304	0.1294	0.1029	0.0994	0.0987	0.0979	0.1134	0.1096	0.1087	0.1079
50	0.0958	0.0948	0.0945	0.0942	0.0854	0.0844	0.0842	0.0839	0.0888	0.0878	0.0876	0.0873
70	0.0735	0.0727	0.0725	0.0724	0.0677	0.0668	0.0667	0.0666	0.0696	0.0688	0.0686	0.0685
100	0.0604	0.0599	0.0599	0.0598	0.0569	0.0565	0.0564	0.0563	0.0581	0.0576	0.0576	0.0575
120	0.0546	0.0543	0.0543	0.0542	0.0519	0.0517	0.0517	0.0516	0.0528	0.0526	0.0525	0.0525

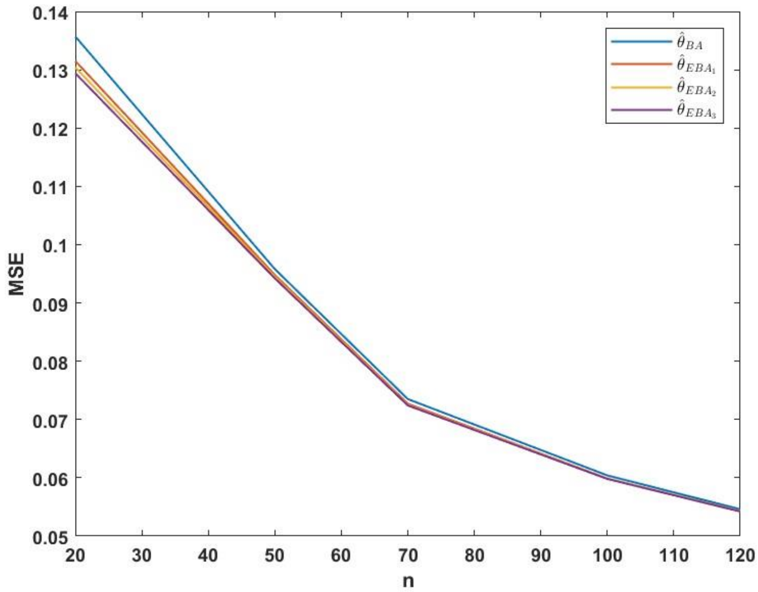

Figure 1: *MSE of Bayesian and E-Bayesian estimates of* θ *under Al-Bayyati loss function.*

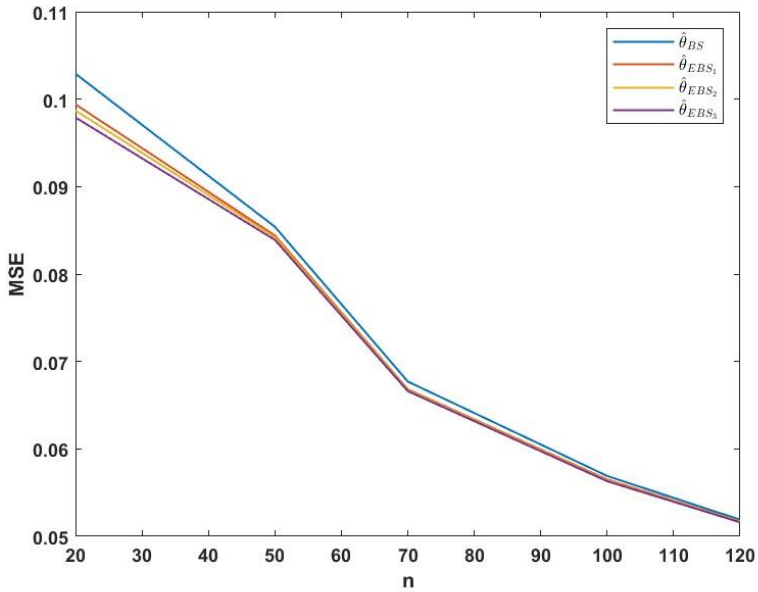

Figure2: *MSE of Bayesian and E-Bayesian estimates of θ under Squared error loss function.*

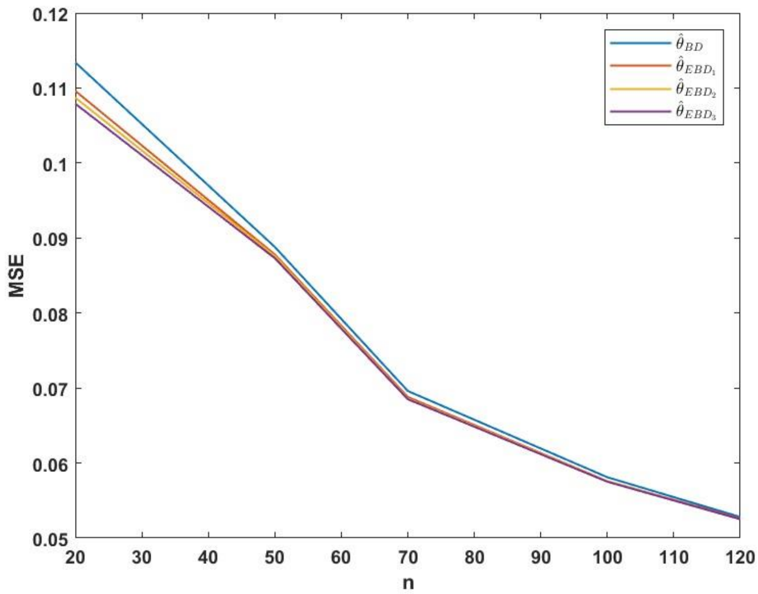

Figure 3: *MSE of Bayesian and E-Bayesian estimates of θ under De-Groot loss function.*

6. Real Data Analysis

The dataset was sourced from [13], comprising monthly actual tax revenues in Egypt spanning fro m January 2006 to November 2010. The data, expressed in 1000 million Egyptian pounds, are as foll ows: 5.9, 20.4, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1, 6.7, 17, 8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10, 4.1, 36, 8.5, 8, 9.2, 26.2, 21.9, 16.7, 21.3, 35.4, 14.3, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 1 1.9, 7, 8.6, 12.5, 10.3, 11.2, 6.1, 8.4, 11, 11.6, 11.9, 5.2, 6.8, 8.9, 7.1, 10.8. The Kolmogorov-Smirnov (K-S) statistic's value is 0.082194, with an associated p-value of 0.8203. This suggests that the Exponentia ted Inverse Rayleigh Distribution (EIRD) is the best fit for this dataset. Based on this data, the Maxi mum Likelihood estimates yield $\hat{\theta}$ = 8.9362 and $\hat{\alpha}$= 9.8028. Figure 4 and 5 displays the histogram an d the estimated Cumulative Distribution Function (CDF) of the EIRD for the dataset, while the esti mated Bayesian and E-Bayesian values are presented in Table 2.

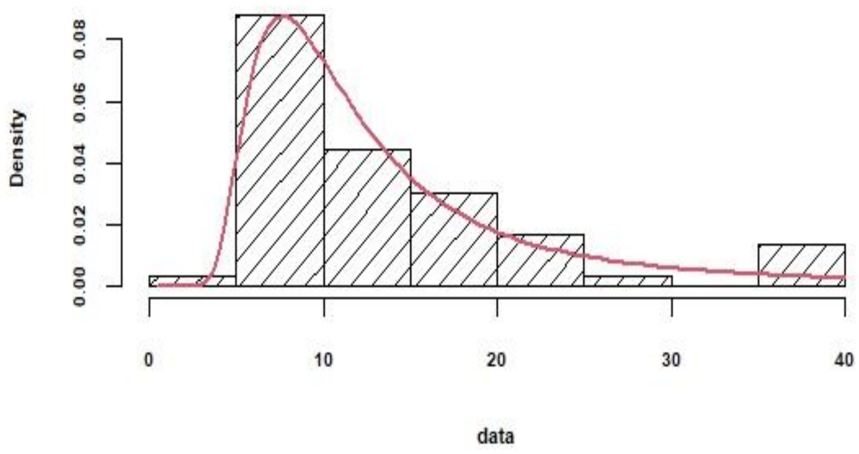

Figure 4: *Histogram and the fitted density for the monthly actual taxes revenue.*

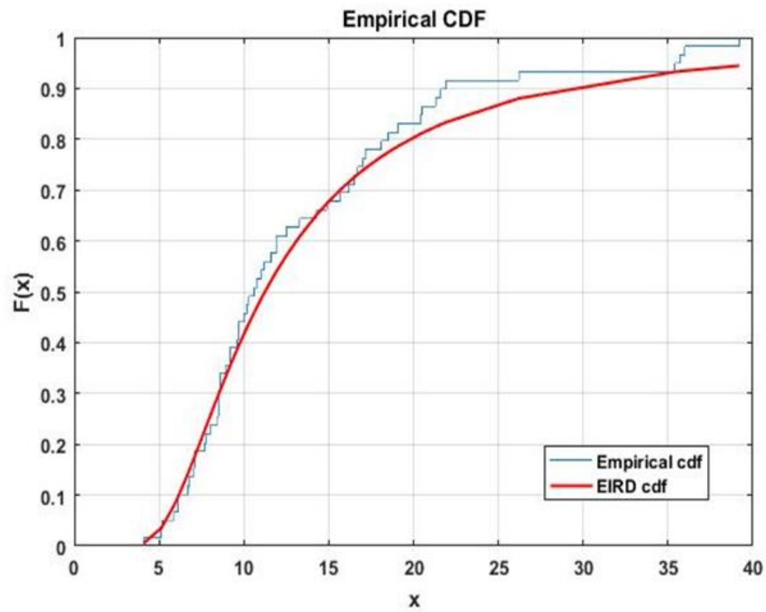

Figure 5: *Plot for the ECDF of the EIRD model.*

Table 2: *Bayesian and E-Bayesian estimates of* θ *based on real dataset.*

$\hat{\theta}_{BA}$	$\hat{\theta}_{EBA_1}$	$\hat{\theta}_{EBA_2}$	$\hat{\theta}_{EBA_3}$	$\hat{\theta}_{BS}$	$\hat{\theta}_{EBS_1}$	$\hat{\theta}_{EBS_2}$	$\hat{\theta}_{EBS_3}$	$\hat{\theta}_{BD}$	$\hat{\theta}_{EBD_1}$	$\hat{\theta}_{EBD_2}$	$\hat{\theta}_{EBD_3}$
94.257	129.236	95.558	71.446	89.769	123.032	90.971	68.017	91.265	125.100	92.500	69.160

7. Conclusion

This paper focuses on employing Bayesian and E-Bayesian methods to estimate the scale parameter of the Exponentiated Inverse Rayleigh distribution (EIRD) through the use of diverse loss functions. Additionally, certain properties of the E-Bayesian estimates are explored. Notably, the results of a simulation study reveal that E-Bayesian estimates exhibit a smaller Mean Squared Error (MSE) in comparison to Bayesian estimates, thereby demonstrating their enhanced efficiency. Among the E-Bayesian estimates, the third one stands out as the most effective.
Moreover, the analysis highlights that the Squared Error loss function outperforms the Al-Bayyati and De-Groot loss functions, exhibiting a smaller MSE. The conclusions drawn from the simulation study are further substantiated by validating the findings through a real-life dataset.

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