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BAYESIAN ANALYSIS OF EXTENDED MAXWELL-BOLTZMANN DISTRIBUTION USING SIMULATED AND REAL-LIFE DATA SETS

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Abstract

The objective of the study is to use Bayesian techniques to estimate the scale parameter of the 2Kth order weighted Maxwell-Boltzmann distribution(KWMBD). This involved using various prior assumptions such as extended Jeffrey's, Hartigan's , Inverse-gamma and Inverse-exponential, as well as different loss functions including squared error loss function (SELF), precautionary loss function (PLF), Al Bayyati's loss function (ALBF), and Stein's Loss Function (SLF).The maximum likelihood estimation (MLE) is also obtained. We compared the performances of MLE and bayesian estimation under each prior and its associated loss functions. And demonstrated the effectiveness of Bayesian estimation through simulation studies and analyzing real-life datasets.

Keywords: 2Kth Order Weighted Maxwell-Boltzmann Distribution, Prior Distribution, Loss Function and Bayesian estimation.

1. Introduction

The Maxwell-Boltzmann distribution, characterizes the probability distribution of speeds for particles in a gas at various temperatures. It provides a statistical framework for understanding the distribution of kinetic energies among particles, which makes it vital for modeling physical systems and predicting their behavior. Because of its practical significance, scientists and engineers closely examine the Maxwell-Boltzmann distribution to attain a deeper understanding of various scientific phenomena and to create precise models of complex systems. Tyagi and Bhattacharya [15] were the first to explore the Maxwell distribution as a lifetime model, and introduced considerations of Bayesian and minimum variance unbiased estimation methods for determining its parameters and reliability function. Chaturvedi and Rani [6] derived classical and Bayesian estimators for the Maxwell distribution by extending it with an additional parameter. Various Statisticians and Mathematicians have carried out the Bayesian paradigm of Maxwell-Boltzmann distribution by using loss functions and prior distributions, See, Spiring and Yeung [14], Rasheed [11], Reshi[13] , and Ahmad and Tripathi[1] .

The 2Kth order weighted maxwell-Boltzmann distribution (KWMBD) is a flexible, symmetric continuous univariate probability distribution suitable for modelling datasets of decreasing-increasing, increasing and constant behaviour. The probability density function (pdf) of KWMBD is given by:

$$f(x) = \frac{x^{2(k+1)}\alpha^{-(3+2k)}e^{-\frac{x^2}{2\alpha^2}}}{2^{k+\frac{1}{2}}\Gamma(k+\frac{3}{2})} \qquad x > 0, \alpha > 0, k \in R. \tag{1}$$

And, the corresponding cummulative distribution function (cdf) of KWMBD is given by:

$$F(x) = 1 - \frac{\Gamma\left((k+\frac{3}{2}), \frac{x^2}{2\alpha^2}\right)}{\Gamma(k+\frac{3}{2})} \qquad x > 0, \alpha > 0, k \in R. \tag{2}$$

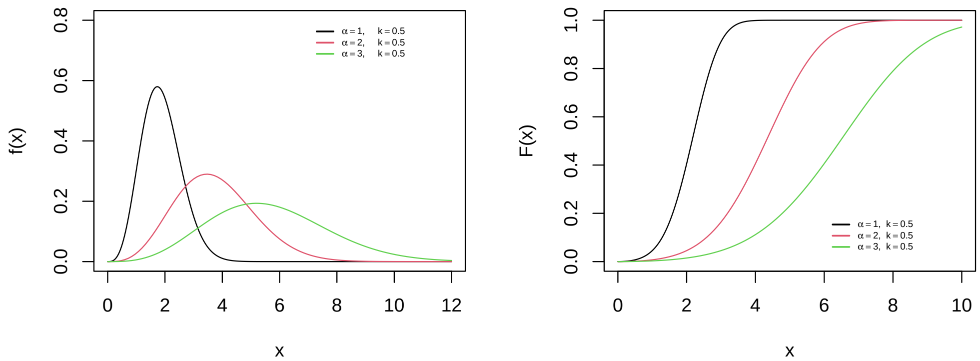

Figure 1: *Probability density plot and cumulative distribution plot of KWMBD for different combinations of parameters.*

2. Methodological Procedure

Bayesian approach utilizes prior beliefs, observed data, and a loss function to make decision in a structured manner, and is considered more reliable for estimating distribution parameters Compared to the classical approach, especially when the prior distribution accurately represents the parameter 's random behavior. In Bayesian analysis, parameters are treated as uncertain variables, allowing prior knowledge to be incorporated into the analysis. This prior information is typically described using a probability distribution known as the prior distribution. Friesl and Hurt[7] noted that employing Bayesian theory is a viable approach for incorporating prior information into the model, potentially improving the inference process and reflects the parameter's behavior. However, there are no strict rules for choosing one prior over another, frequently, prior distributions are selected based on an individual's subjective knowledge and beliefs . When sufficient information about the parameter is available, informative priors are preferred; otherwise, non-informative priors, such as the uniform prior, are used. Aslam [4] demonstrated the application of prior predictive distribution for determining the prior density. In this study, we assume the parameter α follows an extension of Jeffrey's prior proposed by Al-Kutobi[3] and α^2 follows a inverse-gamma prior and are given by:

2.1. Extension of Jeffrey's prior

The prior, known as extension of Jeffrey's prior is given by:

$$g(\alpha) = [I(\alpha)]^{c_1}; \quad c_1 \in R^+$$

where, $I(\alpha) = -nE\{\frac{d^2}{d^2(\alpha)} log f(x)\}$ is fisher-information matrix.
Thus, the resulting extension of Jeffrey's-prior for KWMBD will be:

$$g(\alpha) = \left[\frac{1}{\alpha^2}\right]^{c_1}; \quad c_1 \in R^+ \tag{3}$$

2.2. Inverse-gamma prior

The density of parameter α^2 on assuming it to follow Gamma (β, λ) distribution is given by:

$$g(\alpha^2) = \frac{\lambda^\beta}{\Gamma(\beta)} (\alpha^2)^{-\beta-1} e^{-\frac{\lambda}{\alpha^2}} \tag{4}$$

2.3. Loss functions

The idea of loss functions had been introduced first by Laplace, and later during the mid-20th century it was reintroduced by Weiss[16]. Loss function, serves as a measure of the discrepancy between observed data and the values predicted by a statistical model. Decisions in Bayesian inference, apart from relying on experimental data, are not entirely controlled by the loss function. Moreover, the relationship between the loss function and the posterior probability is significant. The choice of a loss function depends on the specific characteristics of the data and the goals of the analysis. Han[9] pointed out that, in Bayesian analysis choosing the right loss function and prior distribution is essential for making accurate statistical inferences. The Bayesian estimator is directly impacted by the choice of loss function, while the parameters of the prior density function may be affected by hyper parameters. Various symmetric and asymmetric loss functions have been demonstrated to be effective in research conducted by Zellner [17], Reshi [12], and Ahmad [2], among others. In this study, we have explored squared error, precautionary, Al-Bayyati's, and Stein's loss functions to enhance the comparison of Baye's estimators. And are given by:

2.3.1. Squared error loss function

The squared error loss function is given by:

$$l_{sq}(\hat{\alpha}, \alpha) = c(\hat{\alpha} - \alpha)^2; \quad c \in R^+ \tag{5}$$

2.3.2. Precautionary loss function

The Precautionary loss function is given by:

$$l_{pr}(\hat{\alpha}, \alpha) = \frac{c(\hat{\alpha} - \alpha)^2}{\hat{\alpha}} \tag{6}$$

2.3.3. Al-Bayyati's loss function

The Al-Bayyati's loss function is given by:

$$l_{Al}(\hat{\alpha}, \alpha) = \alpha^{c_2}(\hat{\alpha} - \alpha)^2; \quad c_2 \in R^+ \tag{7}$$

2.3.4. Stein's loss function

The Stein's loss function is given by:

$$l_{St}(\hat{\alpha}, \alpha) = \frac{\hat{\alpha}}{\alpha} - log\left(\frac{\hat{\alpha}}{\alpha}\right) - 1 \tag{8}$$

3. Parametric Estimation of KWMBD

In this section, we discuss the various estimation methods for KWMB Distribution.

3.1. Maximum Likelihood Estimation

Let $x_1, x_2, x_3, ..., x_n$ be a random sample of size n from kth Order Weighted Maxwell-Boltzmann Distribution. Therefore the maximum likelihood estimator(MLE) of α is:

$$\hat{\alpha} = \sqrt{\frac{x_i^2}{n(2k+3)}} \tag{9}$$

3.2. Baye's Estimator under Extension of Jeffrey's Prior

The Joint Probability Density Function of x and given α is given by:

$$L(\underline{x}|\alpha) = \frac{\prod_{i=1}^{n} x_i^{2(k+1)} \alpha^{-n(3+2k)} e^{\frac{\sum_{i=1}^{n} x_i^2}{2\alpha^2}}}{\left(2^{k+\frac{1}{2}}\right)^n \left(\Gamma\left(k+\frac{3}{2}\right)\right)^n} \tag{10}$$

The posterior probability density function of α for given data x is given by:

$$\pi_1(\alpha|\underline{x}) \propto L(\underline{x}|\alpha) g(\alpha)$$

$$\pi_1(\alpha|\underline{x}) \propto \frac{\prod_{i=1}^{n} x_i^{2(k+1)} \alpha^{-n(3+2k)} e^{\frac{\sum_{i=1}^{n} x_i^2}{2\alpha^2}}}{\left(2^{k+\frac{1}{2}}\right)^n \left(\Gamma\left(k+\frac{3}{2}\right)\right)^n} \frac{1}{\alpha^{2c_1}}$$

$$\pi_1(\alpha|\underline{x}) = k\alpha^{-n(3+2k)-2c_1} e^{\frac{\sum_{i=1}^{n} x_i^2}{2\alpha^2}}$$

where k is normalising constant independent of α and is given by:

$$k^{-1} = \int_0^\infty \alpha^{-n(3+2k)-2c_1} e^{\frac{\sum_{i=1}^{n} x_i^2}{2\alpha^2}} d\alpha$$

$$k^{-1} = \frac{\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{-n(3+2k)-2c_1+1}{2}} \Gamma\left(\frac{n(3+2k)+2c_1-1}{2}\right)}{2^{\frac{-n(3+2k)-2c_1+3}{2}}}$$

Therefore, the posterior probability density function is:

$$\pi_1(\alpha|\underline{x}) = \frac{2^{\frac{-n(3+2k)-2c_1+3}{2}} \alpha^{-n(3+2k)-2c_1} e^{\frac{\sum_{i=1}^{n} x_i^2}{2\alpha^2}}}{\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{-n(3+2k)-2c_1+1}{2}} \Gamma\left(\frac{n(3+2k)+2c_1-1}{2}\right)} \tag{11}$$

3.2.1. Baye's Estimator under squared error loss function

The Risk Function Under SELF is given by:

$$R_{(sq,ej)}(\hat{\alpha}) = \int_0^\infty c(\hat{\alpha}-\alpha)^2 \pi_1(\alpha|\underline{x})d\alpha$$

$$R_{(sq,ej)}(\hat{\alpha}) = c\hat{\alpha}^2 + \frac{\sum_{i=1}^{n} x_i^2}{(n(3+2k)+2c_1-3)} - 2\hat{\alpha}c\sqrt{\frac{\sum_{i=1}^{n} x_i^2}{2}}\frac{\Gamma\left(\frac{n(3+2k)+2c_1-2}{2}\right)}{\Gamma\left(\frac{n(3+2k)+2c_1-1}{2}\right)} \tag{12}$$

now, the Baye's estimator is obtained by solving

$$\frac{d(R_{(sq,ej)}(\hat{\alpha}))}{d\hat{\alpha}} = 0$$

and, is given by:

$$\hat{\alpha}_{(ej,sq)} = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{2}}\frac{\Gamma\left(\frac{n(3+2k)+2c_1-2}{2}\right)}{\Gamma\left(\frac{n(3+2k)+2c_1-1}{2}\right)} \tag{13}$$

3.2.2. Baye's Estimator under precautionary Loss function

The Risk Function Under PLF is given by:

$$R_{(pre,ej)}(\hat{\alpha}) = \int_0^\infty c\frac{(\hat{\alpha}-\alpha)^2}{\hat{\alpha}} \pi_1(\alpha|\underline{x})d\alpha$$

$$R_{(pre,ej)}(\hat{\alpha}) = c\hat{\alpha} + c\frac{\sum_{i=1}^{n} x_i^2}{\hat{\alpha}(n(3+2k)+2c_1-3)} - 2c\sqrt{\frac{\sum_{i=1}^{n} x_i^2}{2}}\frac{\Gamma\left(\frac{n(3+2k)+2c_1-2}{2}\right)}{\Gamma\left(\frac{n(3+2k)+2c_1-1}{2}\right)} \tag{14}$$

now, the Baye's estimator is obtained by solving

$$\frac{d(R_{(pre,ej)}(\hat{\alpha}))}{d\hat{\alpha}} = 0$$

and, is given by:

$$\hat{\alpha}_{(pre,ej)} = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{(n(3+2k)+2c_1-3)}} \tag{15}$$

3.2.3. Baye's Estimator under Al-Bayyati's loss function

The Risk Function Under Al-Bayyati's loss function is given by:

$$R_{(alb,ej)}(\hat{\alpha}) = \int_0^\infty \alpha^{c_2}(\hat{\alpha}-\alpha)^2 \pi_1(\alpha|\underline{x})d\alpha$$

$$R_{(alb,ej)}(\hat{\alpha}) = \hat{\alpha}^2\left(\frac{\sum_{i=1}^{n} x_i^2}{2}\right)^{\frac{c_2}{2}} + \left(\frac{\sum_{i=1}^{n} x_i^2}{2}\right)^{\frac{c_2-2}{2}}\frac{1}{(n(3+2k)+2c_1-c_2-3)} - 2\hat{\alpha}\left(\frac{\sum_{i=1}^{n} x_i^2}{2}\right)^{\frac{c_2+1}{2}}\frac{\Gamma\left(\frac{n(3+2k)+2c_1-c_2-2}{2}\right)}{\Gamma\left(\frac{n(3+2k)+2c_1-c_2-1}{2}\right)} \tag{16}$$

now, the Baye's estimator is obtained by solving

$$\frac{d(R_{(alb,ej)}(\hat{\alpha}))}{d\hat{\alpha}} = 0$$

and, is given by:

$$\hat{\alpha}_{(alb,ej)} = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{2} \frac{\Gamma\left(\frac{n(3+2k)+2c_1-c_2-2}{2}\right)}{\Gamma\left(\frac{n(3+2k)+2c_1-c_2-1}{2}\right)}} \tag{17}$$

3.2.4. Baye's Estimator under combination of Stein's loss function

The Risk Function Under SLF is given by:

$$R_{(ste,ej)}(\hat{\alpha}) = \int_0^{\infty} \left(\frac{\hat{\alpha}}{\alpha} - log\left(\frac{\hat{\alpha}}{\alpha}\right) - 1\right) \pi_1(\alpha|\underline{x}) d\alpha$$

$$R_{(ste,ej)}(\hat{\alpha}) = \hat{\alpha} \sqrt{\frac{2}{\sum_{i=1}^{n} x_i^2} \frac{\Gamma\left(\frac{n(3+2k)+2c_1}{2}\right)}{\Gamma\left(\frac{n(3+2k)+2c_1-1}{2}\right)}} - log(\hat{\alpha}) - m - 1 \tag{18}$$

where, m is constant of integration.
Now, the Baye's estimator is obtained by solving

$$\frac{d(R_{(ste,ej)}(\hat{\alpha}))}{d\hat{\alpha}} = 0$$

and, is given by:

$$\hat{\alpha}_{(ste,ej)} = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{2} \frac{\Gamma\left(\frac{n(3+2k)+2c_1-1}{2}\right)}{\Gamma\left(\frac{n(3+2k)+2c_1}{2}\right)}} \tag{19}$$

3.3. Baye's Estimator under Inverse-Gamma Prior

The Joint Probability Density Function of x and given α^2 is given by:

$$L(\underline{x}|\alpha^2) = \frac{\prod_{i=1}^{n} x_i^{2(k+1)} (\alpha^2)^{\frac{-n(3+2k)}{2}} e^{\frac{\sum_{i=1}^{n} x_i^2}{2\alpha^2}}}{\left(2^{k+\frac{1}{2}}\right)^n \left(\Gamma\left(k+\frac{3}{2}\right)\right)^n} \tag{20}$$

The posterior probability density function of α^2 for given data x is given by:

$$\pi_2(\alpha^2|\underline{x}) \propto L(\underline{x}|\alpha^2) g(\alpha^2)$$

$$\pi_2(\alpha^2|\underline{x}) \propto \frac{\prod_{i=1}^{n} x_i^{2(k+1)} (\alpha^2)^{\frac{-n(3+2k)}{2}} e^{\frac{\sum_{i=1}^{n} x_i^2}{2\alpha^2}}}{\left(2^{k+\frac{1}{2}}\right)^n \left(\Gamma\left(k+\frac{3}{2}\right)\right)^n} \frac{\lambda^{\beta}}{\Gamma(\beta)} (\alpha^2)^{-\beta-1} e^{-\frac{\lambda}{\alpha^2}}$$

$$\pi_2(\alpha^2|\underline{x}) = k(\alpha^2)^{\frac{-n(3+2k)-2\beta-2}{2}} e^{\left(\frac{\sum_{i=1}^{n} x_i^2}{2} + \lambda\right)\frac{1}{\alpha^2}}$$

where k is normalising constant independent of α and is given by:

$$k^{-1} = \int_0^\infty (\alpha^2)^{\frac{-n(3+2k)-2\beta-2}{2}} e^{\left(\frac{\sum_{i=1}^n x_i^2}{2}+\lambda\right)\frac{1}{\alpha^2}} d\alpha^2$$

$$k^{-1} = \frac{\Gamma\left(\frac{n(3+2k)+2\beta}{2}\right)}{\left(\frac{\sum_{i=1}^n x_i^2}{2}+\lambda\right)^{\frac{n(3+2k)+2\beta}{2}}}$$

Therefore, the posterior probability density function is:

$$\pi_2(\alpha^2|\underline{x}) = \frac{\left(\frac{\sum_{i=1}^n x_i^2}{2}+\lambda\right)^{\frac{n(3+2k)+2\beta}{2}} (\alpha^2)^{\frac{-n(3+2k)-2\beta-2}{2}} e^{\left(\frac{\sum_{i=1}^n x_i^2}{2}+\lambda\right)\frac{1}{\alpha^2}}}{\Gamma\left(\frac{n(3+2k)+2\beta}{2}\right)} \tag{21}$$

3.3.1. Baye's Estimator under squared error loss function

The Risk Function Under SELF is given by:

$$R_{(sq,igp)}(\hat{\alpha^2}) = \int_0^\infty c(\hat{\alpha^2}-\alpha^2)^2 \pi_2(\alpha^2|\underline{x}) d(\alpha^2)$$

$$R_{(sq,igp)}(\hat{\alpha^2}) = c(\hat{\alpha}^2)^2 + \frac{\left(\frac{\sum_{i=1}^n x_i^2}{2}+\lambda\right)^2}{\left(\frac{n(3+2k)+2\beta-2}{2}\right)\left(\frac{n(3+2k)+2\beta-4}{2}\right)} - \hat{\alpha}^2 c \frac{\left(\frac{\sum_{i=1}^n x_i^2}{2}+\lambda\right)}{\left(\frac{n(3+2k)+2\beta-2}{2}\right)} \tag{22}$$

now, the Baye's estimator is obtained by solving

$$\frac{R_{(sq,igp)}(\hat{\alpha^2})}{d(\hat{\alpha^2})} = 0$$

and, is given by:

$$\hat{\alpha}_{(sq,igp)} = \sqrt{\frac{2\left(\frac{\sum_{i=1}^n x_i^2}{2}+\lambda\right)}{(n(3+2k)+2\beta-2)}} \tag{23}$$

3.3.2. Baye's Estimator under precautionary Loss function

The Risk Function Under PLF is given by:

$$R_{(pre,igp)}(\hat{\alpha^2}) = \int_0^\infty c\frac{(\hat{\alpha^2}-\alpha^2)^2}{\hat{\alpha^2}} \pi_2(\alpha^2|\underline{x}) d\alpha^2$$

$$R_{(pre,igp)}(\hat{\alpha^2}) = c\hat{\alpha}^2 + c\frac{1}{\hat{\alpha^2}} \frac{\left(\frac{\sum_{i=1}^n x_i^2}{2}+\lambda\right)^2}{\left(\frac{(n(3+2k)+2\beta-2)(n(3+2k)+2\beta-4)}{4}\right)} - 2c \frac{\left(\frac{\sum_{i=1}^n x_i^2}{2}+\lambda\right)}{\left(\frac{n(3+2k)+2\beta-2}{2}\right)} \tag{24}$$

now, the Baye's estimator is obtained by solving

$$\frac{d(R_{(pre,igp)}(\hat{\alpha^2}))}{d\hat{\alpha^2}} = 0$$

and, is given by:

$$\hat{\alpha}_{(pre,igp)} = \sqrt{\frac{2\left(\frac{\sum_{i=1}^{n} x_i^2}{2} + \lambda\right)}{\sqrt{(n(3+2k)+2\beta-2)(n(3+2k)+2\beta-4))}}} \tag{25}$$

3.3.3. Baye's Estimator under Al-Bayyati's loss function

The Risk Function Under Al-Bayyati's loss function is given by:

$$R_{(alb,igp)}(\hat{\alpha^2}) = \int_0^{\infty} (\alpha^2)^{c_2}(\hat{\alpha^2} - \alpha^2)^2 \pi_2(\alpha^2|\underline{x}) d\alpha^2$$

$$R_{(alb,igp)}(\hat{\alpha^2}) = (\hat{\alpha}^2)^4 \left(\frac{\sum_{i=1}^{n} x_i^2}{2} + \lambda\right)^{c_2} \frac{\Gamma\left(\frac{n(3+2k)+2\beta-2c_2}{2}\right)}{\Gamma\left(\frac{n(3+2k)+2\beta}{2}\right)} + \left(\frac{\sum_{i=1}^{n} x_i^2}{2} + \lambda\right)^{c_2+2} \frac{\Gamma\left(\frac{n(3+2k)+2\beta-2c_2-4}{2}\right)}{\Gamma\left(\frac{n(3+2k)+2\beta}{2}\right)} - 2\hat{\alpha}^2 \left(\frac{\sum_{i=1}^{n} x_i^2}{2} + \lambda\right)^{c_2+1} \frac{\Gamma\left(\frac{n(3+2k)+2\beta-2c_2-2}{2}\right)}{\Gamma\left(\frac{n(3+2k)+2\beta}{2}\right)} \tag{26}$$

now, the Baye's estimator is obtained by solving

$$\frac{d(R_{(alb,igp)}(\hat{\alpha^2}))}{d\hat{\alpha^2}} = 0$$

and, is given by:

$$\hat{\alpha}_{(alb,igp)} = \sqrt{\frac{2\left(\frac{\sum_{i=1}^{n} x_i^2}{2} + \lambda\right)}{(n(3+2k)+2\beta-2)}} \tag{27}$$

3.3.4. Baye's Estimator under combination of Stein's loss function

The Risk Function Under SLF is given by:

$$R_{(s,igp)}(\hat{\alpha^2}) = \int_0^{\infty} \left(\frac{\hat{\alpha^2}}{\alpha^2} - log\left(\frac{\hat{\alpha^2}}{\alpha^2}\right) - 1\right) \pi_2(\alpha^2|\underline{x}) d\alpha^2$$

$$R_{(s,igp)}(\hat{\alpha^2}) = \hat{\alpha^2}\frac{(n(3+2k)+2\beta)}{2\left(\frac{\sum_{i=1}^{n} x_i^2}{2} + \lambda\right)} - log(\hat{\alpha}) - m - 1 \tag{28}$$

where, m is constant of integration.
Now, the Baye's estimator is obtained by solving

$$\frac{d(R_{(s,igp)}(\hat{\alpha^2}))}{d\hat{\alpha^2}} = 0$$

and, is given by:

$$\hat{\alpha}_{(ste,igp)} = \sqrt{\frac{2\left(\frac{\sum_{i=1}^{n} x_i^2}{2} + \lambda\right)}{(n(3+2k)+2\beta)}} \tag{29}$$

Table 1: *Baye's Estimation under Hartigan's Prior Distribution and Different Combinations of Loss Functions.*

Prior	Loss Function	Baye's Estimator
Hartigan's $(i.e. c_1 = 3/2)$	Squared-error	$\sqrt{\frac{\sum_{i=1}^{n} x_i^2}{2} \frac{\Gamma\left(\frac{n(3+2k)+1}{2}\right)}{\Gamma\left(\frac{n(3+2k)+2}{2}\right)}}$
	Precautionary	$\sqrt{\frac{\sum_{i=1}^{n} x_i^2}{(n(3+2k))}}$
	Al-Bayyati's	$\sqrt{\frac{\sum_{i=1}^{n} x_i^2}{2} \frac{\Gamma\left(\frac{n(3+2k)-c_2+1}{2}\right)}{\Gamma\left(\frac{n(3+2k)-c_2+1}{2}\right)}}$
	Stein's	$\sqrt{\frac{\sum_{i=1}^{n} x_i^2}{2} \frac{\Gamma\left(\frac{n(3+2k)+2}{2}\right)}{\Gamma\left(\frac{n(3+2k)+3}{2}\right)}}$

Table 2: *Baye's Estimation under Inverse-Exponential Prior Distributions and Different Combinations of Loss Functions.*

Prior	Loss Function	Baye's Estimator
Inverse-Exponential $(i.e. \beta = 1)$	Squared-error	$\sqrt{\frac{2\left(\frac{\sum_{i=1}^{n} x_i^2}{2}+\lambda\right)}{(n(3+2k))}}$
	Precautionary	$\sqrt{\frac{2\left(\frac{\sum_{i=1}^{n} x_i^2}{2}+\lambda\right)}{\sqrt{(n(3+2k))(n(3+2k)-2))}}}$
	Al-Bayyati's	$\sqrt{\frac{2\left(\frac{\sum_{i=1}^{n} x_i^2}{2}+\lambda\right)}{(n(3+2k))}}$
	Stein's	$\sqrt{\frac{2\left(\frac{\sum_{i=1}^{n} x_i^2}{2}+\lambda\right)}{(n(3+2k)+2)}}$

3.4. Simulation Study

We conducted simulation studies using R software, generated samples of sizes n=10, 50, and 100 to observe the effect of small, medium, and large samples on the estimators of scale parameter α of the 2kth order weighted Maxwell Boltzmann distribution. Each process is replicated 500 times to examine the performance of the MLEs and Bayesian estimators under different priors such as the extension of Jeffrey's prior, Hartigan's prior, inverse-Gamma prior, and inverse-exponential prior, across different loss functions in terms of average estimates, biases, variances, and mean squared errors by considering different parameter combinations.The results are presented in the tables below:

Table 3: *Average estimate, Bias, Variance and Mean Squared Error under Extension of Jeffrey's prior.*

n	α	k	c_1	c_2	Criterion	$\hat{\alpha}_{\text{mle}}$	$\hat{\alpha}_{\text{sq}}$	$\hat{\alpha}_{\text{pre}}$	$\hat{\alpha}_{\text{alb}}$	$\hat{\alpha}_{\text{ste}}$
10	3	-0.5	2	5	Estimate	2.97912	2.87293	2.90732	3.27915	2.80839
					Bias	-0.02088	-0.12707	-0.09268	0.27915	-0.19161
					Variance	0.23825	0.22157	0.22691	0.28866	0.21173
					MSE	0.23869	0.23772	0.23549	0.36658	0.24844
50	3	-0.5	2	5	Estimate	3.00890	2.98656	2.99396	3.06295	2.97196
					Bias	0.00890	-0.01344	-0.00604	0.06295	-0.02804
					Variance	0.04693	0.04624	0.04647	0.04864	0.04579
					MSE	0.04701	0.04642	0.04645	0.05260	0.04658
100	3	-0.5	2	5	Estimate	3.00411	2.99291	2.99663	3.03075	2.98551
					Bias	0.00411	-0.00709	-0.00337	0.03075	-0.01449
					Variance	0.02164	0.02148	0.02153	0.02203	0.02137
					MSE	0.02166	0.02153	0.02154	0.02297	0.02158
10	4	0.1	1.2	3	Estimate	3.96513	3.94644	3.97758	4.14350	3.88675
					Bias	-0.03487	-0.05356	-0.02242	0.14350	-0.11325
					Variance	0.25397	0.25158	0.25557	0.27733	0.24403
					MSE	0.25518	0.25445	0.25620	0.29792	0.25685
50	4	0.1	1.2	3	Estimate	4.00312	3.99937	4.00563	4.03732	3.98695
					Bias	0.00312	-0.00063	0.00563	0.03732	-0.01305
					Variance	0.05165	0.05155	0.05172	0.05254	0.05123
					MSE	0.05166	0.05155	0.05175	0.05393	0.05140
100	4	0.1	1.2	3	Estimate	3.99978	3.99790	4.00103	4.01676	3.99168
					Bias	-0.00022	-0.00210	0.00103	0.01676	-0.00832
					Variance	0.02381	0.02379	0.02382	0.02401	0.02371
					MSE	0.02381	0.02379	0.02382	0.02429	0.02378

Table 4: *Average estimate, Bias, Variance and Mean Squared Error under Hartigan's prior.*

n	α	k	c_1	c_2	Criterion	$\hat{\alpha}_{\text{mle}}$	$\hat{\alpha}_{\text{sq}}$	$\hat{\alpha}_{\text{pre}}$	$\hat{\alpha}_{\text{alb}}$	$\hat{\alpha}_{\text{ste}}$
10	3	-0.5	1.5	5	Estimate	2.98117	2.94416	2.98117	3.38551	2.87491
					Bias	-0.01883	-0.05584	-0.01883	0.38551	-0.12509
					Variance	0.20672	0.20162	0.20672	0.26660	0.19225
					MSE	0.20708	0.20474	0.20708	0.41521	0.20789
50	3	-0.5	1.5	5	Estimate	2.99573	2.98825	2.99573	3.06548	2.97350
					Bias	-0.00427	-0.01175	-0.00427	0.06548	-0.02650
					Variance	0.04357	0.04335	0.04357	0.04562	0.04292
					MSE	0.04359	0.04349	0.04359	0.04991	0.04363
100	3	-0.5	1.5	5	Estimate	2.99912	2.99537	2.99912	3.03344	2.98793
					Bias	-0.00088	-0.00463	-0.00088	0.03344	-0.01207
					Variance	0.02168	0.02163	0.02168	0.02218	0.02152
					MSE	0.02168	0.02165	0.02168	0.02330	0.02167
10	4	0.1	1.5	3	Estimate	3.96310	3.93226	3.96310	4.12731	3.87314
					Bias	-0.03690	-0.06774	-0.03690	0.12731	-0.12686
					Variance	0.25001	0.24614	0.25001	0.27116	0.23879
					MSE	0.25137	0.25073	0.25137	0.28737	0.25489
50	4	0.1	1.5	3	Estimate	3.99271	3.98647	3.99271	4.02426	3.97411
					Bias	-0.00729	-0.01353	-0.00729	0.02426	-0.02589
					Variance	0.04954	0.04939	0.04954	0.05033	0.04908
					MSE	0.04959	0.04957	0.04959	0.05092	0.04975
100	4	0.1	1.5	3	Estimate	4.00132	3.99819	4.00132	4.01704	3.99197
					Bias	0.00132	-0.00181	0.00132	0.01704	-0.00803
					Variance	0.02222	0.02219	0.02222	0.02240	0.02212
					MSE	0.02222	0.02219	0.02222	0.02269	0.02218

$\hat{\alpha}_{mle}$= Estimate under maximum likelihood estimation, $\hat{\alpha}_{sq}$= Bayes estimate under squared error loss function, $\hat{\alpha}_{pre}$= Bayes estimate under precautionary loss function, $\hat{\alpha}_{alb}$= Bayes estimate under Al-Bayyati's loss function, $\hat{\alpha}_{ste}$= Bayes estimate under Stein's loss function.

Table 5: *Average estimate, Bias, Variance and Mean Squared Error under Inverse-Gamma prior.*

n	α	k	β	λ	c_2	Criterion	$\hat{\alpha}_{\text{mle}}$	$\hat{\alpha}_{\text{sq}}$	$\hat{\alpha}_{\text{pre}}$	$\hat{\alpha}_{\text{alb}}$	$\hat{\alpha}_{\text{ste}}$
10	3	-0.5	1.5	3.5	5	Estimate	2.96815	2.95500	3.02987	4.08292	2.82360
						Bias	-0.03185	-0.04500	0.02987	1.08292	-0.17640
						Variance	0.22183	0.20296	0.21337	0.38746	0.18531
						MSE	0.22284	0.20498	0.21426	1.56018	0.21642
50	3	-0.5	1.5	3.5	5	Estimate	3.01586	3.01248	3.02758	3.17368	2.98309
						Bias	0.01586	0.01248	0.02758	0.17368	-0.01691
						Variance	0.04434	0.04357	0.04400	0.04835	0.04272
						MSE	0.04459	0.04372	0.04476	0.07852	0.04301
100	3	-0.5	1.5	3.5	5	Estimate	3.00761	3.00593	3.01346	3.08362	2.99109
						Bias	0.00761	0.00593	0.01346	0.08362	-0.00891
						Variance	0.02039	0.02021	0.02031	0.02127	0.02001
						MSE	0.02045	0.02024	0.02049	0.02826	0.02009
10	4	0.1	1.2	3	3	Estimate	3.96991	3.96910	4.03283	4.39706	3.85199
						Bias	-0.03009	-0.03090	0.03283	0.39706	-0.14801
						Variance	0.24080	0.23493	0.24254	0.28833	0.22127
						MSE	0.24171	0.23589	0.24361	0.44598	0.24318
50	4	0.1	1.2	3	3	Estimate	3.97652	3.97628	3.98878	4.05281	3.95172
						Bias	-0.02348	-0.02372	-0.01122	0.05281	-0.04828
						Variance	0.05037	0.05013	0.05044	0.05207	0.04951
						MSE	0.05092	0.05069	0.05056	0.05486	0.05184
100	4	0.1	1.2	3	3	Estimate	4.00210	4.00195	4.00822	4.03995	3.98952
						Bias	0.00210	0.00195	0.00822	0.03995	-0.01048
						Variance	0.02531	0.02525	0.02533	0.02573	0.02509
						MSE	0.02531	0.02525	0.02534	0.02733	0.02520

Table 6: *Average estimate, Bias, Variance and Mean Squared Error under Inverse-Exponential prior.*

n	α	k	β	λ	c_2	Criterion	$\hat{\alpha}_{\text{mle}}$	$\hat{\alpha}_{\text{sq}}$	$\hat{\alpha}_{\text{pre}}$	$\hat{\alpha}_{\text{alb}}$	$\hat{\alpha}_{\text{ste}}$
10	3	-0.5	1	3.5	5	Estimate	2.93546	2.99603	3.07599	4.23702	2.85660
						Bias	-0.06454	-0.00397	0.07599	1.23702	-0.14340
						Variance	0.22744	0.21819	0.23000	0.43639	0.19836
						MSE	0.23161	0.21821	0.23577	1.96661	0.21892
50	3	-0.5	1	3.5	5	Estimate	2.98571	2.99747	3.01265	3.15961	2.96794
						Bias	-0.01429	-0.00253	0.01265	0.15961	-0.03206
						Variance	0.04084	0.04052	0.04093	0.04502	0.03973
						MSE	0.04105	0.04053	0.04109	0.07050	0.04076
100	3	-0.5	1	3.5	5	Estimate	2.99384	2.9997	3.00724	3.07762	2.98481
						Bias	-0.00616	-0.0003	0.00724	0.07762	-0.01519
						Variance	0.02369	0.0236	0.02372	0.02484	0.02336
						MSE	0.02373	0.0236	0.02377	0.03087	0.02360
10	4	0.1	1	3	3	Estimate	3.96977	3.99370	4.05866	4.43061	3.87445
						Bias	-0.03023	-0.00630	0.05866	0.43061	-0.12555
						Variance	0.25840	0.25532	0.26369	0.31424	0.24030
						MSE	0.25931	0.25536	0.26713	0.49966	0.25606
50	4	0.1	1	3	3	Estimate	3.99208	3.99679	4.00938	4.07391	3.97204
						Bias	-0.00792	-0.00321	0.00938	0.07391	-0.02796
						Variance	0.05112	0.05100	0.05132	0.05298	0.05037
						MSE	0.05118	0.05101	0.05140	0.05845	0.05115
100	4	0.1	1	3	3	Estimate	3.99707	3.99942	4.00569	4.03745	3.98698
						Bias	-0.00293	-0.00058	0.00569	0.03745	-0.01302
						Variance	0.02562	0.02559	0.02567	0.02608	0.02543
						MSE	0.02563	0.02559	0.02570	0.02749	0.02560

From the results of simulation tables 3,4,5, and 6 , conclusions are drawn regarding the performance and behavior of the estimators under different priors, which are summarized below.

- The performances of the Bayesian and MLEs become better when the sample size increases.
- It has been observed that Bayesian estimation, outperforms MLE estimation.

- In terms of MSE,in most cases the bayesian estimation under squared error loss function gives smaller MSEs as compared to other loss functions.

3.5. Fitting of real life data-set

For illustrative purposes, we analyze three different types of real datasets. The dataset I consists of tensile strength measurements (in GPA) from 69 carbon fibers tested under tension at gauge lengths of 20mm. These measurements were initially reported by Bader and Priest [5] . The datasets II consists of an accelerated life test conducted on 59 conductors, with failure times measured in hours. Reported first by Johnston[10] . The dataset III comprises times between arrivals of 25 customers at a facility and reported first Grubbs[8] . Our objective is to evaluate and contrast the performance of KWMBD estimates using mle and baysian estimation.

Table 7: *Average estimate,Mean Squared Error, AIC, BIC for posterior distribution under different priors for dataset I.*

criterion	MLE	Ex-Jeffreys Prior	Hartigan's Prior	I-Gamma Prior	I- Exponential Prior
Estimate	2.5001	2.4390	2.4911	2.4144	2.4819
MSE	0.2440	0.2418	0.24320	0.2430	0.2426
AIC	228.6145	197.1729	198.5274	196.5776	198.2806
BIC	230.8486	199.4070	200.7615	198.8117	200.5147

Table 8: *Estimates and MSE for Extension of Jeffrey's and Inverse-Gamma Priors with different loss functions for dataset I.*

$\hat{\alpha}_{mle}$		priors	$\hat{\alpha}_{sq}$		$\hat{\alpha}_{pre}$		$\hat{\alpha}_{alb}$		$\hat{\alpha}_{ste}$	
Estimate	MSE		Estimate	MSE	Estimate	MSE	Estimate	MSE	Estimate	MSE
2.5001	0.2440	EX-Jeffrey's Prior	2.4390	0.2418	2.4475	0.2417	2.4911	0.2432	2.4224	0.2424
		I-Gamma Prior	2.4391	0.2418	2.456	0.2416	2.5460	0.2506	2.4064	0.2436

Table 9: *Average estimate,Mean Squared Error, AIC, BIC for posterior distribution under different priors for dataset II.*

criterion	MLE	Ex-Jeffreys Prior	Hartegan's Prior	I-Gamma Prior	I- Exponential Prior
Estimate	7.16117	6.957312	7.129853	6.855565	7.077978
MSE	2.59377	2.561495	2.583413	2.576478	2.570564
AIC	319.9468	246.6048	247.7058	246.1869	247.3245
BIC	322.0243	248.6823	249.7833	248.2644	249.4021

Table 10: *Estimates and MSE for Extension of Jeffrey's and Inverse-Gamma Priors with different loss functions for dataset II.*

$\hat{\alpha}_{mle}$		priors	$\hat{\alpha}_{sq}$		$\hat{\alpha}_{pre}$		$\hat{\alpha}_{alb}$		$\hat{\alpha}_{ste}$	
Estimate	MSE		Estimate	MSE	Estimate	MSE	Estimate	MSE	Estimate	MSE
7.1612	2.5938	EX-Jeffrey's Prior	6.9577	2.5615	6.9858	2.5610	7.2538	2.6359	6.9027	2.5670
		I-Gamma Prior	6.9370	2.5628	6.9931	2.5612	7.5631	2.9010	6.8294	2.5837

Table 11: *Average estimate,Mean Squared Error, AIC, BIC for posterior distribution under different priors for dataset III.*

criterion	MLE	Ex-Jeffreys Prior	Hartegan's Prior	I-Gamma Prior	I- Exponential Prior
Estimate	4.0405	3.9242	4.0003	3.8025	3.9433
MSE	0.6053	0.6015	0.6010	0.6264	0.6003
AIC	108.1082	85.9577	86.3621	85.4566	86.0532
BIC	109.3270	87.1765	87.5810	86.6755	87.2721

Table 12: *Estimates and MSE for Extension of Jeffrey's and Inverse-Gamma Priors with different loss functions for dataset III.*

$\hat{\alpha}_{mle}$		priors	$\hat{\alpha}_{sq}$		$\hat{\alpha}_{pre}$		$\hat{\alpha}_{alb}$		$\hat{\alpha}_{ste}$	
Estimate	MSE		Estimate	MSE	Estimate	MSE	Estimate	MSE	Estimate	MSE
4.0405	0.6054	EX-Jeffrey's Prior	3.9242	0.6015	3.9621	0.5998	4.0811	0.6131	3.8522	0.6126
		I-Gamma Prior	3.9070	0.6032	3.9829	0.6001	4.2331	0.6713	3.7699	0.6381

The results of tables 7 , 8 ,9 ,10 ,11 and 12 demonstrate that the estimation of parameters for KWMBD under both priors (Extension of Jeffrey's and Inverse Gamma prior) and precautionary and square error loss function is better compared to the other three loss functions considered and mle estimation, owing to its lower Mean Squared Error (MSE).

4. Conclusion

We compared estimation methods for the scale parameter α of the 2kth order weighted Maxwell-Boltzmann distribution, utilizing both Maximum Likelihood Estimation (MLE) and Bayesian Estimation under various loss functions and prior distributions. This comparison is based on the simulated data and real-life datasets. Results of simulated data reveal that as the sample size increases, MSE decreases. and the Bayesian Estimation outperforms Maximum Likelihood Estimation (MLE). Furthermore, results from the real-life datasets demonstrate that the estimation of parameters of KWMBD under both prior distributions and precautionary loss function and square error loss function yields better performance, with smaller MSE compared to other estimators.

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