ON THE FLEXIBILITY OF TYPE I HALF LOGISTIC EXPONENTIATED FRECHET DISTRIBUTION

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Abstract

In this article, we delve into the modeling and analysis of lifetimes, which hold substantial importance across various scientific and industrial fields. Our focus is on introducing a novel distribution termed the Type I Half-Logistic Exponentiated Frechet (TIHLEtF) Distribution, which is an extension of the Frechet distribution. We have derived a crucial representation of the density function for this distribution. Furthermore, we explore several statistical properties associated with the TIHLEtF distribution. These properties encompass explicit expressions for the quantile function, probability-weighted moments, moments, moments generating function, reliability function, hazard function, and order statistics. To estimate the model parameters, we employ the maximum likelihood estimation technique and present the results of a simulation study. To emphasize the superiority of our newly introduced distribution, we apply it to two real datasets. The outcomes of our analysis reveal that the TIHLEtF distribution outperforms the other considered distributions in terms of fitting the data in these real-world cases.

Keywords Type I Half-Logistic Exponentiated-G, Frechet distribution, Quantile function, Hazard function, Maximum likelihood, Order Statistics

I. Introduction

The Frechet distribution, often referred to as the type II extreme value distribution, plays a crucial role in various fields, including engineering, actuarial science, environmental studies, medical sciences, economics, finance, and insurance. It serves as a fundamental statistical tool in these disciplines. It was introduced by [10] as a means of modeling extreme values in data. Despite the widespread use of traditional probability distributions in these areas, there is a growing need for more flexible forms of these distributions, and the Frechet distribution is one such example that has been proposed to meet this need. It is a flexible tool for modeling data and is widely used in extreme value theory. Various researchers have proposed several modifications to the Frechet distribution in recent literature. The exponentiated Frechet was first introduced by [16], followed by the beta Frechet proposed by [17], and the transmuted Frechet proposed by [13]. The gamma extended Frechet defined by [7], while [12] introduced the Marshall-Olkin Frechet. The Kumaraswamy Frechet proposed by [14], while [9] studied the transmuted exponentiated Frechet. [1] investigated the transmuted Marshall-Olkin Frechet, and [3] proposed the Kumaraswamy Marshall-Olkin Frechet, [2] also studied the Weibull Frechet. A novel distribution family was introduced by [4] in their recent study, which they have named Type I Half-Logistic Exponentiated-G (TIHLEt-G). This distribution family is characterized by two positive shape parameters, denoted by λ and α, and can be applied to any arbitrary cumulative distribution function (cdf) $H\left(x,\vartheta\right)$. The cumulative distribution function (cdf) and the probability density function (pdf) for TIHLEt-G are given by

$$F_{TIHLEt-G}(x;\lambda,\alpha,\vartheta)=\frac{1-[1-H^{\alpha}(x;\vartheta)]^{\lambda}}{1+[1-H^{\alpha}(x;\vartheta)]^{\lambda}},\quad x>0,\ \ \lambda,\alpha>0 \tag{1}$$

and

$$f_{TIHLEt-G}(x;\lambda,\alpha,\vartheta)=\frac{2\lambda\alpha h(x;\vartheta)H^{\alpha-1}_{(x;\vartheta)}[1-H^{\alpha}_{(x;\vartheta)}]^{\lambda-1}}{\left[1+[1-H^{\alpha}_{(x;\vartheta)}]^{\lambda}\right]^{2}},x>0,\ \lambda,\alpha>0 \tag{2}$$

The TIHLEt-G family of distributions is noteworthy due to several factors, as explained in [4]. This family of distributions offers increased flexibility in terms of kurtosis compared to conventional models. It also enables the creation of skewed distributions even for symmetrical ones and can produce heavy-tailed distributions that better fit real data. The TIHLEt-G family can generate symmetric, left-skewed, right-skewed, and reversed-J shaped distributions and allows for special models with different types of hazard rate functions.
The Frechet distribution's cdf and pdf are provided as

$$H(x;\theta,\delta)=e^{-(\frac{\theta}{x})^{\delta}}\quad,\quad x>0,\theta,\delta>0 \tag{3}$$

$$h(x;\theta,\delta)=\delta\theta^{\delta}x^{-\delta-1}e^{-(\frac{\theta}{x})^{\delta}}\quad,\quad x>0,\theta,\delta>0 \tag{4}$$

The rest of the paper is organized into several sections. The second section describes the materials and a method used to drive the pdf and cdf of Type I Half-Logistic Exponentiated Frechet (TIHLEtF) Distribution and presents the expansion of the density. In the third section, we explore the statistical properties of the distribution, including its moment, moment-generating function, probability-weighted moment, reliability function, hazard function, and quantile function. The fourth section derives order statistics. The fifth section describes how maximum likelihood estimation is used to estimate the unknown model parameters and presents the results of a simulation study. The sixth section demonstrates the flexibility of the TIHLEtF distribution using two real data sets. Finally, the paper concludes with a summary of the findings and some closing remarks in the seventh section.

II. The Type I Half-Logistic Exponentiated Frechet (TIHLEtF) Distribution

A new model called the TIHLEtF model is introduced, where a random variable X is said to follow the TIHLEtF model if its cumulative distribution function (cdf) is obtained by using equation (3) in equation (1), which is defined as follows:

$$F_{TIHLEtF}(x;\lambda,\alpha,\theta,\delta)=\frac{1-\left[1-\left(e^{-\alpha(\frac{\theta}{x})^{\delta}}\right)\right]^{\lambda}}{1+\left[1-\left(e^{-\alpha(\frac{\theta}{x})^{\delta}}\right)\right]^{\lambda}}\text{ , } x>0,\lambda,\alpha,\theta,\delta>0 \tag{5}$$

and the pdf corresponding to equation (6) is

$$f_{TIHLEtF}(x;\lambda,\alpha,\theta,\delta)=2\lambda\alpha\delta\theta^{\delta}x^{-\delta-1}e^{-(\frac{\theta}{x})^{\delta}}e^{-(\alpha-1)(\frac{\theta}{x})^{\delta}}\left[1-e^{-\alpha(\frac{\theta}{x})^{\delta}}\right]^{\lambda-1}\left[1+\left[1-e^{-\alpha(\frac{\theta}{x})^{\delta}}\right]^{\lambda}\right]^{-2} \tag{6}$$

where θ is a scale parameter and λ,α,δ are shape parameters.

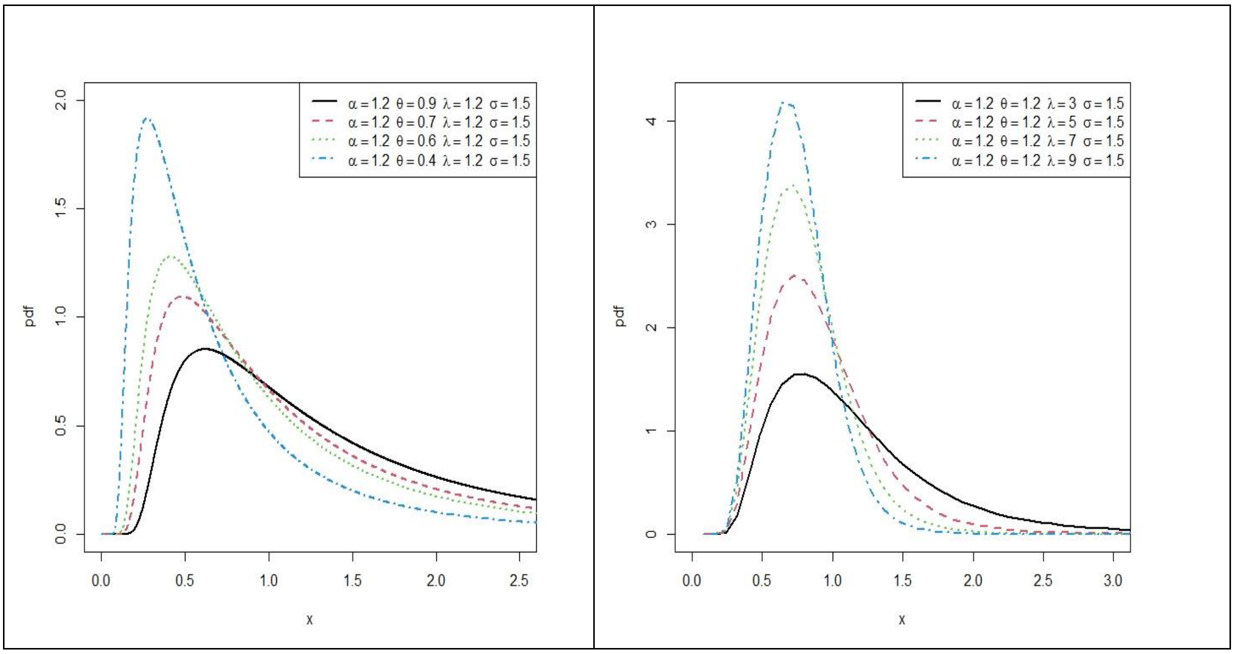

Figure 1: *Plots of pdf of TIHLEtF distribution for different values of parameters*

III. Expansion of Density for TIHLEtF Distribution

In this section, we present an advantageous expansion of the pdf and cdf for the TIHLEtF distribution. As a result of the generalized binomial series being

$$\left(1+z\right)^{-b}=\sum_{i=0}^{\infty}(-1)^{i}\binom{b+i-1}{i}z^{i} \tag{7}$$

Moreover, by applying the binomial theorem from equation (7) to equation (6)

$$f_{TIHLEtF}(x;\lambda,\alpha,\theta,\delta)=2\lambda\alpha\delta\theta^{\delta}x^{-\delta-1}e^{-(\frac{\theta}{x})^{\delta}}e^{-(\alpha-1)(\frac{\theta}{x})^{\delta}}\sum_{i=0}^{\infty}(-1)^{i}\binom{1+i}{i}\left[1-\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{\alpha}\right]^{\lambda(i+1)-1}$$

also

$$\left[1-\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{\alpha}\right]^{\lambda(i+1)-1} = \sum_{j=0}^{\infty}(-1)^{j}\binom{\lambda(i+1)-1}{j}\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{\alpha i}$$

Now, the pdf can be written as

$$f_{TIHLEtF}(x;\lambda,\alpha,\theta,\delta) = \sum_{i,j=0}^{\infty}\eta_{p}\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{\alpha(j+1)} \tag{8}$$

where, $\eta_{p} = 2\lambda\alpha\delta\theta^{\delta}x^{-\delta-1}(-1)^{i+j}\binom{i+1}{i}\binom{\lambda(i+1)-1}{j}$

Furthermore, an expansion for the $\left[F_{TIHLEtF}(x;\lambda,\alpha,\theta,\delta)\right]^{h}$ is produced, with h being an integer, and the binomial expansion is worked out once more.

$$\left[F_{TIHLEtF}(x;\lambda,\alpha,\theta,\delta)\right]^{h} = \sum_{p,m=0}^{h}(-1)^{p+m}\binom{h}{m}\binom{h+p-1}{p}\left[1-\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{\alpha}\right]^{\lambda(p+m)}$$

Consider

$$\left[1-\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{\alpha}\right]^{\lambda(p+m)} = \sum_{z=0}^{\infty}(-1)^{z}\binom{\lambda(p+m)}{z}\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{\alpha z}$$

The cdf can be written as:

$$\left[F_{TIHLEtF}(x;\lambda,\alpha,\theta,\delta)\right]^{h} = \sum_{p,m=0}^{h}\varphi_{t}\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{\alpha z} \tag{9}$$

where, $\varphi_{t} = \sum_{z=0}^{\infty}(-1)^{p+m+z}\binom{h+p-1}{p}\binom{h}{m}\binom{\lambda(m+p)}{z}$

IV. Statistical Properties

We derived some statistical properties of the new distribution.

I. Probability weighted moments

The probability-weighted moments (PWMs) were introduced by [11]. It is used to derive inverse form estimators for the parameters and quantiles of a distribution. The PWMs, is denoted by $K_{r,s}$ which can be derived for a random variable X using the following affiliations.

$$K_{r,s} = E[X^{r}F(x)^{s}] = \int_{0}^{\infty}x^{r}f(x)F(x)^{s}dx \tag{10}$$

The PWMs of TIHLEtF distribution is developed by substituting (8) and (9) into (10), and substituting h with s, as proceed

$$K_{r,s} = \sum_{i,j=0}^{\infty} \sum_{p,m=0}^{s} \eta_p \varphi_t \int_0^{\infty} x^r \left[e^{-(\frac{\theta}{x})^{\delta}} \right]^{\alpha(j+1+z)} dx \tag{11}$$

Consider the integral

$$\int_0^{\infty} x^r \left[e^{-(\frac{\theta}{x})^{\delta}} \right]^{\alpha(j+1+z)} dx$$

$$\text{Let } y = \alpha(j+1+z)(\frac{\theta}{x})^{\delta} \Rightarrow x = \left[\frac{\alpha(j+1+z)\theta^{\delta}}{y} \right]^{\frac{1}{\delta}}; dx = \frac{dy x^{\delta-1}}{\delta\theta^{\delta}\alpha(j+1+z)}$$

Then

$$\int_0^{\infty} \left[\frac{\alpha(j+1+z)\theta^{\delta}}{y} \right]^{\frac{r}{\delta}} e^{-y} \frac{dy x^{\delta-1}}{\delta\theta^{\delta}\alpha(j+1+z)} = \int_0^{\infty} y^{-\frac{r}{\delta}} e^{-y} dy = \Gamma(1-\frac{r}{\delta})$$

The PWMs of TIHLEtF can be written as proceed

$$K_{r,s} = \sum_{i,j=0}^{\infty} \sum_{p,m=0}^{s} (\alpha)^{\frac{r}{\delta}} \theta^r (j+1+z)^{\frac{r}{\delta}-1} \eta_p \varphi_t \Gamma(1-\frac{r}{\delta}) \tag{12}$$

now

$$\varphi_t = \sum_{z,q=0}^{\infty} (-1)^{p+m+z+q} \binom{s+p-1}{p} \binom{s}{m} \binom{\lambda(m+p)}{z}$$

and

$$\eta_p = 2\lambda(-1)^{i+j} \binom{i+1}{i} \binom{\lambda(i+1)-1}{j}$$

II. Moments

Moments are fundamental to all statistical analysis, particularly in applications. So, for the new distribution, we determine the r[th] moment.

$$E(X^r) = \int_0^{\infty} x^r f(x) dx \tag{13}$$

Using the important representation of the pdf as shown in equation (8), we have

$$E(X^r) = \sum_{i,j=0}^{\infty} \eta_p \int_0^{\infty} x^r \left[e^{-(\frac{\theta}{x})^{\delta}} \right]^{\alpha(j+1)} dx \tag{14}$$

Consider the integral

$$\int_0^{\infty} x^r \left[e^{-(\frac{\theta}{x})^{\delta}} \right]^{\alpha(j+1)} dx$$

$$\text{Let } w = \alpha(j+1)(\frac{\theta}{x})^{\delta} \Rightarrow x = \left[\frac{\alpha(j+1)\theta^{\delta}}{y} \right]^{\frac{1}{\delta}}; dx = \frac{dw x^{\delta-1}}{\alpha(j+1)\theta^{\delta}\delta}$$

Then

$$\int_0^{\infty} \left[\frac{\alpha(j+1)\theta^{\delta}}{y} \right]^{\frac{r}{\delta}} e^{-w} \frac{dw x^{\delta-1}}{\alpha(j+1)\theta^{\delta}\delta} = \theta^r \alpha^{\frac{r}{\delta}} (j+1)^{\frac{r}{\delta}-1} \int_0^{\infty} w^{-\frac{r}{\delta}} e^{-w} dw = \int_0^{\infty} w^{-\frac{r}{\delta}} e^{-w} dw = \Gamma(1-\frac{r}{\delta})$$

The r[th] moment of the TIHLEtF distribution can be expressed as follows

$$E(X^r) = \sum_{i,j=0}^{\infty} \eta_p \theta^r \alpha^{\frac{r}{\delta}} (j+1)^{\frac{r}{\delta}-1} \Gamma(1-\frac{r}{\delta}) \tag{15}$$

Now

$$\eta_p = 2\lambda\alpha(-1)^{i+j} \binom{i+1}{i} \binom{\lambda(i+1)-1}{j}$$

The expected value and the spread of the TIHLEtF distribution can be described by the following

$$E(X) = \sum_{i,j=0}^{\infty} \eta_p \theta \alpha^{\frac{1}{\delta}} (j+1)^{\frac{1}{\delta}-1} \Gamma(1-\frac{1}{\delta})$$

and

$$\operatorname{var}(x) = \sum_{i,j=0}^{\infty} \eta_p \theta \alpha^{\frac{1}{\delta}} (j+1)^{\frac{1}{\delta}-1} \Gamma(1-\frac{1}{\delta}) - \left[\sum_{i,j=0}^{\infty} \eta_p \theta \alpha^{\frac{1}{\delta}} (j+1)^{\frac{1}{\delta}-1} \Gamma(1-\frac{1}{\delta}) \right]^2$$

III. Moment-generating function (mgf)

The Moment-Generating Function of x is expressed as follows:

$$M_x(t) = \int_0^{\infty} e^{tx} f(x) dx \tag{16}$$

where the expansion of $e^{tx} = \sum_{m=0}^{\infty} \frac{(tx)^m}{m!}$

The moment-generating function of TIHLEtF distribution can be represented as follows

$$M_x(t) = \sum_{i,j=0}^{\infty} \sum_{m=0}^{\infty} \frac{t^m \eta_p \theta^m \alpha^{\frac{m}{\delta}} (j+1)^{\frac{m}{\delta}-1} \Gamma(1-\frac{m}{\delta})}{m!} \tag{17}$$

IV. Reliability function

The reliability function provides the likelihood that an individual will endure beyond a designated time frame. This function is defined as follows:

$$R(x;\lambda,\alpha,\theta,\delta) = \frac{2\left[1-\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{\alpha}\right]^{\lambda}}{1+\left[1-\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{\alpha}\right]^{\lambda}} \tag{18}$$

V. Hazard function

The hazard function represents the likelihood of the event of interest happening within a relatively brief time period. It can be defined as follows:

$$T(x;\lambda,\alpha,\theta,\delta)=\frac{\lambda\alpha\delta\theta^{\delta}x^{-\delta-1}e^{-(\frac{\theta}{x})^{\delta}}\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{\alpha-1}}{\left[1+\left[1-\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{\alpha}\right]^{\lambda}\right]\left[1-\left[e^{-(\frac{\theta}{x})^{\delta}}\right]^{\alpha}\right]} \tag{19}$$

VI. Quantile Function

The quantile function, also known as the inverse CDF, of the TIHLEtF distribution is obtained by using the CDF in equation (5)

$$x=Q(u)=\frac{\theta}{\left[-\log\left[1-\left[\frac{1-U}{U+1}\right]^{\frac{1}{\lambda}}\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{\delta}}} \tag{20}$$

The median of the TIHLEtF distribution can be derived by substituting U=0.5 in equation (20) as follows:

$$\text{median}=Q(0.5)=\frac{\theta}{\left[-\log\left[1-\left[\frac{1-0.5}{0.5+1}\right]^{\frac{1}{\lambda}}\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{\delta}}} \tag{21}$$

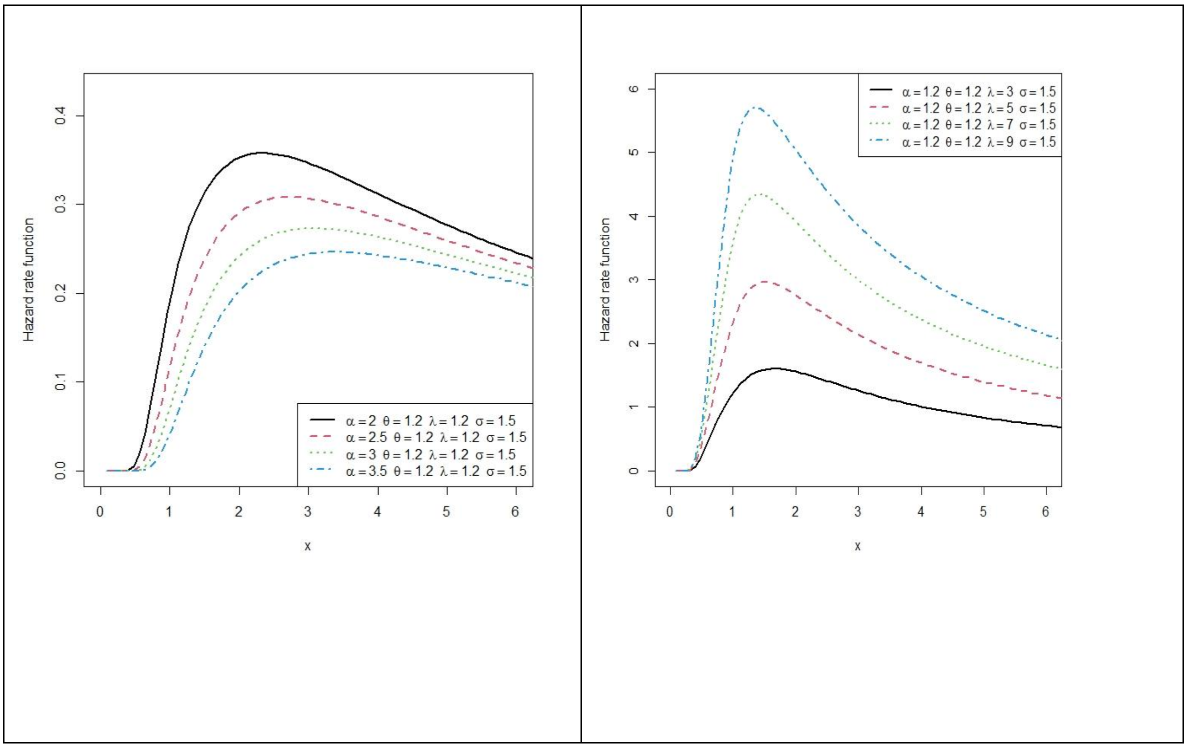

Figure 2: *Plots of the hazard function of the TIHLEtF distribution for different parameter values.*

VII. Order Statistics

Let $X_1, X_2, ..., X_3$ be independent and identically distributed (i.i.d) random variables with their corresponding continuous distribution function $F(x)$. Let $X_{1:n} < X_{2:n} < ... < X_{n:n}$ the corresponding ordered random sample from the TIHLEtF distributions. Let $F_{r:n}(x)$ and $f_{r:n}(x)$, $r = 1, 2, 3, ..., n$ denote the CDF and PDF of the r^{th} order statistics $X_{r:n}$ respectively. The PDF of the r^{th} order statistics of $X_{r:n}$ is given as:

$$f_{r:n}(x;\lambda,\alpha,\theta) = \frac{f(x)}{B(r,n-r+1)} \sum_{v=0}^{n-r} (-1)^v \binom{n-r}{v} \left[F(x)\right]^{v+r-1} \tag{22}$$

The PDF of r^{th} order statistic for TIHLEtF distribution is derived by substituting equation (8) and equation (9) into equation (22). Also replacing h with $v + r - 1$ in equation (9), so we

$$f_{r:n}(x;\lambda,\alpha,\theta,\delta) = \frac{1}{B(r,n-r+1)} \sum_{v=0}^{n-r} \sum_{i,j=0}^{\infty} \sum_{p,m=0}^{r+v-1} (-1)^v \binom{n-r}{v} \eta_p \varphi_t \left[e^{-(\frac{\theta}{x})^{\delta}} \right]^{\alpha(j+i+z)} \tag{23}$$

The PDF of minimum order statistic of the TIHLEtF distribution is obtained by setting r = 1 in equation (23) as

$$\begin{aligned} f_{1:n}(x;\lambda,\alpha,\theta,\delta) = 2n\lambda\alpha\delta\theta^{\delta} x^{-\delta-1} \sum_{v=0}^{n-1} \sum_{i,j=0}^{\infty} \sum_{p,m=0}^{v} (-1)^{i+j} (-1)^{p+m+z} (-1)^v \binom{n-1}{v} \binom{i+1}{i} \\ \binom{\lambda(i+1)-1}{j} \binom{v+p-1}{p} \binom{v}{m} \binom{\lambda(m+p)}{z} \left[e^{-(\frac{\theta}{x})^{\delta}} \right]^{\alpha(j+i+z)} \end{aligned} \tag{24}$$

Also, the PDF of maximum order statistic of the TIHLEtF distribution is obtained by setting r = n in equation (23) as

$$\begin{aligned} f_{n:n}(x;\lambda,\alpha,\theta,\delta) = 2n\lambda\alpha\delta\theta^{\delta} x^{-\delta-1} \sum_{i,j=0}^{\infty} \sum_{p,m=0}^{n+v-1} (-1)^{i+j} (-1)^{p+m+z} (-1)^v \binom{i+1}{i} \binom{\lambda(i+1)-1}{j} \\ \binom{n+v+p-2}{p} \binom{n+v-1}{m} \binom{\lambda(m+p)}{z} \left[e^{-(\frac{\theta}{x})^{\delta}} \right]^{\alpha(j+i+z)} \end{aligned} \tag{25}$$

V. Parameter Estimation

Given complete data, we investigate the maximum likelihood method to estimate the TIHLEtF distribution's unknown parameters. Maximum likelihood estimates (MLEs) are attractive because they can be used to produce confidence intervals and offer straightforward approximations that work well in finite samples. The resulting approximation for MLEs is simple to handle in distribution theory, both analytically and numerically. Let $x_1, x_2, x_3, ..., x_n$ be a random sample of size n from the TIHLEtF distribution. Then, the likelihood function based on the observed sample for the vector of the parameter $(\lambda, \alpha, \theta, \delta)^T$ is given by

$$\log L = n\log(2) + n\log(\lambda) + n\log(\alpha) + n\log(\delta) + n\delta\log(\theta) - (\delta-1)\sum_{i=1}^{n}\log(x_i)$$
$$-\alpha\sum_{i=1}^{n}(\frac{\theta}{x_i})^{\delta} + (\lambda-1)\sum_{i=1}^{n}\log\left[1-\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha}\right] - 2\sum_{i=1}^{n}\log\left[1+\left[1-\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha}\right]^{\lambda}\right] \tag{26}$$

The components of the score vector $\Delta L(\phi) = \left(\frac{\partial L(\phi)}{\partial\lambda}, \frac{\partial L(\phi)}{\partial\alpha}, \frac{\partial L(\phi)}{\partial\theta}, \frac{\partial L(\phi)}{\partial\delta}\right)^{T}$ are given as

$$\frac{\partial \log L}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\log\left[1-\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha}\right] - 2\sum_{i=1}^{n}\frac{\left[1-\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha}\right]^{\lambda}\log\left[1-\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha}\right]}{1+\left[1-\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha}\right]^{\lambda}} = 0 \tag{27}$$

$$\frac{\partial \log L}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}(\frac{\theta}{x_i})^{\delta} + (\lambda-1)\sum_{i=1}^{n}\frac{\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha}\log\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]}{1-\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha}}$$
$$+2\sum_{i=1}^{n}\frac{\lambda\left[1-\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha}\right]^{\lambda-1}\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha}\log\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]}{1+\left[1-\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha}\right]^{\lambda}} = 0 \tag{28}$$

$$\frac{\partial \log L}{\partial\delta} = \frac{n}{\delta} + n\log(\theta) + \sum_{i=1}^{n}\log(x_i) - \alpha\sum_{i=1}^{n}(\frac{\theta}{x_i})^{\delta}\log(\frac{\theta}{x_i})$$
$$+(\lambda-1)\sum_{i=1}^{n}\frac{\alpha\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha-1} e^{-(\frac{\theta}{x_i})^{\delta}}(\frac{\theta}{x_i})^{\delta}\log(\frac{\theta}{x_i})}{1-\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha}}$$
$$-2\sum_{i=1}^{n}\frac{\lambda\left[1-\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha}\right]^{\lambda-1}\alpha\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha-1} e^{-(\frac{\theta}{x_i})^{\delta}}(\frac{\theta}{x_i})^{\delta}\log(\frac{\theta}{x_i})}{1+\left[1-\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha}\right]^{\lambda}} = 0 \tag{29}$$

$$\frac{\partial \log L}{\partial \theta}=\frac{n\delta}{\theta}-\alpha\delta\sum_{i=1}^{n}(\frac{\theta}{x_i})^{\delta-1}\frac{1}{x_i}+(\lambda-1)\sum_{i=1}^{n}\frac{\alpha\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha-1}e^{-(\frac{\theta}{x_i})^{\delta}}\delta(\frac{\theta}{x_i})^{\delta-1}\frac{1}{x_i}}{1-\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha}}$$

$$-2\sum_{i=1}^{n}\frac{\lambda\left[1-\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha}\right]^{\lambda-1}\alpha\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha-1}e^{-(\frac{\theta}{x_i})^{\delta}}\delta(\frac{\theta}{x_i})^{\delta-1}\frac{1}{x_i}}{1+\left[1-\left[e^{-(\frac{\theta}{x_i})^{\delta}}\right]^{\alpha}\right]^{\lambda}}=0 \quad (30)$$

The MLEs are obtained by setting $\frac{\partial L(\phi)}{\partial \lambda},\frac{\partial L(\phi)}{\partial \alpha},\frac{\partial L(\phi)}{\partial \theta}$ and $\frac{\partial L(\phi)}{\partial \delta}$ to zero and solving these equations simultaneously. These equations cannot be solved analytically, so we have to appeal to numerical method.

I. Simulation Study

In this section, a numerical analysis will be conducted to evaluate the performance of MLE for TIHLEtF Distribution.

Table 1: *MLEs, biases and RMSE for some values of parameters*

		(1.5,1,2,1.5)			(2,1,3,2.5)		
n	Parameters	Estimated Values	Bias	RMSE	Estimated Values	Bias	RMSE
	λ	1.5650	0.0650	0.7475	2.1548	0.1548	0.9708
20	α	1.0471	0.0471	0.5045	1.0385	0.0385	0.4666
	θ	2.0348	0.0348	0.3533	3.0465	0.0465	0.3439
	δ	1.7878	0.2878	0.7086	2.8275	0.3275	0.9009
	λ	1.5507	0.0507	0.5664	2.1074	0.1074	0.7485
50	α	1.0106	0.0106	0.3523	1.0073	0.0073	0.3337
	θ	2.0310	0.0310	0.2845	3.0360	0.0360	0.2589
	δ	1.6328	0.1328	0.4472	2.6418	0.1418	0.5924
100	λ	1.5251	0.0251	0.4358	2.0546	0.0546	0.5630
	α	1.0024	0.0024	0.2587	1.0024	0.0024	0.2520
	θ	2.0271	0.0271	0.2265	3.0456	0.0456	0.1937
	δ	1.5729	0.0729	0.2982	2.5830	0.0830	0.4288
250	λ	1.5391	0.0391	0.3064	2.0524	0.0524	0.3855
	α	1.0018	0.0018	0.1615	1.0011	0.0011	0.1426
	θ	2.0280	0.0280	0.1695	3.0313	0.0313	0.1479
	δ	1.5153	0.0153	0.1928	2.5129	0.0129	0.2808
500	λ	1.5269	0.0269	0.2273	2.0285	0.0285	0.2806
	α	1.0004	0.0004	0.1024	1.0002	0.0002	0.0983
	θ	2.0284	0.0284	0.1306	3.0251	0.0251	0.1162
	δ	1.5034	0.0034	0.1372	2.5062	0.0062	0.2051
1000	λ	1.5215	0.0215	0.1693	2.0275	0.0275	0.2059

α	1.0001	0.0001	0.0729	1.0000	0.0000	0.0693
θ	2.0182	0.0182	0.1014	3.0206	0.0206	0.0868
δ	1.5012	0.0012	0.1005	2.5013	0.0013	0.1446

The presented table demonstrates that the biases and RMSE values converge towards zero. As the sample size increases, the estimations approach the original (true) values, indicating the efficiency and reliability of the estimates.

VI. Applications to Real Data

We fit the TIHLEtF distribution to two real data sets and give a comparative study with the fits to the Exponentiated Half-Logistic Frechet (EHLF) distribution by [6], Kumaraswamy Exponentiated Frechet (KExF) distribution by [8], Gompertz Frechet (GoFr) distribution by [18], Exponentiated Frechet (ExFr) distribution by [16], and Frechet distribution by [10] as comparator distributions for illustrative purposes.

The EHLF distribution by [6]

$$f(x;\alpha,\lambda,\theta,\beta)=2\alpha\lambda\theta\beta^{\theta}x^{-(\theta+1)}e^{\left[-\left(\frac{\beta}{x}\right)^{\theta}\right]}\left[1-e^{\left[-\left(\frac{\beta}{x}\right)^{\theta}\right]}\right]^{\lambda-1}\left[1-\left[1-e^{\left[-\left(\frac{\beta}{x}\right)^{\theta}\right]}\right]^{\lambda}\right]^{\alpha-1}\left[1+\left[1-e^{\left[-\left(\frac{\beta}{x}\right)^{\theta}\right]}\right]^{\lambda}\right]^{-(\alpha+1)} \tag{31}$$

The KExF distribution by [8]

$$f(x;\alpha,\sigma,\lambda,\theta,\beta)=\alpha\lambda\sigma\beta\theta^{\lambda}x^{-(1+\lambda)}e^{-(\frac{\theta}{x})^{\lambda}}\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha-1}\left[1-\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha}\right]^{\sigma-1}\left[1-\left[1-\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha}\right]^{\sigma}\right]^{\beta-1} \tag{32}$$

The GoFr distribution by [18]

$$f(x;\alpha,\lambda,\theta,\beta)=\theta\beta\alpha^{\beta}x^{-\beta-1}e^{\left[-(\frac{\alpha}{x})^{\beta}\right]}\left[e^{\left[-(\frac{\alpha}{x})^{\beta}\right]}\right]^{\lambda-1}e^{\left[\frac{\theta}{\lambda}\left(1-\left[1-e^{\left[-(\frac{\alpha}{x})^{\beta}\right]}\right]\right)^{-\lambda}\right]} \tag{33}$$

The ExFr distribution by [16]

$$f(x;\alpha,\lambda,\sigma)=\alpha\lambda\sigma^{\lambda}\left[1-e^{\left(-(\frac{\sigma}{x})^{\lambda}\right)}\right]^{\alpha-1}x^{-(1+\lambda)}e^{\left(-(\frac{\sigma}{x})^{\lambda}\right)} \tag{34}$$

The Fr distribution by [10]

$$f(x;\theta,\sigma)=\delta\theta^{\sigma}x^{-\sigma-1}e^{-(\frac{\theta}{x})^{\sigma}} \tag{35}$$

The two datasets used to illustrate the application offer practical proof of the adaptability and appropriateness of the new proposed distribution. This distribution is shown to be the optimal selection for modeling the datasets, distinct from the comparator distributions mentioned earlier. All calculations were conducted using the R programming language.

Data set 1

The first data set shown below represents the strength of carbon fibers tested under tension at gauge lengths of 10mm, previously used by [5]:

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

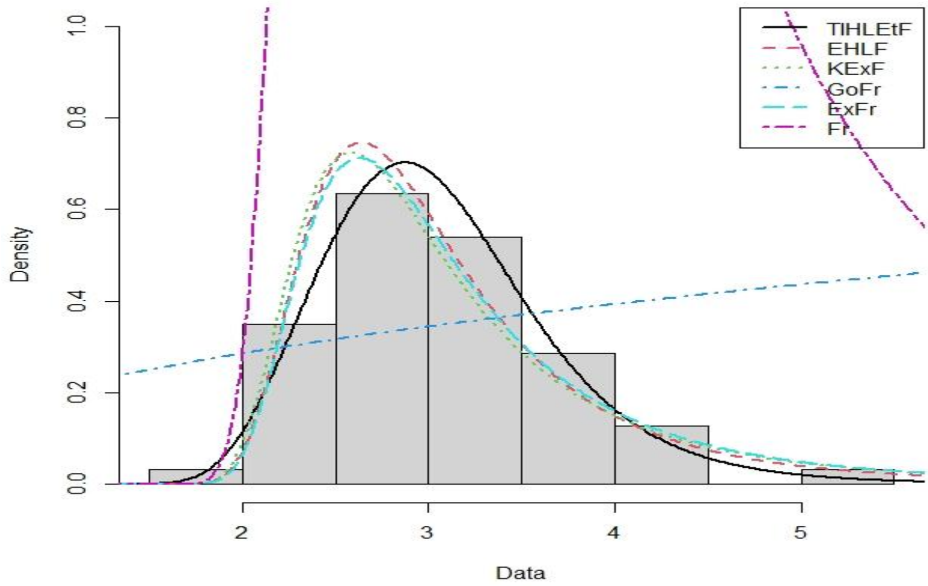

Figure 3: *Fitted pdfs for the TIHLEtF, EHLF, KExF, GoFr, ExFr and Fr distributions to the data set 1*

Table 2: *The MLEs, log-likelihoods, and goodness of fit statistics of the models are based on the strength of carbon fibers tested under tension at gauge lengths of 10 mm (Data set 1).*

Distributions	α	λ	θ	δ	β	LL	AIC
TIHLEtF	1.2456	3.7093	3.0756	3.1438		-56.4277	120.8555
EHLF	22.4675	0.2832	19.1580		1.3491	-58.9236	125.8472
KExF	2.4724	3.7429	2.0495	9.6236	0.4923	-60.2204	130.4407
GoFr	5.3750	2.7756	3.3750		6.3750	-67.2387	142.4774
ExFr	0.7937	5.8763		2.6529		-69.9499	145.8998
Fr			2.7215	5.4345		-79.5116	155.0232

Maximum likelihood was employed to estimate the parameters of both the newly developed distribution and five other comparable distributions. The outcomes are presented in Table 2. Extremely, the newly proposed distribution exhibited the lowest AIC value based on the goodness of fit measure, closely followed by EHLF. Furthermore, the superiority of the proposed distribution over its competitors is reinforced by visually assessing the fit, as illustrated in Figure 3. This underscores the fact that the newly recommended distribution is the most appropriate choice for accurately representing the carbon fiber dataset among the mentioned distributions.

Data set 2

The second data set shown below represents the survival times of one hundred and twenty-one (121) patients with breast cancer obtained from a large hospital in a period from 1929 to 1938, previously used by [19]:

0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 1.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0.

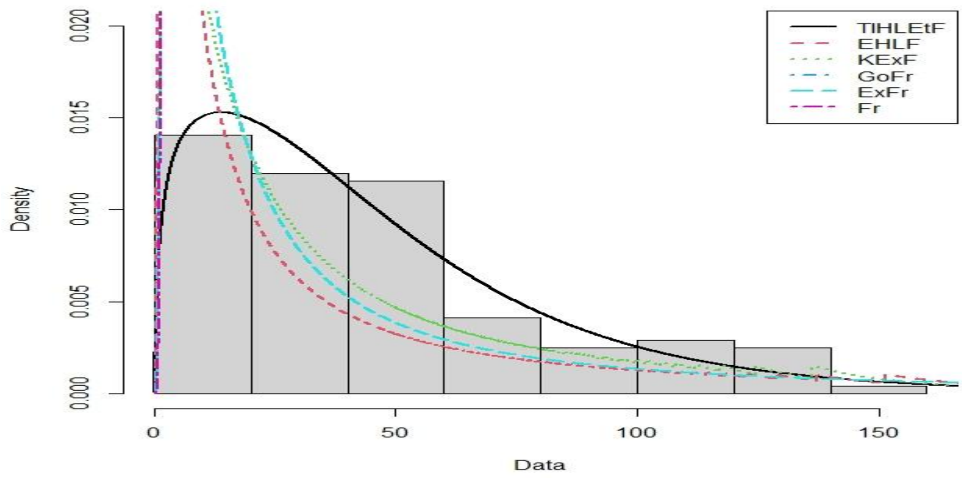

Figure 4: *Fitted pdfs for the TIHLEtF, EHLF, KExF, GoFr, ExFr and Fr distributions to the data set 2*

Table 3: *The MLEs, log-likelihoods, and goodness of fit statistics of the models based on the survival time of patients with breast cancer (Data 2).*

Distributions	α	λ	θ	δ	β	LL	AIC
TIHLEtF	4.6372	6.6628	3.7129	0.4224		- 599.0163	1206.033
EHLF	6.0551	0.1075	5.2159		0.1429	- 644.6529	1297.306
KExF	0.0343	5.5835	0.2036	5.9722	13.1474	- 619.2298	1248.46
GoFr	41.0479	59.0871	0.2511		0.5019	-1311.34	2630.68
ExFr	1.3538	0.6216		18.0606		- 632.1718	1270.344
Fr			16.8934	0.6524		-695.0586	1394.117

The parameters of the TIHLEtF distribution and five similar distributions were estimated using the maximum likelihood method, and the outcomes are detailed in Table 3. It's worth highlighting that the new distribution displayed the lowest AIC value when compared to the other options, which signifies that it is the most appropriate choice for modeling the survival time of patients with breast cancer based on the goodness of fit measure AIC. Additionally, a visual examination of the fit, as illustrated in Figure 4, reinforces the idea that the new distribution outperforms its rivals.

VII. CONCLUSION

In our research, we have introduced and explored a novel statistical distribution known as the Type I Half-Logistic Exponentiated Frechet Distribution, building upon the distribution family introduced by [4]. This study delved into various statistical aspects of this newly introduced distribution. These aspects encompassed explicit quantile functions, probability-weighted moments, moments, moments generating functions, reliability functions, hazard functions, and order statistics. For parameter estimation, we utilized the maximum likelihood method. Our analysis also included simulations to assess the effectiveness of this newly proposed distribution. To underscore the significance and versatility of this innovative distribution, we compared it with well-established models using two authentic datasets. The outcomes of our investigation underscore the superiority of the new distribution in comparison to the other models under consideration. This implies that the proposed distribution shows promise for effectively modeling data across various applications.

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