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A MODIFIED AILAMUJIA DISTRIBUTION: PROPERTIES AND APPLICATION

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Abstract

This study presents a modified one-parameter Ailamujia distribution called the Entropy Transformed Ailamujia distribution (ETAD) is introduced to handle both symmetric and asymmetric lifetime data sets. The ETAD properties like order and reliability statistics, entropy, moment and moment generating function, quantile function, and its variability measures were derived. The maximum likelihood estimation (MLE) method was used in estimating the parameter of ETAD and through simulation at different sample sizes, the MLE was found to be consistent, efficient, and unbiased for estimating the ETAD parameter. The flexibility of ETAD was shown by fitting it to six different real lifetime data sets and compared it alongside seven competing one-parameter distributions. The goodness of fit (GOF) results from Akaike information criteria, Bayesian information criteria, corrected Akaike information criteria, and Hannan-Quinn information criteria show that the ETAD was the best fit amongst all the seven competing distributions across all the six data sets.

Keywords: Ailamujia distribution, Entropy Transformation, Maximum likelihood estimation, Goodness of fit, Information criteria

I. Introduction

Modelling, organizing, as well as analyzing real life phenomenon is essential in the field of applied sciences such as engineering, medical sciences, finance, architecture, amongst others. Recently, progressive hybrid censoring scheme is becoming popular in a life testing problems and reliability analysis. Since there are so many data generation processes characterized under different systems and environments, no single probability model can perfectly be used to describe and model all the phenomena. Therefore, the consistency and accuracy of statistical analysis in highly influenced by the probability distribution or model adopted {[1], [2], [3], and [4]}. One remarkable method is accomplished by adding parameter(s) to an existing/traditional distribution [5]. Also, [6] introduced a new dimension of distribution, which discussed a new family of distribution, namely the Exponentiated Exponential distribution. Kumaraswamy-G proposed by [7] while [8], [9], and [10] introduced Logistic-X, New Sine Inverse Rayleigh, and New Sine Inverted Exponential distributions, respectively. In each of the methods listed above, at least one parameter is added to the existing distribution which thus adds to the complexity in obtaining the estimates of the parameters for the new model. Considering the associated complexities, transformation techniques were recommended which do not require the addition of parameters to the baseline distribution to modify its flexibility and robustness [11]. In line with this system of modification, [12] proposed a probability distribution called the Entropy Transformed Weibull distribution and Entropy Transformed Rayleigh distribution was derived by [13]. In the same system of transformation, this paper introduced an Entropy Transformed Ailamujia Distribution (ETAD) which is a modification of the Ailamujia Distribution introduced by [14].

II. Methods

In this section, the ETAD and its application are presented. The Probability Density Function (PDF), Cumulative Distribution Function (CDF), reliability functions, variability measures and distribution shape, measures of uncertainty, and the estimation of the ETAD parameter using the maximum likelihood estimation technique are presented. Simulation and six (6) real life data sets are fitted to the ETAD and compared to seven (7) other competing one parameter distributions using the technique of information criteria.

I. Entropy Transformed Ailamujia Distribution (ETAD)

Soleha & Sewilam [15] considered a random variable X which represents the appropriate runtime for any component, and they introduced the following expression

$$g(x) = F(x) + R(x) lnR(x),\ x \ge 0 \tag{1}$$

where, $F(x)$ and $R(x)$ are the cumulative and reliability function, respectively, of a positive continuous random variable X. They called $g(x)$ an "Entropy Transformation" and this possibility is because the term $R(x)\ln R(x)$ is similar to the associated entropy expression for the density function of a continuous random variable X

$$H(f) = -\int_0^\infty f(x)\ln f(x)dx\,. \tag{2}$$

Differentiating equation (1) with respect to x the PDF is obtained, its integral gives the CDF, and fixing the range validates the PDF:

$$\int_0^\infty g'(x)dx = \int_0^\infty R'(x)\ln R(x)dx = 1 \tag{3}$$

The CDF of the random variable X of Ailamujia distribution (AD) is given by [14]:

$$F(x) = 1-(1+2\theta x)e^{-2\theta x},\ x>0, \theta>1 \tag{4}$$

The reliability function $R(x)$ of AD is given as

$$R(x) = 1-F(x) = (1+2\theta x)e^{-2\theta x},\ x>0, \theta>1\,. \tag{5}$$

Substituting equation (4) and (5) into equation (1) and take the derivative w.r.t x gives

$$g(x) = 1-(1+2\theta x)e^{-2\theta x} + (1+2\theta x)e^{-2\theta x}\ln\left((1+2\theta x)e^{-2\theta x}\right) \tag{6}$$

However, from equation (3) it can be easily seen that the differentiation of equation (6) can be easily obtained as

$$g'(x) = \left(\frac{d}{dx}\left[(1+2\theta x)e^{-2\theta x}\right]\right)\ln\left((1+2\theta x)e^{-2\theta x}\right) = \left[2\theta e^{-2\theta x} + (1+2\theta x)e^{-2\theta x}(-2\theta)\right]\ln\left((1+2\theta x)e^{-2\theta x}\right)$$

$$= \left(e^{-2\theta x}[2\theta - 2\theta - 4\theta^2 x]\right)\ln\left((1+2\theta x)e^{-2\theta x}\right) = -4\theta^2 x e^{-2\theta x}\ln\left((1+2\theta x)e^{-2\theta x}\right) \tag{7}$$

Let $f(x)$ be $g'(x)$ then

$$f(x) = -4\theta^2 x e^{-2\theta x}\ln\left((1+2\theta x)e^{-2\theta x}\right)\quad x>0, \theta>1$$

is the PDF of the ETAD. The corresponding CDF after integrating (7) is obtained as

$$F(x) = (2\theta x+1)e^{-2\theta x}\left(\ln(2\theta x+1) - 2\theta x - 1\right). \tag{8}$$

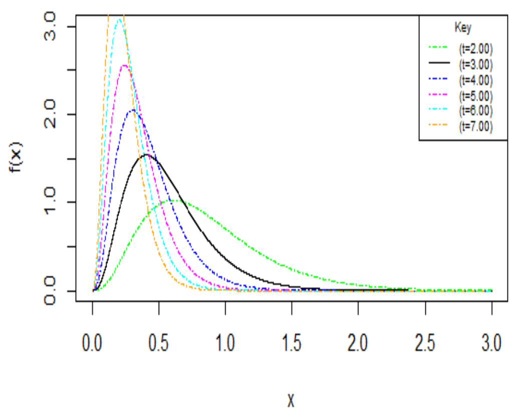
The PDF and CDF plots at different values of θ are presented in Figure 1 and Figure 2, respectively.

II. Linear Representation of ETAD

The ETAD is linearly represented for simplicity using the Taylor series expansion theorem at order four (4). Expanding the ln part of $f(x)$ and let it be represented by $t(x)$, thus

$$\frac{d}{dx}\left[t(x)\right]=t'(x)=\frac{d}{dx}\left[\ln\left((2\theta x+1)e^{-2\theta x}\right)\right]=\frac{1}{(2\theta x+1)e^{-2\theta x}}\frac{d}{dx}\left[(2\theta x+1)e^{-2\theta x}\right] \quad (9)$$

$$=\frac{\left(2\theta e^{-2\theta x}-2(2\theta x+1)\theta e^{-2\theta x}\cdot 1\right)e^{2\theta x}}{(2\theta x+1)e^{-2\theta x}}=\frac{\left(2\theta e^{-2\theta x}-2\theta(2\theta x+1)e^{-2\theta x}\right)}{2\theta x+1} \quad (10)$$



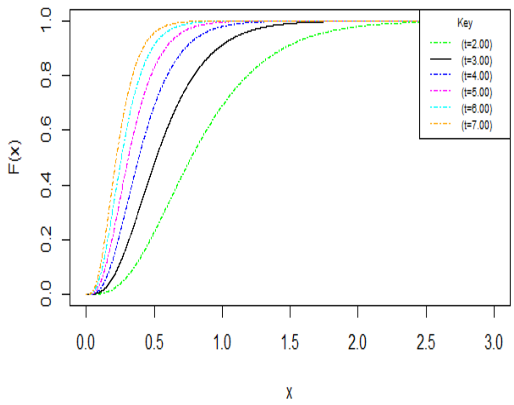


Figure 1: *PDF Plots of ETAD* **Figure 2:** *CDF Plots of ETAD*

By expansion equation (10) becomes

$$t'(x)=-\frac{4\theta^2 x}{2\theta x+1}=0 \quad (11)$$

$$\frac{d^2}{dx}=t''(x)=\frac{d}{dx}\left[-\frac{4\theta^2 x}{2\theta x+1}\right]=-4\theta^2\frac{d}{dx}\left[\frac{x}{2\theta x+1}\right]=\frac{4\theta^2\left((2\theta x+1)-2(2\theta)\right)}{(2\theta x+1)^2} \quad (12)$$

By expansion equation (12) becomes

$$t''(0)=\frac{4\theta^2}{(2\theta x+1)^2}=4\theta^2 \quad (13)$$

$$\frac{d^3}{dx^3}=t'''(x)=\frac{d}{dx}\left[\frac{-4\theta^2}{(2\theta x+1)^2}\right]=-4\theta^2\frac{d}{dx}\left[\frac{1}{(2\theta x+1)^2}\right]=-4\theta^2(-2)(2\theta x+1)^{-3}2\theta \quad (14)$$

$$=\frac{16\theta^3}{(2\theta x+1)^3} \quad (15)$$

By expansion equation (15) becomes

$$t'''(0)=16\theta^3 \quad (16)$$

$$\frac{d^4}{dx^4}=t''''(x)=\frac{d}{dx}\left[\frac{16\theta^3}{(2\theta x+1)^3}\right]=16\theta^3\frac{d}{dx}\left[\frac{1}{(2\theta x+1)^3}\right] \quad (17)$$

$$=16\theta^3(-3)(2\theta x+1)^{-4}2\theta=-\frac{96\theta^4}{(2\theta x+1)^4} \quad (18)$$

By expansion equation (18) becomes

$$t''''(\theta) = 96\theta^4 \tag{19}$$

Therefore,

$$t(0) = \frac{0x^0}{0!} + \frac{0x^1}{1!!} - 4\theta^2\frac{x^2}{2!} + 16\theta^3\frac{x3}{3!} - 96\theta^4\frac{x^4}{4!} + \cdots$$

$$\sum_{n=2}^{\infty} t(0) = -2\theta^2x^2 + 2\cdot 67\theta^3x^3 - 4\theta^4x^4 + \cdots \tag{20}$$

The approximate and nth series expansion of (20) are given respectively as

$$\sum_{n=2}^{\infty} t(0) = -2\theta^2x^2 + 3\theta^3x^3 - 4\theta^4x^4 + \cdots$$

and

$$\sum_{n=2}^{\infty} t(0) = \sum_{n=2}^{\infty}(2)(1.5)^{(n-2)}(-1)^{(n+1)}\theta^n x^n .$$

Therefore, the linear representation of ETAD is given as

$$f(x) = -4\theta^2 xe^{-2\theta x}\left[\sum_{n=2}^{\infty}(2)(1.5)^{(n-2)}(-1)^{(n+1)}\theta^n x^n\right] = -4\theta^2 xe^{-2\theta x}\left[G_n\theta^n x^n\right] = -4\theta^{2+n}G_n x^{1+n}e^{-2\theta x} \tag{21}$$

where, $G_n = \sum_{n=2}^{\infty}(2)(1.5)^{(n-2)}(-1)^{(n+1)}$.

III. Some Properties of ETAD

The survival $[S(x)]$, hazard $[h(x)]$, and odd $[O(x)]$ functions are presented as well as their respective plots follows.

$$S(x) = 1 - F(X \le x) = 1 - (2\theta x + 1)e^{-2\theta x}\left(\ln(2\theta x + 1) - 2\theta x - 1\right) \tag{22}$$

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{f(x)}{S(x)} = \frac{-4\theta^2 x\left[\ln(2\theta x + 1)e^{-2\theta x}\right]}{1 - (2\theta x + 1)e^{-2\theta x}\left(\ln(2\theta x + 1) - 2\theta x - 1\right)} \tag{23}$$

$$\textit{Odds function}: O(x) = \frac{F(x)}{1 - F(x)} = \frac{F(x)}{S(x)} = \frac{(2\theta x + 1)e^{-2\theta x}\left(\ln(2\theta x + 1) - 2\theta x - 1\right)}{1 - (2\theta x + 1)e^{-2\theta x}\left(\ln(2\theta x + 1) - 2\theta x - 1\right)} \tag{24}$$

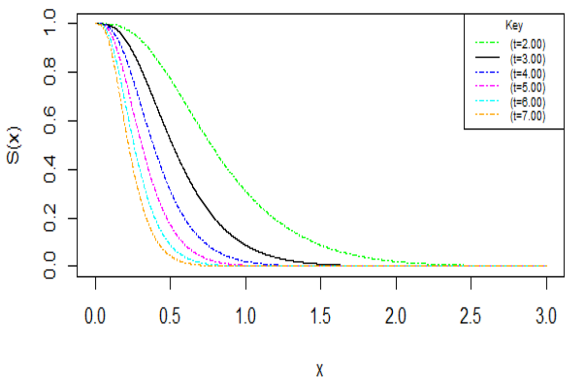


Figure 3: *Plot of S(x) at Different Parameter Values*

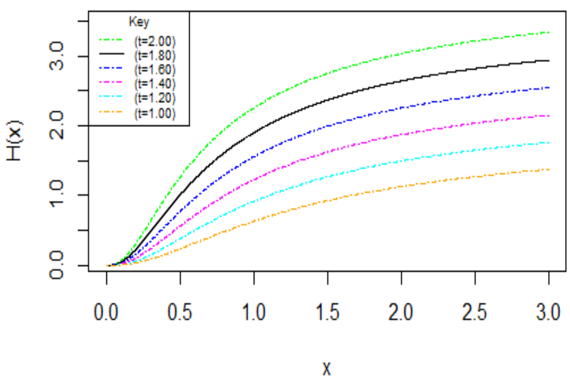


Figure 4: *Plot of h(x) at Different Parameter Values*

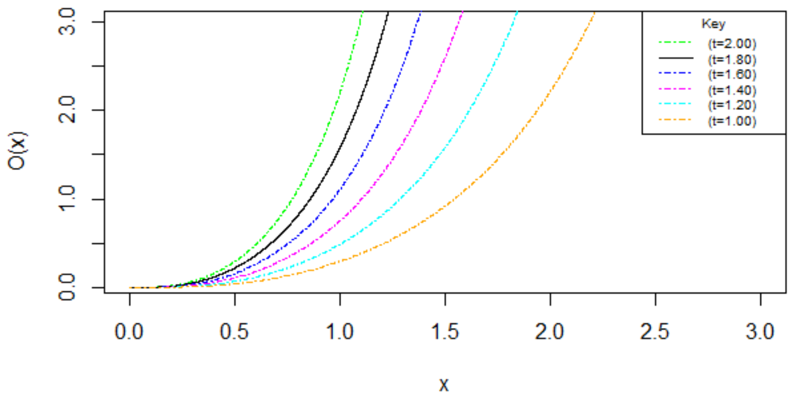


Figure 5: *Plot of O(x) at Different Parameter Values*

Quantile Function: the inverse of the CDF of ETAD gives the quantile function of ETAD, through the Lambert W function transformation it is expressed as

$$x = \frac{-W_{-1}\left[-10W_0\left(\frac{u}{10}\right)e^{-1}\right]-1}{2\theta} \tag{25}$$

IV. Moments and Moment Generating Function

Moments: This is an important property of a distribution that is used to getting some measures comprising of the mean, variance, etc. Suppose the random variable X ~ ETAD(x, θ), the r^{th} moment (μ_r), can be obtained as

$$\mu^r = E(x^r) = \int_0^\infty x^r f(x,\theta)dx \tag{26}$$

$$f(x) = -4\theta^{2+n}G_n x^{1+n}e^{-2\theta x}$$

$$\mu_r = \int_0^\infty x^r \cdot -4\theta^{2+n}x^{n+1}G_n e^{-2\theta x}dx = -4\theta^{2+n}G_n\int_0^\infty x^{n+1+r}e^{-2\theta x}dx$$

Let $u = 2\theta x$; $\frac{du}{dx} = 2\theta \Rightarrow du = \frac{1}{2\theta}du$; $dx = \frac{du}{2\theta}$. Where $x = 0$, $u = 2\theta \times 0 = 0$; As $x \to \infty$, $u \to \infty$.

By substitution the following is obtained

$$u_r = -4\theta^{2+n}G_n\int_0^\infty \left(u/2\theta\right)^{n+1+r}e^{-u}\frac{du}{2\theta} = -4G_n\int_0^\infty \theta^{2+n}(2\theta)^{-[n+1+r+1]}u^{n+1+1}e^{-u}du$$

$$= -4G_n\theta^{2+n}(2\theta)^{-[n+1+r+2]}\int_0^\infty u^{n+r+1}e^{-u}du = -4G_n\theta^{2+n}(2\theta)^{-[n+1+r+2]}\int_0^\infty u^{n+r+1+1-1}e^{-u}du$$

$$\mu_r = -4G_n\theta^{2+n}[2\theta]^{-(n+r+2)}\Gamma(n+r+2). \tag{27}$$

The *mean* of ETAD, that is, E[x] is given as

$$\mu_r = -4G_n\theta^{2+n}[2\theta]^{-(n+r+2)}\Gamma(n+r+2)$$

$$E(x) = \mu_1 = -4G_n\theta^{2+n}[2\theta]^{-[n+1+2]}\Gamma(n+1+2) = -\left[\frac{G_n\Gamma(n+3)}{2^{(n+1)}\theta}\right] \tag{28}$$

The population *variance*, σ^2 of ETAD can be obtained as follows,

$$Var(x)=E\left[x^2\right]-\left[E(x)\right]^2 \tag{29}$$

$$E[x]^2=\mu_2=-4G_n\theta^{2+n}\left[2\theta\right]^{-(n+4)}$$

$$=-4G_n\theta^{2+n}2^{-(n+4)}\theta^{-(n+4)}\Gamma(n+4)=-2^2\cdot 2^{-(n+4)}G_n\theta^{2+n}\theta^{-(n+4)}\Gamma(n+4)$$

$$=-\left[2^{-(n+2)}G_n\theta^{-2}\Gamma(n+4)\right]=-\left[\frac{G_n\Gamma(n+4)}{2^{(n+2)}\theta^2}\right]=-\frac{G_n\Gamma(n+4)}{2^{(n+2)}\theta^2}-\left[\frac{G_n\Gamma(n+3)}{2^{(n+1)}\theta}\right]^2$$

$$Var(x)=\frac{G_n\Gamma(n+3)}{2^{(n+1)}\theta}-\frac{G_n\Gamma(n+4)}{2^{(n+2)}\theta^2}=\frac{G_n\Gamma(n+3)}{2^{(n+1)}\theta}\left[1-\frac{(n+3)}{2\theta}\right] \tag{30}$$

The *Coefficient of Variance* (CV) of ETAD can be obtained as follows

$$CV=\frac{\sigma}{\mu}100\%=\frac{\sqrt{\frac{G_n\Gamma(n+3)}{2^{(n+1)}\theta}\left[1-\frac{(n+3)}{2\theta}\right]}}{\frac{-G_n\Gamma(n+3)}{2^{(n+1)}\theta}}=\left[\frac{G_n\Gamma(n+3)}{2^{(n+1)}\theta}\left[1-\frac{(n+3)}{2\theta}\right]\right]^{\frac{1}{2}}\frac{\left(-2^{(n+1)}\theta\right)}{G_n\Gamma(n+3)} \tag{31}$$

$$=\left[\frac{G_n\Gamma(n+3)}{2^{(n+1)}\theta}\right]^{\frac{1}{2}}\left[\frac{-2^{(n+1)}\theta}{G_n\Gamma(n+3)}\right]\left[1-\frac{(n+3)}{2\theta}\right]^{\frac{1}{2}}=-\left[G_n\Gamma(n+3)\right]^{\frac{1}{2}}\left[2^{(n+1)}\theta\right]\left[G_n\Gamma(n+3)\right]^{-1}\left[1-\frac{(n+3)}{2\theta}\right]^{\frac{1}{2}}$$

$$=-\left[G_n\Gamma(n+3)\right]^{\frac{1}{2}}\left[2^{(n+1)}\theta\right]^{\frac{1}{2}}\left[1-\frac{(n+3)}{2\theta}\right]^{\frac{1}{2}}=-\left[\frac{2^{(n+1)}\theta}{G_n\Gamma(n+3)}\right]^{\frac{1}{2}}\left[1-\frac{(n+3)}{2\theta}\right]^{\frac{1}{2}}$$

$$CV=-\left[\frac{2^{(n+1)}\theta}{G_n\Gamma(n+3)}\left[1-\frac{(n+3)}{2\theta}\right]\right]^{\frac{1}{2}}\times 100\% \tag{32}$$

The *Harmonic Mean* (HM) of ETAD is obtained as follows

$$HM(X)=\int_0^\infty\frac{1}{x}f(x)dx \tag{33}$$

$$=\int_0^\infty\frac{1}{x}\left(-4G_n\theta^{2+n}x^{1+n}e^{-2\theta x}\right)dx=-4G_n\theta^{2+n}\int_0^\infty x^{-n}x^{1+n}e^{-2\theta x}dx=-4G_n\theta^{2+n}\int_0^\infty x^n e^{-2\theta x}dx$$

$$=-4G_n\theta^{2+n}\left[\frac{n!}{(2\theta)^{n+1}}\right]=-\frac{-4G_n\theta^{2+n}n!}{2^{n+1}\theta^{n+1}}=-2^2\cdot 2^{-(n+1)}G_n\theta^{2+n}\theta^{-(n+1)}n!=2^{(1-n)}G_n\theta n! \tag{34}$$

The *Mode* (M) of ETAD can be obtained as follows

$$M(x)=\frac{d}{dx}\ln\left(f(x,\theta)\right) \tag{35}$$

$$=\frac{d}{dx}\ln\left[-4G_n\theta^{2+n}x^{1+n}e^{-2\theta x}\right]=\frac{1}{-4G_n\theta^{2+n}x^{1+n}e^{-2\theta x}}\frac{d}{dx}\left[-4G_n\theta^{2+n}x^{1+n}e^{-2\theta x}\right]$$

$$=\frac{\left[-4G_n\theta^{2+n}\frac{d}{dx}\left[x^{1+n}e^{-2\theta x}\right]\right]}{-4G_n\theta^{2+n}x^{1+n}e^{-2\theta x}}=\frac{(n+1)x^ne^{-2\theta x}+x^{n+1}\cdot e^{-2\theta x}(-2\theta)}{x^{n+1}e^{-2\theta x}}$$

$$=\frac{(n+1)x^ne^{-2\theta x}-2x^{n+1}\cdot e^{-2\theta x}\theta}{x^{n+1}e^{-2\theta x}}=\frac{x^ne^{-2\theta x}\left[(n+1)-2x\theta\right]}{x^n\cdot x\cdot e^{-2\theta x}}=\frac{(n+1)-2\theta x}{x}=\left[(n+1)-2\theta x\right]x^{-1}$$

$$M(x)=(n+1)x^{-1}-2\theta \tag{36}$$

The *Median* can be derived by using equation (25) which is the derived quantile function of ETAD and fixing $u = 0.5$, the median of ETAD is obtained as follows:

$$x = \frac{-W_{-1}\left[-10W_0\left(\frac{0.5}{10}\right)e^{-1}\right]-1}{2\theta} = \frac{-W_{-1}\left[-10W_0\left(0.05\right)e^{-1}\right]-1}{2\theta} \tag{37}$$

Skewness [SK]: The skewness of a probability distribution is a measure of symmetry and the lack of symmetry of the probability distribution. The ETAD SK is derived from the 3rd moment of the mean.

$$SK = \frac{E(x^3) - 3\mu\sigma^2 - \mu^3}{\sigma^3} \tag{38}$$

$$SK = \frac{2^{\left[\frac{3n+7}{2}\right]}\left[\frac{G_n\Gamma n+5}{2^{(n+2)}} + \frac{3\left[G_n\Gamma n+3\right]^2\left[2(n+3)+G_n\Gamma n+3\right]}{2^{2n+3}} - \frac{\left[G_n\Gamma n+3\right]^3}{2^{3n+2}}\right]}{\sqrt{i\left[G_n\Gamma n+3\left[2(n+3)+G_n\Gamma n+3\right]\right]^{3/2}}} \tag{39}$$

Kurtosis [KT]: The kurtosis measure whether or not the probability distribution is heavy-tailed. For ETAD the KT is derived from the 4th moment and is obtained as

$$KT = \frac{E\left(x^4\right) - 4\mu E\left[x^3\right] + 6\sigma^2 E\left[x^2\right] - 3E\left[x^4\right]}{\sigma^4} \tag{40}$$

$$KT = \frac{3G_n\Gamma(n+4)\left[2(n+3)+G_n\Gamma(n+3)\right] - 4G_n\Gamma(n+5) + 2^{(n+2)}(n+5)(n+4)(n+3)}{G_n\Gamma(n+3)\left[2(n+3)+G_n\Gamma(n+3)\right]^2} \tag{41}$$

Dispersion Index [DI]: It is simply defined as the variance divided by the mean. It tells how dispersed the mean is from the variance. The DI for the ETAD is obtained as

$$DI = \frac{\sigma^2}{\mu} = \frac{\frac{G_n\Gamma(n+3)}{2^{(n+1)}\theta}\left[1 - \frac{(n+3)}{2\theta}\right]}{\frac{G_n\Gamma(n+3)}{2^{(n+1)}\theta}} = 1 - \frac{(n+3)}{2\theta} \tag{42}$$

Moment Generating Function: The moment generating function of X, say $M_X(t)$, is

$$M_x(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f(x,\theta)dx = \int_0^\infty e^{tx} \cdot -4\theta^{2+n} G_n x^{1+n} e^{-2\theta x} dx\ ;\ e^{tx} = \sum_{j=0}^{\infty} \frac{(tx)^m}{m!} \tag{43}$$

By substitution equation (43) becomes

$$M_x(t) = \int_0^\infty \sum_{j=0}^{\infty} \frac{(tx)^m}{m!} \cdot -4\theta^{2m} G_n x^{1+n} e^{-2\theta x} dx \tag{44}$$

$$= \int_0^\infty \sum_{j=0}^{\infty} \frac{t^m}{m!} x^m \cdot -4\theta^{2+n} G_n x^{1+n} e^{-2\theta x} dx = \phi \int_0^\infty x^{(1+n+m)} \theta^{2+n} e^{-2\theta x} dx \tag{45}$$

Let $u = 2\theta x \Rightarrow x = \frac{u}{2\theta}\ ;\ \frac{d}{dx} = 2\theta \Rightarrow dx = \frac{1}{2\theta}dx$. Where, $x = 0, u = 0, x \to \infty, u \to \infty$

Substituting into (45) to get

$$= \phi \int_0^\infty \left(\frac{u}{2\theta}\right)^{(1+n+m)} \theta^{2+n} e^{-u} \frac{du}{2\theta} = \phi \int_0^\infty u^{(1+n+m)} (2\theta)^{-[1+n+m]} \theta^{2+n} e^{-u}\, 2\theta^{-1} du$$

$$M_x(t) = \varphi(2\theta)^{-[2+n+m]}\,\theta^{2+n}\int_0^\infty u^{[1+n+m]-1}e^{-u}du = \phi[2\theta]^{-[2+n+m]}\,\theta^{2+n}\Gamma[2+n+m] \quad (46)$$

V. Entropy

Suppose a random variable X follows the ETA distribution, then the Renyi entropy (Renyi, 1961) which measures the uncertainty of information is expressed as

$$I_R(c) = \frac{1}{1-c}\log\int_0^\infty f^c(x)dx = \frac{1}{1-c}\log\left[\int_0^\infty\left\{-4\theta^2 xe^{-2\theta x}\left[\log\left(e^{-2\theta x}(2\theta x+1)\right)\right]\right\}\right]^c dx \quad (47)$$

$$= \frac{1}{1-c}\log\left[\int_0^\infty\left(-4\theta^2\right)^c x^c e^{-2\theta cx}\left[\log\left(e^{-2\theta cx}[2\theta x+1]\right)\right]^c\right]dx = \frac{1}{1-c}\log\left[\int_0^\infty\left[-4\theta^{2+n}G_n x^{1+n}e^{-2\theta x}\right]^c dx\right]$$

$$= \frac{1}{1-c}\log\left[\int_0^\infty\left[-4\theta^2 G_n\right]^c x^{c(1+n)}e^{-2\theta cx}dx\right] = \frac{1}{1-c}\log\left[\left[-4\theta^{2n}G_n\right]^c\int_0^\infty x^{c(1+n)}e^{-2\theta x}dx\right]$$

Let $u = 2\theta cx, x = \frac{u}{2\theta c}$. $\frac{d}{dx} = 2\theta c, du = \frac{du}{2\theta c}$. Where $x = 0, u = 0, x \to \infty, u \to \infty$. Therefore,

$$I_R(c) = \frac{1}{1-c}\log\left[\left(-4\theta^{2+n}G_n\right)^c\int_0^\infty\left(\frac{u}{2\theta c}\right)^{c(1+n)}e^{-u}\frac{du}{2\theta c}\right]$$

$$= \frac{1}{1-c}\log\left[\left(-4\theta^{2+n}G_n\right)^c\int_0^\infty u^{c(1+n)}(2\theta c)^{-(n+1)}(2\theta c)^{-1}e^{-u}du\right]$$

$$= \frac{1}{1-c}\log\left[\left(-4\theta^{2+n}G_n\right)^c(2\theta c)^{-(n+1)c-1}\int_0^\infty u^{c(1+n)}e^{-u}du\right]$$

$$= \frac{1}{1-c}\log\left[\left(-4\theta^{2+n}G_n\right)^c(2\theta c)^{-(n+1)c-1}\int_0^\infty u^{c(1+n)+1-1}e^{-u}du\right]$$

$$= \frac{1}{1-c}\log\left[\left(-4\theta^{2+n}G_n\right)^c(2\theta c)^{-c(n+1)-1}\Gamma(c[1+n]+1)\right]$$

$$= \frac{1}{1-c}\log\left[\left(-4\theta^{2+n}G_n\right)^c\right] + \log\left[(2\theta c)^{-c(n+1)-1}\right] + \log\left[\Gamma(c(n+1)+1)\right] \quad (48)$$

VI. ETAD Order statistics

Given an independence characteristic distribution of a random sample $X_1; X_2 \cdots; X_n$, this sample can be in ordered form as $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ or in a better notation, $X_{(1;n)} \le X_{(2;n)} \le \cdots \le X_{(n;n)}$. These representations are called order statistics. The 1^{st} order statistics, $X_{(1)}$ is the minimum, the 2^{nd} order statistics, $X_{(2)}$ is the second while the n^{th} order statistics, $X_{(n)}$ is the maximum.

PDF of the k^{th} order statistics of ETAD: Suppose a random sample $X_1, X_2 \cdots, X_n$ from the ETAD is ordered as $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$, then the PDF $f_{(n;n)}(X)$ of the k^{th} order statistics can be define as:

$$f_{(k,n)}(x) = \frac{n!}{(k-1)!(n-k)!} f(x) \times F(x)^{k-1} \times [1 - F(x)]^{n-k} \tag{49}$$

where, $F(X)$ and $f(X)$ are the CDF and PDF respectively of the ETA distribution. Hence, for easier simplification, the binomial expansion of $[1-F(x)]^{n-k}$ was used as

$$[1 - F(x)]^{n-k} = \sum_{c_5}^{\infty} \binom{n-k}{c_5} (-1)^{c_5} [F(x)]^{c_5} \tag{50}$$

Substituting equation (50) into (49) to obtain

$$f_{(k,n)}(x) = \frac{n!}{(k-1)!(n-k)!} f(x) \cdot F(x)^{k-1} \sum_{c_5}^{\infty} \binom{n-k}{c_5} (-1)^{c_5} [F(x)]^{c_5} \tag{51}$$

$$= \sum_{c_5}^{\infty} \frac{n!}{(k-1)!(n-k)!} f(x) \binom{n-k}{c_5} (-1)^{c_5} [F(x)]^{c_5} \tag{52}$$

Substituting the PDF and CDF of ETA distribution into (52) we have

$$= \sum_{c_5}^{\infty} \frac{n!(-1)^{c_5}}{(k-1)!(n-k-c_5)! c_5!} \left(-4\theta^2 x e^{-2\theta x} \ln\left[(2\theta x + 1) e^{-2\theta x}\right]\right) \binom{n-k}{c_5} (-1)^{c_5} \left[e^{-2\theta x}(2\theta x + 1)\left\{\ln[2\theta x + 1]^{-2\theta x}\right\} - 1\right]^{c_5 + k - 1} \tag{53}$$

PDF of the Smallest and Largest Ordered Statistic

To obtain the 1^{st} or minimum order statistic, let $k = 1$ be substituted into equation (53) to give

$$= \sum_{c_5}^{\infty} \frac{n!(-1)^{c_5}}{(n-1-c_5)! c_5!} \left\{-4\theta^2 x e^{-2\theta x} \ln\left[(2\theta x + 1)^{-2\theta x}\right]\right\} \left[e^{-2\theta x}(2\theta x + 1)\left\{\ln\left[(2\theta x + 1)^{-2\theta x}\right] - 1\right\}\right]^{c_5} \tag{54}$$

Similarly, the n^{th} order statistic or maximum order statistic is obtained by substituting $k = n$, to give

$$= \sum_{c_5}^{\infty} \frac{n!(-1)^{c_5}}{(n-1)! - c_5! c_5!} \left\{-4\theta^2 x e^{-2\theta x} \ln\left[(2\theta x + 1) e^{-2\theta x}\right]\right\} \left[e^{-2\theta x}(2\theta x + 1)\left\{\ln\left[(2\theta x + 1) e^{-2\theta x}\right] - 1\right\}\right]^{c_5 + n - 1} \tag{55}$$

VII. Parameter Estimation using Maximum Likelihood Estimation

Suppose $X_1, X_2, \ldots, X_n$ is a random sample with size n drawn from ETA distribution. The likelihood function

$$L(x_1, x_2, \cdots, x_n; \theta) = \prod_{i=1}^{n} f(x; \theta) \tag{56}$$

$$= \prod_{i=1}^{n} \left[-4\theta^2 x e^{-2\theta x} \left[\ln\left[(2\theta x + 1) e^{-2\theta x}\right]\right]\right] = \left[-4\theta^2\right]^n \prod_{i=1}^{n} x \prod_{i=1}^{n} e^{-2\theta x} \prod_{i=1}^{n} \left[\ln\left[(2\theta x + 1) e^{-2\theta x}\right]\right]$$

The log-likelihood for our PDF becomes

$$\log L(x_1, x_2, \cdots, x_n; \theta) = \log\left[\left(-4\theta^2\right)^n \prod_{i=1}^{n} x \prod_{i=1}^{n} e^{-2\theta x} \prod_{i=1}^{n} \left[\ln\left[(2\theta x + 1) e^{-2\theta x}\right]\right]\right] = Z$$

$$= n\left(-4\theta^2\right) + \sum_{i=1}^{n} \log(x_i) - 2\theta \sum_{i=1}^{n} x_i + \log\left[\prod_{i=1}^{n} \left[\ln\left[(2\theta x + 1) e^{-2\theta x}\right]\right]\right]$$

$$= -4n\theta^2 + \sum_{i=1}^{n} \log(x_i) - 2\theta \sum_{i=1}^{n} x_i + \left[\sum_{i=1}^{n} \log\left[\ln\left[(2\theta x + 1) e^{-2\theta x}\right]\right]\right] \tag{57}$$

The parameter estimate $\hat{\theta}$ of ETAD is obtained by differentiating equation (57) partially with respect to θ and equate to zero. This implies,

$$\frac{\partial Z}{\partial \theta} = -8n\theta - 2\sum_{i=1}^{n} x_i - 4\sum_{i=1}^{n}\left[\frac{\theta^2 x}{(2\theta x+1)\ln\left[\left((2\theta x+1)e^{-2\theta x}\right)\right]}\right] = 0 \quad (58)$$

$$\sum_{i=1}^{n} x_i = -4n\theta - 2\sum_{i=1}^{n}\left[\frac{\theta^2 x}{(2\theta x+1)\ln\left[\left((2\theta x+1)e^{-2\theta x}\right)\right]}\right]$$

$$\hat{\theta} = -\left(4\bar{x}_i + \frac{1}{2n}\sum_{i=1}^{n}\left[\frac{\hat{\theta}^2 x_i}{(2\hat{\theta} x_i+1)\ln\left[\left((2\hat{\theta} x_i+1)e^{-2\hat{\theta} x_i}\right)\right]}\right]\right) \quad (59)$$

Equation (59) clearly reveal that there is no close form expression for θ and this implies it can only be estimated through numerical methods using software like SAS, R, Maple, etc., for this research the R open source software was used.

VIII. Goodness of Fit Test

The goodness of fit tests used are, Akaike IC (AIC), Bayesian IC (BIC), Corrected AIC (CAIC), and Hannan-Quinn IC (HQIC). Their respective formulas are presented as follows.

$$AIC = -2(ll) + 2k \quad (60)$$

$$BIC = -(2ll) + (k(\ln(n))) \quad (61)$$

$$CAIC = AIC + 2k(k+1) \div (n-k-1) \quad (62)$$

$$HQIC = -(2(ll)) + (2 \cdot k \cdot \ln(\ln(n))) \quad (63)$$

where, ll is the log-likelihood, n is the sample size, and k is the number of parameter to be estimated.

III. Results

I. Simulation Study

The usefulness of simulation is to determine the efficiency and consistency of the estimate of MLE method. Using different parameters values and sample sizes (20-1500), the estimation methods used for the comparing are based on absolute bias (AB), variance (S^2), standard error (SE), and root mean square error (RMSE).

$$AB = \frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{\theta}_i - \theta_i\right) \quad (64)$$

$$RMSE = \sqrt{\frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{\theta}_i - \theta_i\right)^2} \quad (65)$$

$$S^2 = \frac{\sum (x_i - \bar{x})^2}{n-1} \quad (66)$$

$$SE = \frac{\sigma}{\sqrt{n}} \quad (67)$$

Table 1 presents the simulation results for up to 1500 simulations at different parameter values. The results demonstrate that the MLE method is consistent and efficient. This is due to the fact that as the sample size increases from 20 to 1500 the variance and RMSE decreases to the minimum.

Table 1: *Absolute Bias, Variance, Standard Error, and Root Mean Square Error*

Sample Size	Parameter Value	Estimate	ǀBiasǀ	Variance	Standard Error	RMSE
20	0.1	0.1613	0.0613	0.0000	0.0000	0.0038
20	0.3	0.5012	0.2012	0.0003	0.0039	0.0408
20	0.5	0.8389	0.3389	0.0008	0.0063	0.1157
20	0.7	1.1764	0.4764	0.0015	0.0087	0.2285
40	0.1	0.1609	0.0609	0.0000	0.0000	0.0037
40	0.3	0.4911	0.1911	0.0001	0.0016	0.0366
40	0.5	0.8204	0.3204	0.0004	0.0032	0.1031
40	0.7	1.1497	0.4497	0.0008	0.0045	0.2030
80	0.1	0.1606	0.0606	0.0000	0.0000	0.0037
80	0.3	0.4859	0.1859	0.0001	0.0011	0.0347
80	0.5	0.8108	0.3108	0.0002	0.0016	0.0968
80	0.7	1.1357	0.4357	0.0004	0.0022	0.1902
100	0.1	0.1608	0.0608	0.0000	0.0000	0.0037
100	0.3	0.4856	0.1856	0.0001	0.0010	0.0345
100	0.5	0.8102	0.3102	0.0002	0.0014	0.0964
100	0.7	1.1347	0.4347	0.0003	0.0017	0.1893
150	0.1	0.1606	0.0606	0.0000	0.0000	0.0037
150	0.3	0.4839	0.1839	0.0000	0.0000	0.0338
150	0.5	0.8071	0.3071	0.0001	0.0008	0.0944
150	0.7	1.1303	0.4303	0.0002	0.0012	0.1854
250	0.1	0.1606	0.0606	0.0000	0.0000	0.0037
250	0.3	0.4831	0.1831	0.0000	0.0000	0.0335
250	0.5	0.8055	0.3055	0.0000	0.0000	0.0933
250	0.7	1.1279	0.4279	0.0001	0.0006	0.1832
500	0.1	0.1606	0.0606	0.0000	0.0000	0.0037
500	0.3	0.4824	0.1824	0.0000	0.0000	0.0333
500	0.5	0.8042	0.3042	0.0000	0.0000	0.0925
500	0.7	1.1260	0.4260	0.0000	0.0000	0.1815
750	0.1	0.1605	0.0605	0.0000	0.0000	0.0037
750	0.3	0.4820	0.1820	0.0000	0.0000	0.0331
750	0.5	0.8035	0.3035	0.0000	0.0000	0.0921
750	0.7	1.1250	0.4250	0.0000	0.0000	0.1806
1000	0.1	0.1605	0.0605	0.0000	0.0000	0.0037
1000	0.3	0.4819	0.1819	0.0000	0.0000	0.0331
1000	0.5	0.8033	0.3033	0.0000	0.0000	0.0920
1000	0.7	1.1246	0.4246	0.0000	0.0000	0.1803
1500	0.1	0.1605	0.0605	0.0000	0.0000	0.0037
1500	0.3	0.4818	0.1818	0.0000	0.0000	0.0331
1500	0.5	0.8030	0.3030	0.0000	0.0000	0.0918
1500	0.7	1.1243	0.4243	0.0000	0.0000	0.1800

II. Real Life Data Analysis

This research fitted six (6) real data sets to the ETAD and seven other competing distributions [Inverted Exponential Distribution (IED), Sine Exponential Distribution (SED), Ram Awadh Distribution (RAD), Prakaamy Distribution (PD), N-Sine Exponential Distribution (NSED), Exponential Distribution (ED), Ailamujia Distribution (AD)] for comparison. The data sets are nicotine measurement from different cigarettes brands [16], Carbon Fiber Tensile Strength of Length 20mm and 50mm [17], Survival Times of Growth Hormone Medication [18], Lung Cancer

Patients Tumours Size [19], and Glass Fiber Strength of 1.5cm [20]. The descriptive statistics, MLE, and GoF results are presented.

Table 2: *Descriptive Statistics*

Variable	Data 1	Data 2	Data 3	Data 4	Data 5	Data 6
Sample size	346	69	66	35	76	63
Maximum value	2	2.585	4.9	13.7	11.18	2.24
Minimum Value	0.1	0.312	0.39	2.15	0.96	0.55
Mean	0.8525	1.451	2.751	5.298	3.529	1.507
Median	0.0041	1.49	2.035	4.51	2.7	1.54
Variance	0.1192	0.245	0.7948	8.509	6.595	0.1051

Table 3: *The MLE Estimates for the Six Data Sets*

Distribution	MLE 1	MLE 2	MLE 3	MLE 4	MLE 5	MLE 6
IED	0.839	1.81	1.064	1.993	0.345	1.999
SED	0.672	0.396	0.208	0.108	0.161	0.383
RAD	1.993	0.755	1.763	1.114	1.548	1.997
PD	1.927	0.401	1.879	1.117	1.589	2
NSED	1.174	0.217	0.441	0.778	1.62	1.205
ED	1.173	0.688	0.362	0.188	0.283	0.664
AD	1.985	0.689	0.362	1.665	0.283	0.664
ETAD	1.985	1.167	0.614	0.3204	0.32	1.122

Table 4: *Log-likelihood and Information Criteria for Data Set I*

	ll	AIC	BIC	CAIC	HQIC	Rank
IED	-467.823	937.646	941.492	937.657	939.177	5
SED	-273.346	548.692	552.539	548.704	550.224	3
RAD	-5076.81	10155.617	10159.464	10155.629	10157.149	6
PD	-693.204	1388.407	1392.254	1388.419	1389.939	7
NSED	-1267.3	2536.593	2540.440	2536.605	2538.125	8
ED	-290.826	583.652	587.498	583.663	585.183	4
AD	-193.354	388.708	392.554	388.720	390.240	2
ETAD	-144.56	291.119	294.965	291.130	292.650	1

Table 5: *Log-likelihood and Information Criteria for Data Set II*

	ll	AIC	BIC	CAIC	HQIC	Rank
IED	-89.1475	180.295	182.529	180.354	181.181	4
SED	-90.9535	183.907	186.141	183.967	184.793	5
RAD	-386.964	775.927	778.161	775.987	776.813	7
PD	-608.715	1219.429	1221.663	1219.488	1220.315	8
NSED	-60.315	122.630	124.864	122.690	123.517	2
ED	-94.7015	191.403	193.637	191.463	192.289	6
AD	-73.095	148.190	150.424	148.250	149.076	3
ETAD	-59.69	121.380	123.614	121.440	122.267	1

Table 6: *Log-likelihood and Information Criteria for Data Set III*

	ll	AIC	BIC	CAIC	HQIC	Rank
IED	-400.249	802.497	804.686	802.559	803.362	8
SED	-129.339	260.677	262.867	260.740	261.542	5
RAD	-104.968	211.936	214.126	211.999	212.801	4
PD	-100.044	202.088	204.278	202.151	202.953	2
NSED	-181.793	365.585	367.774	365.647	366.450	7

ED	-132.995	267.989	270.178	268.051	268.854	6
AD	-112.004	226.008	228.197	226.070	226.873	3
ETAD	-98.697	199.394	201.584	199.457	200.259	1

Table 7: *Log-likelihood and Information Criteria for Data Set IV*

	ll	AIC	BIC	CAIC	HQIC	Rank
IED	-555.075	1112.150	1113.705	1112.271	1112.686	8
SED	-91.9045	185.809	187.364	185.930	186.346	5
RAD	-81.9755	165.951	167.506	166.072	166.487	4
PD	-81.9295	165.859	167.414	165.980	166.396	3
NSED	-307.259	616.518	618.073	616.639	617.055	2
ED	-93.4075	188.815	190.370	188.936	189.352	6
AD	-480	961.999	963.554	962.120	962.536	7
ETAD	-80.142	162.284	163.839	162.405	162.821	1

Table 8: *Log-likelihood and Information Criteria for Data Set V*

	ll	AIC	BIC	CAIC	HQIC	Rank
IED	-3948.32	7898.632	7900.963	7898.686	7899.563	8
SED	-169.826	341.651	343.982	341.705	342.582	3
RAD	-184.068	370.136	372.466	370.190	371.067	6
PD	-183.404	368.807	371.137	368.861	369.738	5
NSED	-910.383	1822.766	1825.097	1822.820	1823.698	7
ED	-171.901	345.802	348.133	345.856	346.734	4
AD	-159.121	320.241	322.572	320.295	321.173	2
ETAD	-80.142	162.284	163.839	162.405	162.821	1

Table 9: *Log-likelihood and Information Criteria for Data Set VI*

	Ll	AIC	BIC	CAIC	HQIC	Rank
IED	-70.438	142.876	145.019	142.942	143.719	3
SED	-84.9795	171.959	174.102	172.024	172.801	5
RAD	-110.009	222.017	224.160	222.082	222.860	7
PD	-107.312	216.623	218.767	216.689	217.466	6
NSED	-253.889	509.777	511.920	509.843	510.620	8
ED	-88.8305	179.661	181.804	179.726	180.504	4
AD	-66.3175	134.635	136.778	134.700	135.477	2
ETAD	-49.5355	101.071	103.214	101.136	101.914	1

From the results presented in Table 4 to Table 9 clearly shows that the ETAD is more flexible and a better fit distribution compared to the other one-parameter lifetime distributions because its *ll* values is larger than the other distributions *ll* values and it's AIC, BIC, CAIC, and HQIC values are smaller than that of the other seven competing distributions. To buttress these results, the fitted plots for the ETAD and the other competing distributions are presented below. Due to space, only very close competing distributions and our proposed distribution plots are presented in Figure 6 to Figure 8.

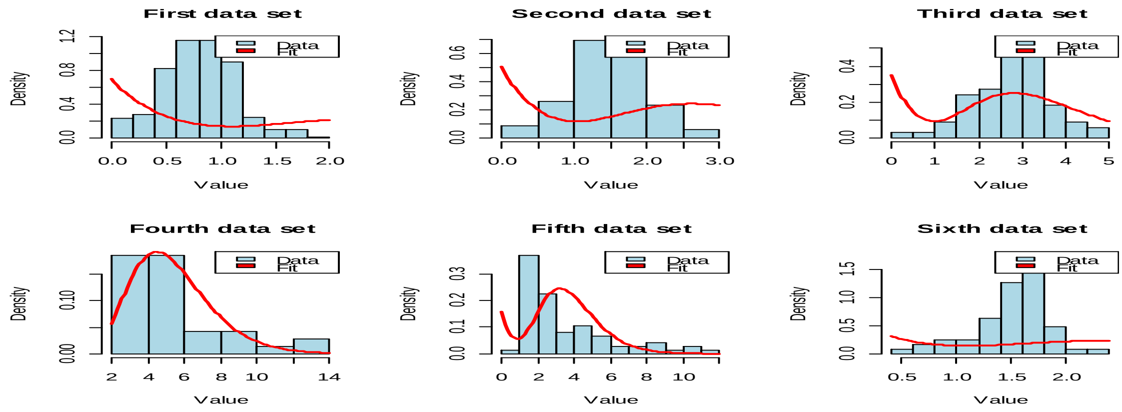


Figure 6: *Density Plot for Sine Exponential Distribution*

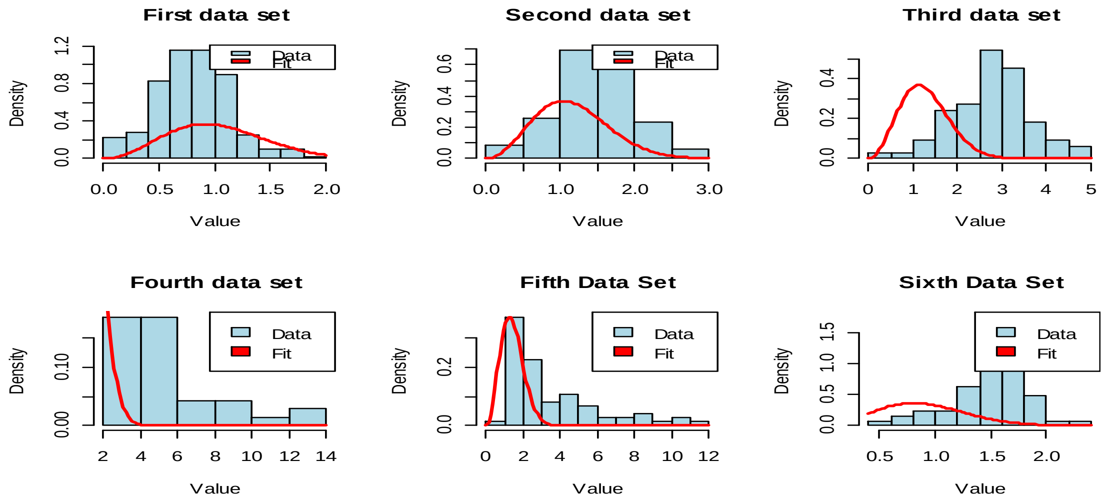


Figure 7: *Density Plot for Ailamujia Distribution*

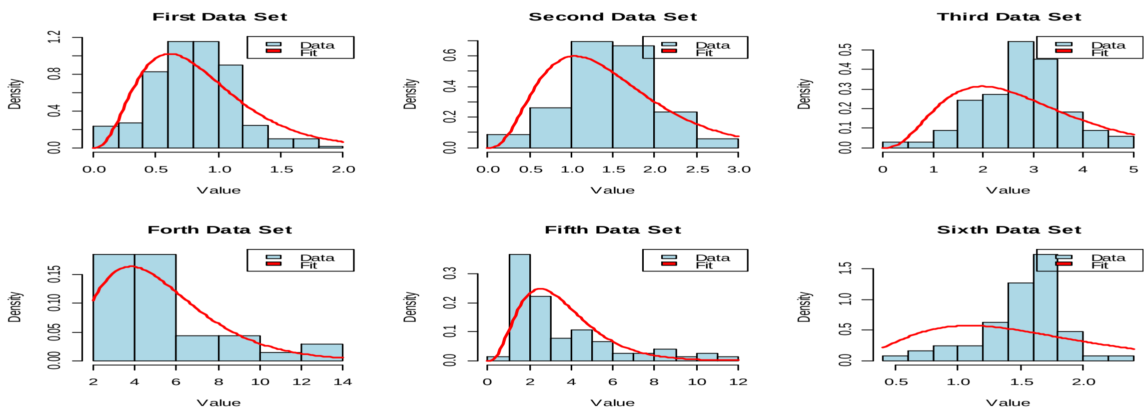


Figure 8: *Density Plot for Entropy Transformed Ailamujia Distribution*

From the plots presented, it clearly shows that the ETAD can flexibly fit right skewed and symmetric data sets well compared to the other seven competing distributions.

IV. Conclusion

This study has been able to utilize a novel approach known as the entropy transformation (ET) in modifying a distribution without adding any extra parameter to the original distribution. Illustratively, the Ailamujia distribution was modified using the ET to form ETAD and it was found to perform exceedingly better than the Ailamujia distribution and other six one-parameter competing distributions. This was observed from the information criteria results and the fitted plots to six different data sets from literature. The MLE was used to estimate the parameter of the distribution which is consistent in estimating the parameter as observed from the simulation results.

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