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A NOVEL HYBRID DISTRIBUTED INNOVATION EGARCH MODEL FOR INVESTIGATING THE VOLATILITY OF THE STOCK MARKET

[1]Mubarak M.T., [2]Adubisi O.D., and [3]Abbas U.F.

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[1,3]School of Science and Technology, Gombe State Polytechnic, Bajoga, Gombe State, Nigeria.
[2]Department of Mathematics and Statistics, Federal University Wukari, Nigeria.
mubarak@gspb.edu.ng[1], adubisiobinna@fuwukari.edu.ng[2], ufabbas@gspb.edu.ng[3]

Abstract

When calculating risk and making decisions, investors and financial institutions heavily rely on the modeling of asset return volatility. For the exponentiated generalized autoregressive conditional heteroscedasticity (EGARCH) model, we created a unique innovation distribution in this study called the type-II-Topp-Leone-exponentiated-Gumbel ($TIITL_{EGU}$) distribution. The key mathematical characteristics of the distribution were determined, and Monte Carlo experiments were used to estimate the parameters of the novel distribution using maximum likelihood estimation (MLE) procedure. The performance of the EGARCH (1,1) model with $TIITL_{EGU}$ distributed innovation density in relation to other innovation densities in terms of volatility modeling is examined through applications using two Nigerian shock returns. The results of the diagnostic tests indicated that, with the exception of the EGARCH (1,1)-Johnson (SU) reparametrized (JSU) innovation density, the fitted models have been sufficiently specified. The parameters for the EGARCH (1,1) model with different innovation densities are significant at various levels. Furthermore, in out-of-sample prediction, the fitted EGARCH (1,1)-$TIITL_{EGU}$ innovation density performed better than the EGARCH (1,1)-existing innovation densities. As a result, it is decided that the EGARCH-$TIITL_{EGU}$ model is the most effective for analyzing Nigerian stock market volatility.

Keywords: EGARCH, Innovations density, Maximum likelihood estimation, Simulation.

I. Introduction

The Nigerian stock market has grown in terms of the number of stock exchanges and other financial intermediaries, the number of listed stocks, trading volumes, market capitalization, investor population, stock exchange turnover, and stock price indexes over time. The stock market's performance is a key measure of a country's progress and development. Because it reflects the potential viability and financial strength of corporate entities registered on the stock exchange, the stock market is one of the yardsticks for assessing an economy's growth and development. However, the state of investor confidence in different economic sectors is reflected in the stock market. It shows hopes for the stability of the financial system and reflects the strength of the producing sector [1-2]. A survey of pertinent literature reveals that while predicting stock market volatility, academics have neglected to account for the contributions of alternate innovation distributions. Since financial time series have leptokurtic and autocorrelation characteristics, mis-specification may result from applying the incorrect innovation density in the EGARCH volatility model. Additionally, an erroneous innovation density specification can result in a significant loss of efficiency for the relevant estimators, an invalid risk assessment, incorrectly priced options, and an improper valuation of Value-at-Risk (VaR).

In making financial decisions such as portfolio selection, risk management, and option pricing, financial return volatility is an important metric to consider. Therefore, in order to model and anticipate the volatility of asset returns, it is imperative to create a model with strongly driven conditional innovation density. According to [3], a suitable volatility model is one that both accurately represents the disturbance term's heteroscedasticity and reflects the stylized facts that are present in stock return series. Financial institutions usually employ generalized autoregressive conditional heteroskedasticity (GARCH) models to predict return volatility for stocks, bonds, and market indexes, while they can also be used to analyze other types of financial data, such as macroeconomic data. In order to aid in their decisions about asset allocation, hedging, risk management, and portfolio optimization, they use the information that is produced to estimate the returns of current investments, help decide pricing, and assess which assets may yield larger returns [4]. However, these returns exhibit leverage effects, heavy tail, volatility clustering, significant skewness and excess leptokurtic behaviours which the symmetric GARCH model with normal distributed innovations in most cases fail to capture [5-7]. Despite the skewed form and appealing characteristics of the student-t in GARCH models, the tail behaviour is still too short to adequately describe skewness and fat tails in asset returns [8-9]. According to [10], volatility models that do not allow for conditional variance asymmetry typically result in inaccurate volatility estimations and projections. To address this flaw in GARCH's treatment of financial time series, the asymmetric exponential GARCH model was developed using the generalized error distributed innovation. Specifically, to accommodate asymmetric impacts between asset returns that are positive and negative [10-11]. Few studies had been done on returns volatility modeling utilizing the EGARCH model with common innovation densities, which raises questions about the choice of innovation densities [11-17].

Quite a few academics have focused on establishing novel distributions for volatility model innovations, and have conducted various studies in this area of altering the innovation density of the EGARCH volatility model. [18] advocated the use of the beta-student-t distribution for the EGARCH model in the estimation of volatility. [19] discovered that more flexible GARCH-type models are sufficiently acceptable in predicting volatility for all density assumptions. Using simulated and actual data, the Bayesian analysis of a stochastic model with generalized hyperbolic skew Student-t distribution. [20] proposed the EGARCH model with the beta-skew Student-t density for predicting daily volatility. [21] discovered that using leptokurtic distributions in GARCH-type models helped them produce more accurate volatility projections. [22-23] evaluated the daily volatility of stock index returns using a new generalization of the skew Student-t distribution, and demonstrated that it performed better than some innovation densities. [2] and [24] proposed the exponentiated half-logistic skew-t and generalized odd generalized skew-t densities for evaluating the daily volatility of bitcoin, Nigeria inflation and first bank Nigeria stock returns, and found that it performed better than other innovation densities in the GARCH-type models. [7] proposed the GARCH model with the exponentiated Gumbel density for predicting daily return volatility of the S&P 500 index. [9] proposed the odd generalized exponential Laplace density for evaluating the daily volatility of the Nigeria stock exchange, and found that it performed better than other innovation densities in the GARCH-type models. These models informed entrepreneur and investors on volatile nature of stock prices and bitcoin rates. However, developing robust distributions remains critical in improving the accuracy of the monetary risk system. Therefore, this research set out to propose a new innovation density for the asymmetric EGARCH model by introducing a novel distribution. This research is focused on the distinctiveness of the structural properties of the novel distribution and modification of the distributional assumption of the innovations of the EGARCH volatility model.

The article is arranged as follows: Section 2 presents the developed novel distribution and its standardized form. Section 3 presents the properties of the novel distribution. Section 4 presents the maximum likelihood estimation procedure and Monte-Carlo simulation process. Section 5 presents the EGARCH model with the novel innovation density function including methods for selecting models and appraising predictions. Section 6 reports the empirical results of both estimation and forecast assessment, and conclusion in Section 7.

2. Distribution Genesis

The cumulative distribution function (cdf) of the Type II Topp Leone (TIITL-G) family of distributions developed by [25] is specified as

$$F(x) = 1 - [1 - G^2(x)]^{\theta} \tag{1}$$

and the corresponding probability density function (pdf) is

$$f(x) = 2\theta g(x)G(x)[1 - G^2(x)]^{\theta-1} \tag{2}$$

where $\theta > 0$ is the shape parameter, $G(x)$ and $g(x)$ are the baseline cdf and pdf, respectively. The cdf of the baseline distribution titled the exponentiated Gumbel (EGU) distribution introduced by [26] is specified as

$$G(x) = 1 - \left\{1 - e^{\left[-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right]}\right\}^{\alpha}, \tag{3}$$

and the pdf is given as

$$g(x) = \frac{\alpha}{\sigma}\left\{1 - e^{\left[-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right]}\right\}^{\alpha-1} e^{\left[-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right]} e^{\left(-\frac{x-\mu}{\sigma}\right)}, \tag{4}$$

where $\alpha > 0,\ \sigma > 0,\ \mu \in \mathcal{R}$ are shape, scale and location parameters, and $x \in \mathcal{R}$.

The cdf and pdf of the type II Topp Leone exponentiated Gumbel (TIITLEGU) model derived by inserting Equations (3) and (4) into Equations (1) and (2), respectively, are specified as

$$F(x;\alpha,\theta,\mu,\sigma) = 1 - \{1 - [\Theta(x)]^2\}^{\theta}, \tag{5}$$

$$f(x;\alpha,\theta,\mu,\sigma) = 2\theta\Phi(x)\{1 - [\Theta(x)]^2\}^{\theta-1}, \tag{6}$$

Hither, the pdf and cdf of the EGU are represented with $\Phi(x)$ and $\Theta(x)$. $\alpha > 0, \theta > 0, \sigma > 0$ are the shape and scale parameters, and $\mu \in \mathcal{R}$ is the location parameter. More so, the survival and hazard rate functions are specified as follows

$$S(x;\alpha,\theta,\mu,\sigma) = \{1 - [\Theta(x)]^2\}^{\theta} \tag{7}$$

$$h(x;\alpha,\theta,\mu,\sigma) = 2\theta\Phi(x)\Theta(x)\{1 - [\Theta(x)]^2\}^{-1}, \tag{8}$$

Figures 1 depicts that the TIITLEGU density function can be very useful in describing symmetric, heavy-tailed, unimodality, leptokurtic and skew patterns of most data sets. Hence, a viable alternative innovation density for increasing the accuracy of the EGARCH volatility model prediction.

2.1 Standardized TIITLEGU Model

The standardized TIITLEGU model is obtained via the transformation $\varepsilon_t = z\sqrt{h_t^2}$, where $E(z_t) = 0$ and $var(z_t) = 1$. The random variable z_t can be expressed as $z_t = \frac{(x_t-\mu)}{\sqrt{h_t^2}} = \frac{\varepsilon_t}{\sqrt{h_t^2}}$, then $\frac{dz_t}{d\varepsilon_t} = \frac{1}{\sqrt{h_t^2}}$. Therefore, the standardized TIITLEGU density function is

$$f(\varepsilon_t;\alpha,\theta,h_t) = \frac{2\alpha\theta}{(h_t^2)^{\frac{1}{2}}}\eta^{\alpha-1}(1-\eta)e^{\left(-\frac{\varepsilon_t}{h_t^2}\right)}(1-\eta^{\alpha})[1-(1-\eta^{\alpha})^2]^{\theta-1}\frac{1}{(h_t^2)^{\frac{1}{2}}} \tag{9}$$

where $\eta = 1 - e^{-e^{\left(-\frac{\varepsilon_t}{h_t^2}\right)}}$, μ and $\sqrt{h_t^2}$ denote the mean and standard deviation. The standardized

TIITLEGU is the novel hybrid distributed innovation function for the EGARCH volatility model.

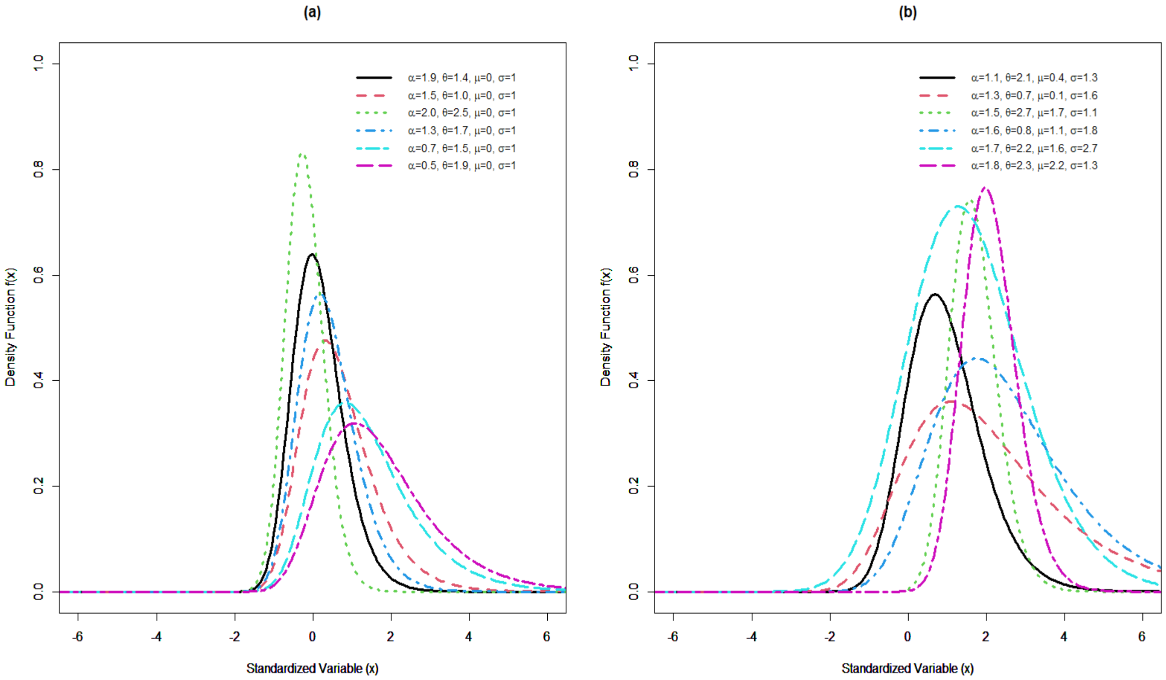

Figure 1: *The TIITLEGU density function (pdf) plots with selected parameter values.*

3. Properties of the TIITLEGU Model

3.1 Quantile and Median Functions

The quantile function (qf) of the TIITLEGU model derived by inverting Eq. (5) is specified as

$$Q(u) = -\sigma \log\left[-\log\left(1-\left\{1-\left[1-(1-u)^{\frac{1}{\theta}}\right]^{\frac{1}{2}}\right\}^{\frac{1}{\alpha}}\right)\right]+\mu \tag{10}$$

where $u \sim \text{Uniform}(0,1)$. The median (M) of the TIITLEGU model is specified as

$$M = -\sigma \log\left[-\log\left(1-\left\{1-\left[1-(0.5)^{\frac{1}{\theta}}\right]^{\frac{1}{2}}\right\}^{\frac{1}{\alpha}}\right)\right]+\mu \tag{11}$$

By means of the quantile function in Eq (10), various quantile measures can be estimated.

3.2 Raw-Moment and Moment-Generating-Function

The raw-moment (rm) of the TIITLEGU model is derived using the expansion series approach. Let $X \sim TIITL_{EGU}(x;\alpha,\theta,\mu,\sigma)$, the rth raw-moment of X is specified as

$$\acute{\mu}_r = \int_0^{+\infty} x^r f(x;\alpha,\theta,\mu,\sigma)dx \tag{12}$$

The expanded form of the TIITLEGU pdf using the series expansion is

$$f(x) = \sum_{m=0}^{\infty} \omega_m e^{-(m+1)\frac{x-\mu}{\sigma}} \tag{13}$$

where $\omega_m = \frac{2\alpha\theta}{\sigma}\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j+k+m}(k+1)^m\binom{\theta-1}{i}\binom{2i-1}{j}\binom{\alpha(j+1)-1}{k}}{m!}$

By inserting the expanded form in Eq. (13) into Eq. (12), we have

$$\acute{\mu}_r = \sum_{m=0}^{\infty} \omega_m \int_0^{+\infty} x^r\, e^{-(m+1)\frac{x-\mu}{\sigma}}dx, \tag{14}$$

Let,

$z = (m+1)\frac{x-\mu}{\sigma} \Rightarrow x = \frac{z}{-\frac{\mu}{\sigma}(m+1)}$,

$\frac{dx}{dz} = \frac{1}{-\frac{\mu}{\sigma}(m+1)} \Rightarrow dx = \frac{1}{-\frac{\mu}{\sigma}(m+1)}dz$,

Hence, the rm of the TIITLEGU is specified as

$$\acute{\mu}_r = \sum_{m=0}^{\infty} \omega_m \int_0^{+\infty} \left(\frac{z}{-\frac{\mu}{\sigma}(m+1)}\right)^r e^{-z} \frac{1}{-\frac{\mu}{\sigma}(m+1)} dz, \tag{15}$$

Simplifying Eq. (15) leads to

$$\acute{\mu}_r = \sum_{m=0}^{\infty} \omega_m \left(-\frac{\mu}{\sigma}(m+1)\right)^{-r-1} \int_0^{+\infty} (z)^r e^{-z} dz, \tag{16}$$

Using the gamma integral representation $\Gamma(\alpha+1) = \int_0^{+\infty} (y)^\alpha e^{-y} dy$. Therefore, the rm of the TIITLEGU is specified as

$$\acute{\mu}_r = \sum_{m=0}^{\infty} \omega_m \frac{\Gamma(r+1)}{\left(-\frac{\mu}{\sigma}(m+1)\right)^{r+1}} \tag{17}$$

where $\omega_m = \frac{2\alpha\theta}{\sigma} \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k+m}(k+1)^m \binom{\theta-1}{i}\binom{2i-1}{j}\binom{\alpha(j+1)-1}{k}}{m!}$.

Table 1 reports the first four raw-moments, standard-deviation (SD), dispersion index (DI) and coefficient of variation (CV) for the TIITLEGU model utilizing Eq. (17) with fixed $\mu = 0,\ \sigma = 1$.

Table 1: *Summary statistics for the TIITLEGU model*

$\acute{\mu}_r$	$\alpha = 1.5, \theta = 1.5$	$\alpha = 2.5, \theta = 2.0$	$\alpha = 3.3, \theta = 3.5$	$\alpha = 3.7, \theta = 3.7$
$\acute{\mu}_1$	0.4810	0.0969	0.0057	0.0022
$\acute{\mu}_2$	0.5850	0.0577	0.0016	0.0006
$\acute{\mu}_3$	0.9590	0.0472	0.0006	0.0002
$\acute{\mu}_4$	1.9600	0.0483	0.0003	0.0000
SD	0.5950	0.2200	0.0396	0.0244
DI	0.7350	0.4990	0.2750	0.2710
CV	1.2400	2.2700	6.9500	11.1000

Additionally, the moment generating function (mgf) of the TIITLEGU is specified as

$$M_r(t) = \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \omega_m \frac{\Gamma(r+1)t^r}{\left(-\frac{\mu}{\sigma}(m+1)\right)^{r+1} r!} \tag{18}$$

where $\omega_m = \frac{2\alpha\theta}{\sigma} \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k+m}(k+1)^m \binom{\theta-1}{i}\binom{2i-1}{j}\binom{\alpha(j+1)-1}{k}}{m!}$.

3.3 Order Statistics

Let $x_1, x_2, \dots, x_n$ denote a random sample from the TIITLEGU and $x_{1:n} < x_{2:n} < \cdots < x_{n:n}$ be the order statistics obtained from the sample. The p[th] order statistics (os) is specified as

$$f_{p:n}(x) = \frac{f(x)}{B(p,n-p+1)} [F(x)]^{p-1} [1-F(x)]^{n-p} \tag{19}$$

By inserting Eqs. (5) and (6) into Eq. (19), we have

$$f_{p:n}(x) = \frac{2\theta\Phi(x)\{1-[\Theta(x)]^2\}^{\theta-1}}{B(p,n-p+1)} \left[1 - \{1-[\Theta(x)]^2\}^\theta\right]^{p-1} \left[1 - \left(1 - \{1-[\Theta(x)]^2\}^\theta\right)\right]^{n-p} \tag{20}$$

Simplifying Eq. (20), the os of the TIITLEGU is specified as

$$f_{p:n}(x) = \frac{2\theta\Phi(x)\{1-[\Theta(x)]^2\}^{\theta(n-p+1)-1}}{B(p,n-p+1)} \left[1 - \{1-[\Theta(x)]^2\}^\theta\right]^{p-1} \tag{21}$$

The minimum and maximum order statistics is derived by inserting $p = 1$ and p = n into Eq. (21).

4. Estimation and Simulation Study

4.1 Maximum Likelihood Estimation

Let $x_1, x_2, \dots, x_n$ denote the observed random-values from the $TIITL_{EGU}(x; \alpha, \theta, \mu, \sigma)$. Assuming that

$\mu = 0,\ \sigma = 1$, without loss of generality, the log-likelihood-function (LL) is specified as

$$LL(\theta,\alpha) = n\log 2 + n\log\theta + n\log\alpha + (\alpha-1)\sum_{i=1}^{n}\log\{\eta_i\} + \sum_{i=1}^{n}\log\{1-\eta_i\} + \sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\log(1-\eta_i^{\alpha}) + (\theta-1)\sum_{i=1}^{n}\log[1-(1-\eta_i^{\alpha})^2] \quad (22)$$

where $\eta_i = 1 - e^{[-e^{(-x)}]}$. Differentiating Eq. (22) with respect to θ and α gives the follows:

$$\frac{\partial LL}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\log[1-(1-\eta_i^{\alpha})^2]$$

$$\frac{\partial LL}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\{\eta_i\} - \sum_{i=1}^{n}\frac{\eta_i^{\alpha}\log\{\eta_i\}}{1-\eta_i^{\alpha}} + (\theta-1)\sum_{i=1}^{n}\frac{2(1-\eta_i^{\alpha})\eta_i^{\alpha}\log\{\eta_i\}}{1-(1-\eta_i^{\alpha})^2}$$

In this study, R-programming (optim function) is used in finding the maximum likelihood (ML) estimates $(\hat{\theta},\hat{\alpha})$ of the TIITLEGU parameters.

4.2 Simulation-Study

The simulation process for the ML is obtainable as follows: N = 10,000 samples (replicates) are generated from the TIITLEGU model with sizes n′ = 20,50, 150, 250, 500 and 1000 using R-programming. The precision of the ML estimates is evaluated via the mean estimates (MEs), absolute bias (Absbias), mean square errors (MSEs) and root mean square roots (RMSEs). The MLE is a suitable technique for estimating the TIITLEGU parameters Based on the simulation study. The results reported in Table 2 indicate that the parameter estimates are quite stable and very close to the true parameter values for the various sample sizes. The ME tend to be closer to the true values of the parameter with minimum MSEs, and RMSEs values as the sample size increases.

Table 2: *Numerical values of MEs, Absbias, MSEs and RMSEs*

		$\alpha = 1.5, \theta = 1.5$						$\alpha = 3.5, \theta = 3.3$			
n'	Par.	ME	Absbias	MSE	RMSE	n'	Par.	ME	Absbias	MSE	RMSE
20	α	1.7238	0.2238	2.1842	1.4779	20	α	3.7210	0.4210	8.8988	2.9831
	θ	3.1475	1.6475	10.9000	3.3015		θ	7.2017	3.7017	51.1596	7.1526
50	α	1.6239	0.1239	1.0649	1.0320	50	α	3.5275	0.2275	4.4887	2.1187
	θ	2.4156	0.9156	4.5588	2.1351		θ	5.7094	2.2094	23.6443	4.8625
150	α	1.5312	0.0312	0.3634	0.6029	150	α	3.3703	0.0703	1.8493	1.3599
	θ	1.8985	0.3985	1.3260	1.1515		θ	4.6138	1.1138	8.9932	2.9989
250	α	1.5156	0.0156	0.2146	0.4633	250	α	3.3349	0.0349	1.1736	1.0833
	θ	1.7484	0.2484	0.7015	0.8375		θ	4.2532	0.7532	5.2725	2.2962
500	α	1.5089	0.0089	0.1105	0.3323	500	α	3.3155	0.0155	0.6437	0.8023
	θ	1.6267	0.1267	0.3019	0.5495		θ	3.9323	0.4323	2.6383	1.6243
1000	α	1.5033	0.0033	0.0561	0.2368	1000	α	3.3028	0.0028	0.3303	0.5747
	θ	1.5653	0.0653	0.1336	0.3655		θ	3.7293	0.2293	1.1773	1.0850

		$\alpha = 2.0, \theta = 2.5$						$\alpha = 4.1, \theta = 4.5$			
n'	Par.	ME	Absbias	MSE	RMSE	n'	Par.	ME	Absbias	MSE	RMSE
20	α	2.8662	0.3662	6.1123	2.4723	20	α	5.0225	0.5225	15.9168	3.9896
	θ	4.2505	2.2505	19.2609	4.3887		θ	8.4532	4.3532	69.0055	8.3070
50	α	2.6997	0.1997	2.9340	1.7129	50	α	4.7706	0.2706	8.0638	2.8397
	θ	3.2939	3.2939	8.5372	2.9218		θ	6.8341	2.7341	35.0702	5.9220
150	α	2.5439	0.0439	1.0106	1.0053	150	α	4.5916	0.0916	3.5820	1.8926
	θ	2.5848	0.5848	2.7012	1.6435		θ	5.5217	1.4217	13.8556	3.7223
250	α	2.5253	0.0253	0.6185	0.7865	250	α	4.5407	0.0407	2.2462	1.4987
	θ	2.3669	0.3669	1.4303	1.1959		θ	5.0808	0.9808	8.5140	2.9179
500	α	2.5110	0.0110	0.3252	0.5702	500	α	4.5209	0.0209	1.2639	1.1243
	θ	2.1968	0.1968	0.6399	0.7999		θ	4.6527	0.5527	4.0301	2.0075
1000	α	2.5028	0.0028	0.1635	0.4043	1000	α	4.5031	0.0031	0.6643	0.8150
	θ	2.1011	0.1011	0.2767	0.5260		θ	4.4001	0.3001	1.8519	1.3608

5. EGARCH-Model

The asymmetric EGARCH model introduced by [10] is considered as a variate of the ARCH/GARCH model introduced by [3] and [27] for modeling time-varying volatility. The EGARCH differ from the symmetric GARCH variance structure given that the natural log variance is used in other for the parameters to be unrestricted and can take negative values while guaranteeing a positive conditional variance. Moreso, the EGARCH model includes the asymmetric impact of positive and negative shocks on volatility. The return of daily prices of assets is represented by r_t and the EGARCH (1,1) model is defined as:

$$r_t = \mu + \varepsilon_t,$$
$$\varepsilon_t = z_t\sqrt{h_t^2},\ \ z_t \sim i.i.d,$$
$$\log h_t^2 = \lambda_0 + \beta_1 \log h_{t-1}^2 + \lambda_1 \frac{\varepsilon_{t-1}}{\sqrt{h_{t-1}^2}} + \gamma_1 \left|\frac{\varepsilon_{t-1}}{\sqrt{h_{t-1}^2}}\right| \tag{23}$$

where $\lambda_0 > 0,\ \lambda_1 > 0,\ \beta_1 > 0$ are the model parameters, z_t is the conditional innovation density with $E(z_t) = 0$ and $var(z_t) = 1$, μ_t is the conditional mean, γ_1 is the leverage parameter, $\log h_t^2$ is conditional log variance at present day t, ε_{t-1} and $\log h_{t-1}^2$ are the error and conditional log variance at preceding day $t-1$, respectively. The commonly utilized conditional innovation densities are well described in the literature.

5.1 The Novel Conditional Innovation Density

The log-likelihood (LL) function of the standardized TIITLEGU model presented in Eq. (9), is specified as

$$LL(\vartheta) = n\log 2 + n\log\theta + n\log\alpha - \frac{n}{2}\log h_t^2 + (\alpha-1)\sum_{t=1}^{n}\log\{\eta_t\} + \sum_{t=1}^{n}\log\{1-\eta_t\} + \sum_{t=1}^{n}\log\left[e^{\left(\frac{\varepsilon_t}{h_t^2}\right)}\right] + \sum_{t=1}^{n}\log(1-\eta_t^{\alpha}) + (\theta-1)\sum_{t=1}^{n}\log[1-(1-\eta_t^{\alpha})^2] - \frac{n}{2}\log h_t^2 \tag{24}$$

where $\vartheta = (\alpha, \theta, h_t)$, α, θ are the shape parameters and h_t is the EGARCH volatility model with vector parameters and $\eta_t = 1 - e^{\left[-e^{\left(-\frac{\varepsilon_t}{h_t^2}\right)}\right]}$.

5.2 Model Selection Criteria

The modified Akaike information criteria (AIC) and Bayesian information criteria (BIC) proposed by [28] are utilized in selecting the best model under the conditional innovation densities. The modified AIC and BIC criteria are given by

$$AIC = \frac{2k}{N} - \frac{2LL}{N} \tag{25}$$
$$BIC = \frac{k\log_e(N)}{N} - \frac{2LL}{N} \tag{26}$$

where k is the total number of estimated parameters, the estimated log-likelihood value and sample size are denoted by LL and N, respectively. The EGARCH model with the least AIC and BIC values is regarded as the most suitable model under the specified innovation density.

5.3 Forecasts Performance

The forecasts performance of the EGARCH models is appraised using the mean square error (MSE), root mean square root (RMSE), and mean absolute error (MAE). The performance measures for the volatility forecasts are given by

$$MSE = \frac{1}{N}\sum_{t=1}^{N}\left(\widehat{h_t} - h_t\right)^2 \tag{27}$$

$$RMSE = \sqrt{\frac{1}{N}\sum_{t=1}^{N}\left(\widehat{h_t} - h_t\right)^2} \tag{28}$$

$$MSE = \frac{1}{N}\sum_{t=1}^{N}\left|\widehat{h_t} - h_t\right| \tag{29}$$

Where $\widehat{h_t}$ and h_t represent the volatility forecast and realized volatility, and N is the sample size. The model with the least performance measures is regarded as the most appropriate for predicting the volatility of the daily log-returns.

6. Empirical Results

6.1 Data Report

To appraise the performance of the novel distributed innovation density in the EGARCH model, the United bank of Africa (UBA) and Total energies Nigeria (TEN) stock prices log-returns are utilized. The UBA dataset consists 5468 daily log-returns from 1/2/2000 to 5/1/2024 and TEN datasets consist 5510 daily returns from 2/1/2001 to 5/1/2024. The estimation process is executed using 5218 daily log-returns from 1/2/2000 to 30/12/2022 for the UBA while 5268 daily returns from 2/1/2001 to 30/12/2022 for the TEN. The forecast evaluation of the models is carried-out with 250 daily returns from 3/1/2023 to 5/1/2024 for both UBA and TEN. The summary statistics of the daily returns for the estimation processes are reported in Table 2. More so, the graphical plots of the daily returns, squared-returns and absolute-returns with their respective sample autocorrelation function (ACF) for both UBA and TEN are depicted in Figures 3 and 4.

Tables 2 reports positive skewness and high excess kurtosis, leading to large Jarque-Bera (JB) statistic ($p < 0.001$) signifying that the daily returns for the estimation process have non-normality characteristics. Figure 2 displays the density function of a normal distribution that has the same mean and standard deviation as those of the UBA and TEN return series. The plots provide a visual check of the normality assumption for the daily returns. The deviation between the solid (return series) and dashed line (Normal distribution) indicates that the daily returns are not normally distributed.

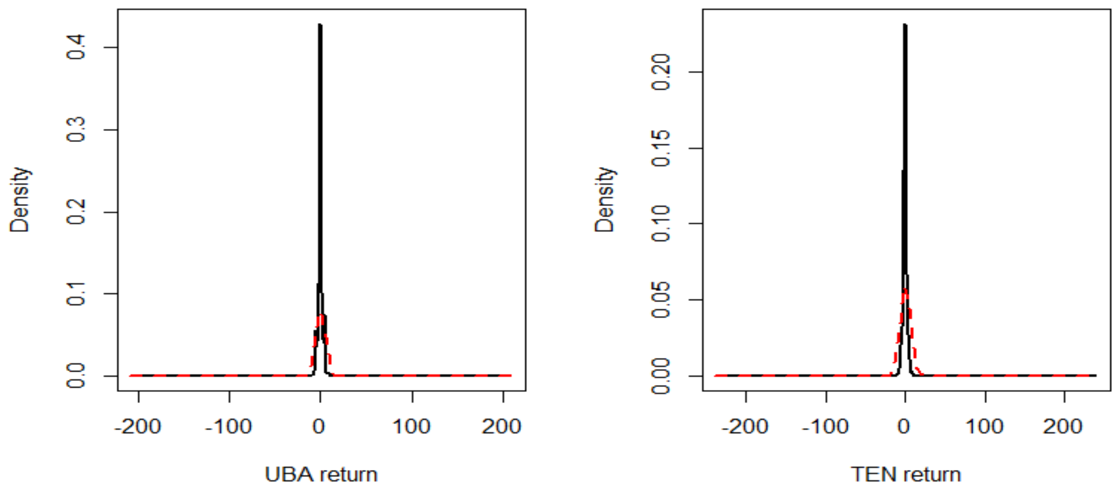

Figure 2: *Empirical density function of the UBA and TEN daily returns.*

Further, the ARCH Lagrange-multiplier (LM) and Ljung Box-Q tests at lag 10, indicates the incidence of conditional heteroscedasticity and autocorrelation in the returns while the Augmented Dickey-Fuller (ADF) test with its p-value indicates that the returns for the UBA and TEN are stationary.

Table 2: *Summary statistics for the daily returns*

UBA returns (Estimation process)					
Number of observations	Mean	Median	Minimum	Maximum	Std Dev.
5218	-0.004	0.000	-204.521	206.978	5.005
Skewness	Kurtosis	Jarque-Bera	ARCH (10)	Q (10)	ADF test
0.087	1096.432	26157 (< 0.0001)	894.89 (< 0.0001)	901.99 (< 0.0001)	-24.7 (<0.01)
TEN returns (Estimation process)					
Number of observations	Mean	Median	Minimum	Maximum	Std Dev.
5268	0.022	0.000	-231.941	235.569	6.790
Skewness	Kurtosis	Jarque-Bera	ARCH (10)	Q (10)	ADF test
0.999	1027.144	23175 (< 0.0001)	107.55 (< 0.0001)	117.84 (< 0.0001)	-27.5 (<0.01)

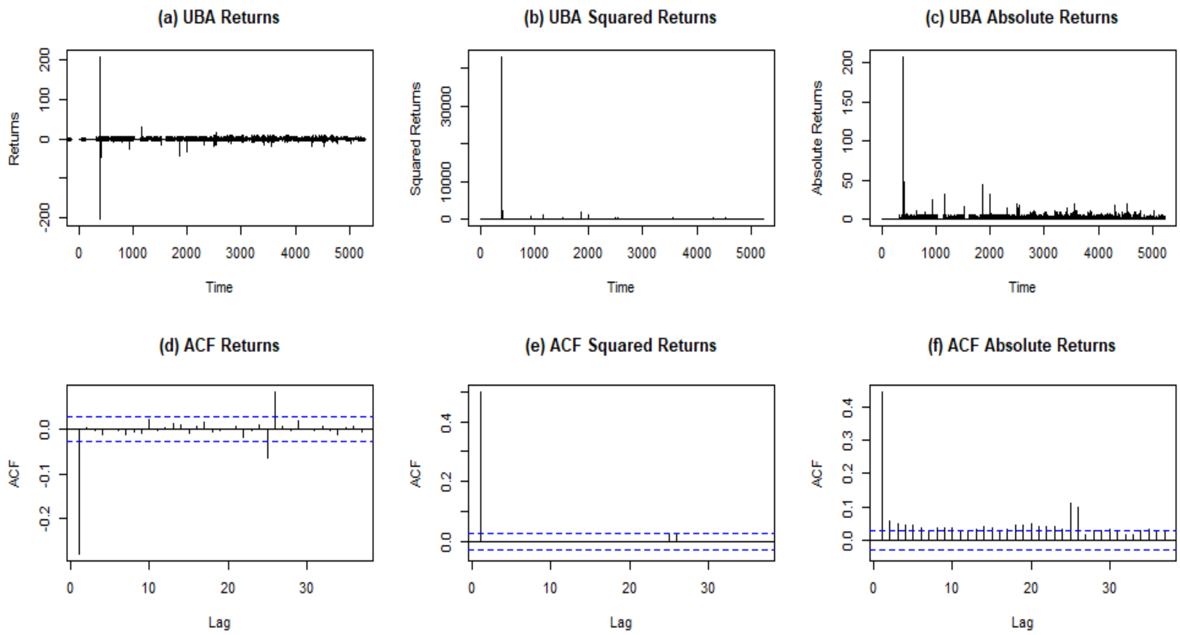

Figure 3: *UBA daily returns, squared returns, absolute returns and sample autocorrelations.*

6.2 Parameters Estimation of the EGARCH (1,1) Model

The EGARCH (1,1) model specified in Eq. (22) is estimated under ten different innovation densities: normal (NORM), student-t (ST), generalized error (GE), skew normal (SNORM), skew student-t (SST), skew generalized error (SGE), generalized hyperbolic (GHYP), Johnson (SU) reparametrized (JSU), Normal inverse Gaussian (NIG) and the novel type II Topp Leone exponentiated Gumbel (TIITLEGU). Tables 3 and 4 reports the estimated parameters of the EGARCH (1,1) models. The *rugarch* package in R-programming is used in estimating the parameters of the EGARCH-NORM, EGARCH-ST, EGARCH-GE, EGARCH-SNORM, EGARCH-SST, EGARCH-SGE, EGARCH-GHYP, EGARCH-JSU and EGARCH-NIG while the *Optim* function in R-programming is utilized to maximize the log-likelihood function of EGARCH-TIITLEGU. As reported in Table 5, the EGARCH-TIITLEGU model has the highest log-likelihood (LL) value and exhibits superior fit to the standardized residuals compare to others for both return series.

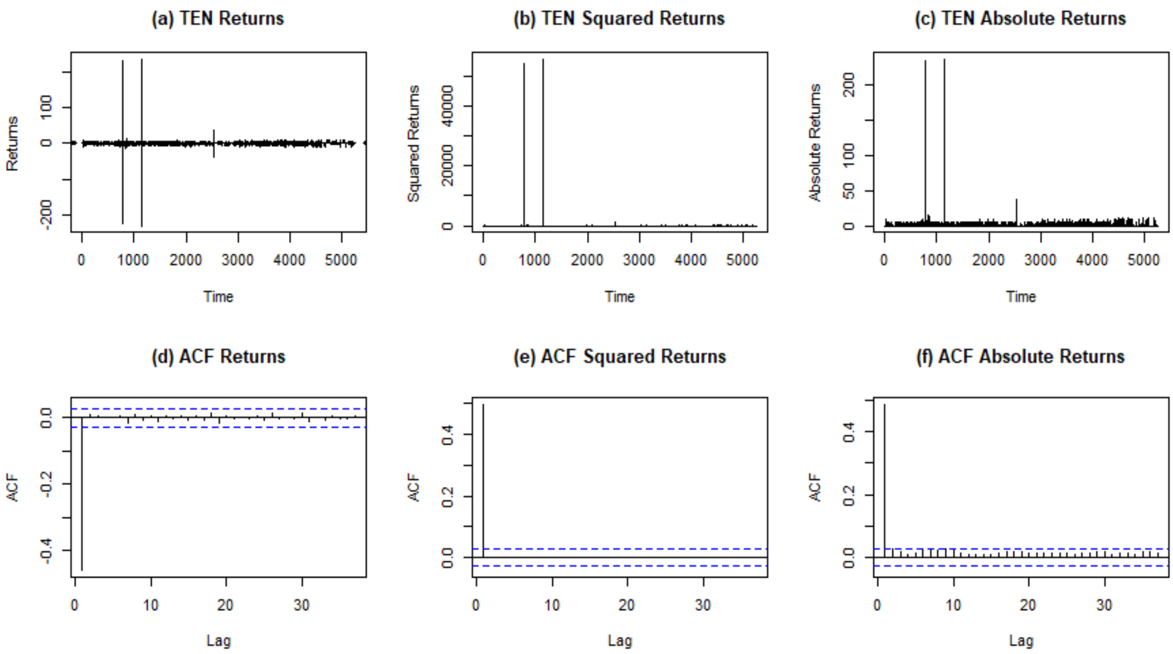

Figure 4: *TEN daily returns, squared returns, absolute returns and sample autocorrelations.*

Table 3: *EGARCH Model Parameter Estimates with Innovation Densities (UBA returns).*

Model	Cond. Distr.	λ_0	λ_1	β_1	γ_1	Ghlambda	Skew	Shape1	Shape2
EGARCH (1,1)	NORM	0.5707‘***’	-0.0581‘***’	0.7492‘***’	0.4881‘***’		-	-	-
	STD	0.4320‘***’	0.0960‘’	0.888‘***’	2.1239‘***’		-	2.1000‘***’	-
	GED	8.5262‘***’	1.2433	0.1171	6.5026‘***’		-	0.1019‘***’	-
	SNORM	0.5680‘***’	-0.0488‘***’	0.7488‘***’	0.4867‘***’		0.9781‘***’	-	-
	SSTD	0.6725‘***’	0.2855‘’	0.8914‘***’	6.3057‘***’		1.0003‘***’	2.0100‘***’	-
	SGED	1.6151‘***’	0.7086‘***’	0.3730‘***’	3.6658‘***’		1.0005‘***’	0.3956‘***’	-
	GHYP	0.0240‘***’	0.0524‘***’	0.9858‘***’	0.2100‘***’	-0.9677‘***’	-0.0547	0.2500‘***’	-
	JSU	0.2856‘’	0.0571‘***’	0.8828‘***’	1.1730‘***’		0.0001	0.8269‘***’	-
	NIG	0.7949‘***’	0.3175‘***’	0.9532‘***’	4.3156‘***’		0.0172	0.0100‘***’	-
	TIITLEGU	2.327e-10‘’	3.929e-10‘’	0.9828‘***’	0.7799‘***’		-	1.9433‘***’	3.5128‘***’

Significance levels: 0 ‘***’, 0.001 ‘**’, 0.01 ‘*’, 0.05 ‘.’, 0.1 ‘’, 1

Table 4: *EGARCH Model Parameter Estimates with Innovation Densities (TEN returns).*

Model	Cond. Distr.	λ_0	λ_1	β_1	γ_1	Ghlambda	Skew	Shape1	Shape2
EGARCH (1,1)	NORM	0.4283‘***’	-0.0869‘***’	0.7781‘***’	0.2932‘***’		-	-	-
	STD	-0.0587	0.2583‘***’	0.8234‘***’	0.6745‘***’		-	2.1000‘***’	-
	GED	1.3745‘***’	0.0974	-0.1137‘***’	1.1712‘***’	-	-	0.1024‘***’	-
	SNORM	0.4296‘***’	-0.0858‘***’	0.7765‘***’	0.3024‘***’		0.9552‘***’	-	-
	SSTD	0.2775‘***’	0.7821‘***’	0.8283‘***’	1.8532‘***’		1.0011‘***’	2.0100‘***’	-
	SGED	0.6894‘***’	0.0739‘***’	0.2451‘***’	0.0659‘***’		1.1175‘***’	0.3868‘***’	-
	GHYP	-0.0226‘***’	-0.0294‘***’	0.9880‘***’	0.0905‘***’	-02818‘***’	0.0025	0.2500‘***’	-
	JSU	0.4089‘***’	-0.0000	0.3658‘***’	-0.0000		0.2953‘***’	0.2616‘***’	-
	NIG	-0.0434‘***’	0.0071‘***’	1.0000‘***’	0.0177‘***’		-0.0071	0.0100‘***’	-
	TIITLEGU	0.0552‘***’	0.0579‘***’	0.5195‘***’	1.5234‘***’		-	5.2086‘***’	9.8771‘***’

Significance levels: 0 ‘***’, 0.001 ‘**’, 0.01 ‘*’, 0.05 ‘.’, 0.1 ‘’, 1

Tables 3 and 4 reports that the parameter estimates of the EGARCH conditional variance specifications are highly statistically significant and γ_1 is highly significant which shows that the daily log-returns have leverage effect. Therefore, the impact of the shocks is asymmetric in nature that is, the impact of positive shocks on volatility are higher than negative shocks of the similar size. The AIC and BIC values also reported in Table 5 suggest that the EGARCH-TIITLEGU model is best for investigating the volatility of the Nigerian stock market.

Table 5: *Comparison of the Innovation Densities for estimated models.*

Model	Innovation Dist.	Log-Likelihood	AIC	BIC	Log-Likelihood	AIC	BIC
		(UBA Returns).			(TEN Returns).		
EGARCH (1,1)	NORM	-12581.400	4.8242	4.8305	-11134.030	4.4296	4.461
	STD	-11070.810	4.2456	4.2532	-8291.421	3.3071	3.3149
	GED	-5747.029	2.2051	2.2126	19540.740	-7.7859	-7.7781
	SNORM	-12578.510	4.8235	4.8310	-11124.140	4.4361	4.4439
	SSTD	-11046.030	4.2365	4.2453	-8173.160	3.2603	3.2694
	SGED	-10738.730	4.1187	4.1275	-4397.584	1.7555	1.7646
	GHYP	-10923.350	4.1899	4.1999	-8299.449	3.3111	3.3215
	JSU	-11066.350	4.2443	4.2531	9042.654	-3.6013	-3.5922
	NIG	-10092	3.8709	3.8797	-4979.278	1.9874	1.9965
	TIITLEGU	**13546.100**	**-5.1894**	**-5.1806**	**25184.930**	**-9.5588**	**-9.5501**

Note(s): *Bolded values indicate the highest log-likelihood value, and the least AIC and BIC values*

Tables 6 and 7 reports the diagnostic tests results for the EGARCH (1,1) model under the various innovation densities. From Table 6, the Ljung Box-Q statistic ($p > 0.05$) specifies that the squared standardized residuals from the EGARCH-TIITLEGU model exhibit no sign of serial-correlation. Likewise, the ARCH-LM statistic ($p > 0.05$) indicates that the standardized residuals from the EGARCH-TIITLEGU model exhibit no additional conditional heteroscedasticity, that is, the conditional variance equation are correctly specified. Therefore, the results disclose that standardized TIITLEGU density is an improved distributed innovation function for the EGARCH (1,1) model.

Table 6: *Estimated EGARCH (1,1) models diagnostic tests (UBA returns).*

Model	Innovation Dist.	Ljung-Box Q-Statistic	p-value	ARCH-LM Statistic	p-value
EGARCH (1,1)	NORM	0.007	0.999	0.007	0.999
	STD	0.008	0.999	0.008	0.999
	GED	0.010	0.999	0.010	0.999
	SNORM	0.007	0.999	0.007	0.999
	SSTD	0.008	0.999	0.008	0.999
	SGED	0.007	0.999	0.007	0.999
	GHYP	0.066	0.999	0.067	0.999
	JSU	0.008	0.999	0.008	0.999
	NIG	0.021	0.999	0.021	0.999
	TIITLEGU	0.013	0.999	0.013	0.999

Note(s): *Significance level:* $\alpha = 0.05$.

Table 7: *Estimated EGARCH (1,1) models diagnostic tests (TEN returns).*

Model	Innovation Dist.	Ljung-Box Q-Statistic	p-value	ARCH-LM Statistic	p-value
EGARCH (1,1)	NORM	0.021	0.999	0.021	0.999
	STD	0.091	0.999	0.092	0.999
	GED	5.936	0.981	5.923	0.981
	SNORM	0.021	0.999	0.020	0.999
	SSTD	0.088	0.999	0.088	0.999
	SGED	0.010	0.999	0.010	0.999
	GHYP	0.045	0.999	0.046	0.999
	JSU	1251.5	2.2E-16	2317.7	2.2E-16
	NIG	0.477	0.999	0.474	0.999
	$TIITL_{EGU}$	0.039	0.999	0.039	0.999

Note(s): *Significance level:* $\alpha = 0.05$.

6.3 Forecasts Evaluation of the EGARCH Models

The forecast evaluation metrics for the out-of-sample are reported in Table 8, and the least MSE, RMSE and MAE values belong to the EGARCH-$TIITL_{EGU}$ model. Therefore, the EGARCH-$TIITL_{EGU}$ model is statistically efficient and displays superior capability in forecasting the volatility of the Nigerian stock market relative to other models.

Table 8: *Forecasts evaluation metrics of the estimated EGARCH (1,1) models.*

Model	Innovation Dist.	MSE	RMSE	MAE	MSE	RMSE	MAE
		(UBA returns)			(TEN returns)		
EGARCH (1,1)	NORM	8.155	2.856	1.752	1.731	1.316	0.213
	STD	8.151	2.855	1.751	1.733	1.316	0.197
	GED	8.151	2.855	1.751	1.733	1.316	0.197
	SNORM	8.121	2.849	1.752	1.730	1.315	0.267
	SSTD	8.151	2.855	1.751	1.733	1.316	0.197
	SGED	8.150	2.855	1.751	1.777	1.332	0.354
	GHYP	8.159	2.856	1.753	1.733	1.316	0.197
	JSU	8.151	2.855	1.751	1.733	1.316	0.198
	NIG	8.151	2.855	1.751	1.733	1.316	0.197
	$TIITL_{EGU}$	**8.050**	**2.837**	**1.736**	**1.547**	**1.244**	**0.182**

Note(s): *Bolded values indicate the conditional distribution with the least MSE, RMSE and MAE.*

7. Conclusion

The estimation of the $TIITL_{EGU}$ model parameters using the MLE procedure, and introduction as a novel distributed innovation function for the EGARCH-volatility model is considered. The density and cumulative functions, failure rate function, quantile function, standardized density function and other mathematical properties are derived. Monte-Carlo experiments are carried out to study the performance of the MLE procedure. The experiments results indicate that the MLE is asymptotically unbiased and consistent given that the ME tend to be closer to the true values of the parameter with minimum MSEs, and RMSEs as the sample size increases.

Additionally, the standardized $TIITL_{EGU}$ density is presented as a novel distributed innovation function for the EGARCH volatility model for investigating the volatility of the Nigerian stock market via the UBA and TEN returns. The empirical findings showed that the EGARCH-TIITL$_{EGU}$ model has the highest log-likelihood, and least AIC and BIC values. Equally, the EGARCH-TIITL$_{EGU}$ model has the least forecast evaluation metrics among other models. In conclusion, the EGARCH-TIITL$_{EGU}$ model is best for investigating the volatility of the Nigerian stock market.

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