INVERTED DAGUM DISTRIBUTION: PROPERTIES AND APPLICATION TO LIFETIME DATASET.

Abdulhameed A. Osi*, Shamsuddeen A. Sabo and Ibrahim Z. Musa

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Department of Statistics, Aliko Dangote University of Science and Technology, Wudil, Nigeria.
abuammarosi@gmail.com, aaosi@kust.edu,ng

Abstract

This article presents the introduction of a novel univariate probability distribution termed the inverted Dagum distribution. Extensive analysis of the statistical properties of this distribution, including the hazard function, survival function, Renyi's entropy, quantile function, and the distribution of the order statistics, was conducted. Parameter estimation of the model was performed utilizing the maximum likelihood method, with the consistency of the estimates validated through Monte Carlo simulation. Furthermore, the applicability of the proposed distribution was demonstrated through the analysis of two real datasets.

Keywords: Dagum distribution, Inverse transformation, Maximum likelihood estimation, Monte Carlo Simulation, COVID-19 data

1. INTRODUCTION

Dagum distribution was named after Camilo Dagum who proposed it in the 1970s to fit wealth and income data as well as accommodate heavy-tailed models. Dagum distribution can be of two forms (the three-parameter and the four-parameter) respectively referred to as Dagum type I distribution [1] and Dagum type II distribution [2].

Definition: A random variable is said to have type I Dagum distribution with parameter if its cumulative density function and probability density function are given by:

$$F_{\alpha_{DD}}(x;\theta,\alpha,\lambda) = \left[1+\lambda x^{-\alpha}\right]^{-\theta} x > 0 \tag{1}$$

$$f_{\alpha_{DD}}(x;\theta,\alpha,\lambda) = \theta\alpha\lambda x^{-\alpha-1}\left(1+\lambda x^{-\alpha}\right)^{-\theta-1} \tag{2}$$

Where $\alpha,\theta,\lambda > 0$. With λ is the scale parameter and α, θ are the two shape parameters. The latter controls the tail weight and the former controls the size of the distribution. It can be observed that for $\theta = 1$, Eq. 1 becomes the log-logistic distribution proposed by [3]. It can also be observed that Eq. 1 is a Burr III distribution with an additional λ . One crucial characteristic possessed by Dagum distribution is that, in addition to its flexibility, its hazard function can be decreased, up-side down, bathtub, and then up-side bathtub shaped [4]. A lot of researchers utilized this behavior to study the Dagum distribution in several fields. An extensive review of the Dagum distribution and its application was detailed in [6] and [7]. The parameters of the Dagum distribution with censored samples was studied by [5] while [8] utilized TL-moments for similar purpose. Some classs of weighted Dagum and related distributions were proposed by [10] and the five-parameter beta-Dagum by [4]. Considering the properties of McDonald, Kumaraswamy and Dagum distribution, two new hybrid distributions called Mc-Dagum and Kum-Dagum distributions were proposed by [10]. Numerous ways of extending well-known distributions were suggested by applied statisticians. An extended Dagum, distribution was proposed by [17]. The distribution of the reciprocal of any well-known distribution are their corresponding inverted distributions. This had received numerous attentions from researchers in the field of distribution theory. Few of these works are; inverted gamma by [12], inverted exponential by [19], inverted Weibull by [20] to mention a few. In this paper, we proposed an inverted Dagum distribution. Some statistical properties of the proposed distribution were derived. A simulation study is further performed to check the flexibility and usefulness of the proposed distribution.

2. METHODS

2.1. Proposed Distribution

Motivated by [10] and the literature cited therein, we proposed an Inverted Dagum Distribution (IDD). A random variable X is said to have an Inverted Dagum Distribution, if the following transformation is applicable $X = \frac{1}{Y}$. Where Y is the Dagum distribution random variable with pdf and cdf expressed respectively in Eq. 1 and Eq. 2. By applying cdf technique:

$$F_Y(y) = P(Y \leq y) = P\left(X > \frac{1}{y}\right) = 1 - P\left(X \leq \frac{1}{y}\right) \tag{3}$$

The pdf and cdf of the proposed distribution can be expressed as;

$$F_{IDD}(x;\theta,\alpha,\lambda) = 1 - [1 + \lambda x^{\alpha}]^{-\theta} \tag{4}$$

$$f_{IDD}(x;\theta,\alpha,\lambda) = \theta\alpha\lambda x^{\alpha-1}(1 + \lambda x^{\alpha})^{-\theta-1} \quad x > 0 \tag{5}$$

2.2. Some Mathematical properties

Here, some important Mathematical and Statistical properties of the proposed inverted Dagum distribution like quantile function, hazard function, moments, moment generating function, Renyi's entropy were presented.

2.2.1 Quantile Function

Let X be a random variable with pdf given in Eq.(5). The quantile function of X (the proposed distribution) can be expressed as;

$$Q(u;\theta,\alpha,\lambda) = \left\{\lambda^{-1}\left[(1-u)^{-\frac{1}{\theta}} - 1\right]\right\}^{\frac{1}{\alpha}} \tag{6}$$

2.2.2 Moments

The moments of any distribution tell a lot about its features. Characteristics like tendency, skewness, dispersion, kurtosis etc can be observed through moments. If the random variable X has an inverted Dagum distribution, then its r^{th} moment about zero can be expressed as;

$$E\left(X^r\right) = \int_0^{\infty} x^r f(x)dx \tag{7}$$

$$= \theta\alpha\lambda^{2-r}\int_0^{\infty} (\lambda x^{\alpha})^{r-1}(1 + \lambda x^{\alpha})^{-((r+1)+(\theta-r))} \tag{8}$$

$$= \theta\alpha\lambda^{2-r}B(1+r, \theta - r) \tag{9}$$

From Eq. (13), the mean and the variance of IDD can be expressed respectively as;

$$E(X) = \theta\alpha\lambda B(2, \theta - 1) \tag{10}$$

$$\text{Variance} = E\left(X^2\right) - [E(X)]^2 \tag{11}$$

$$= \theta\alpha B(3, \theta - 2) - [\theta\alpha\lambda B(2, \theta - 1)]^2 \tag{12}$$

2.2.3 Moment Generating Function

Many important features and characteristics of a distribution can be observed through its moment-generating function (mgf). Let X be a random variable having an inverted Dagum distribution with pdf given in Eq.(5), using the definition of moment generating function of X utilizing Eq, (5) we have;

$$\begin{aligned} M_X(t) = E\left[e^{tx}\right] &= \int_0^\infty e^{tx} f(x)dx \\ &= \theta\lambda\alpha \int_0^\infty e^{tx} x^{\alpha-1}(1+\lambda x^\alpha)^{-\theta-1}dx \\ &= \theta\lambda\alpha \int_0^\infty \sum_{k=0}^\infty \frac{t^k x^k}{k!} x^{\alpha-1}(1+\lambda x^\alpha)^{-\theta-1}dx \end{aligned}$$

$$= \theta\alpha \sum_{k=0}^\infty \frac{\lambda^{2-k}t^k}{k!} \int_0^\infty (\lambda x^\alpha)^{k-1}(1+\lambda x^\alpha)^{-((r+1)+(\theta-r)}dx \tag{13}$$

Eq. (13) can be expressed in a more compact form as;

$$M_X(t) = \theta\alpha M_k(\lambda)B((r+1), (\theta - r)) \tag{14}$$

Where; $M_k(\lambda) = \sum_{k=0}^\infty \frac{\lambda^{2-k}t^k}{k!}$

2.2.4 Survival Function

Let X be a random variable having an inverted Dagum distribution, its survival function is given by;

$$R_X(x; \theta, \alpha, \lambda) = 1 - F_I DD(x; \theta, \alpha, \lambda) \tag{15}$$

$$= [1 + \lambda x^\alpha]^{-\theta} \tag{16}$$

2.2.5 Hazard Function

The reliability characteristics of a system can be checked through its hazard rate function. The hazard rate function of the inverted Dagum distribution is given by;

$$h_X(x; \theta, \alpha, \lambda) = \frac{f(x; \theta, \alpha, \lambda)}{1 - F_X(x; \theta, \alpha, \lambda)} \tag{17}$$

$$= \theta\alpha\lambda x^{\alpha-1}(1+\lambda x^\alpha)^{-1} \tag{18}$$

2.2.6 Renyi's Entropy

This is the measure of the variation of the uncertainty of the distribution. A large value of entropy indicates the greater uncertainty in the data. Renyi's Entropy is defined as:

$$\tau_R(\gamma) = \frac{1}{1-\gamma} \log \left(\int_0^\infty f^\gamma(x) dx \right) \tag{19}$$

From Eq. (5) we have:

$$f^\gamma(x) = (\theta\alpha)^\gamma \lambda^\gamma x^{\gamma(\alpha-1)} \left[(1+\lambda x^\alpha)^{-\gamma(1+\theta)} \right] \tag{20}$$

2.2.7 Order Statistics

Suppose $X_1, X_2, ..., X_n$ are random samples having an inverted Dagum distribution. Let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ denote the order statistics corresponding to the samples. The pdf and cdf of the order statistics are given as;

$$f_{r:n}(t) = \frac{\theta\alpha\lambda x^{\alpha-1} n!}{(r-1)!(n-r)!} \sum_{i=k}^{n} (-1)^n \begin{pmatrix} n-r \\ u \end{pmatrix} (1+\lambda x_i^\alpha)^{-\theta-1} [1-(1-\lambda x^\alpha)]^{r-1+u} \tag{21}$$

$$F_{r:n}(x) = \sum_{i=k}^{n} \sum_{u=0}^{n-r} (-1)^u \begin{pmatrix} n \\ l \end{pmatrix} \begin{pmatrix} n-r \\ u \end{pmatrix} \left[1-(1+\lambda x^\alpha)^{-\theta} \right]^{l+u} \tag{22}$$

2.2.8 Estimation

Here method of maximum likelihood method is used to estimate the parameters of the proposed inverted Dagum distribution. The likelihood function is given by;

$$L(\theta, \lambda, \alpha) = (\theta\lambda\alpha)^n \prod_{i=1}^{n} x_i^{\alpha-1} (1+\lambda x_i^\alpha)^{-\theta-1} \tag{23}$$

The log-likelihood function is;

$$\ln L(\theta, \lambda, \alpha) = n \log\theta + n \log\alpha + n \log\lambda + (\alpha-1) \sum_{i=1}^{n} \log x_i - (\theta+1) \sum_{i=1}^{n} \log(1+\lambda x_i^\alpha) \tag{24}$$

The log-likelihood function is maximized by differentiating Eq. (24) w.r.t to the parameters which yields;

$$\frac{\partial}{\partial\theta} \ln L(\theta, \lambda, \alpha) = \frac{n}{\theta} - \sum_{i=1}^{n} \ln(1+\lambda x_i^\alpha) = 0 \tag{25}$$

$$\frac{\partial}{\partial\alpha} \ln L(\theta, \lambda, \alpha) = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln x_i - (\theta+1) \sum_{i=1}^{n} \frac{\alpha\lambda x_i^{\alpha-1}}{1+\lambda x_i^\alpha} = 0 \tag{26}$$

$$\frac{\partial}{\partial\lambda} \ln L(\theta, \lambda, \alpha) = \frac{n}{\lambda} - (\theta+1) \sum_{i=1}^{n} \frac{x_i^\alpha}{1+\lambda x_i^\alpha} = 0 \tag{27}$$

The maximum likelihood estimates of the parameters can be obtained by solving the non-linear equations (25), (26), and (27) numerically.

Fig. 1 and Fig.3 display the density and CDF plots of IDD as it can be observed the density exhibits right-skewed and reverse J-shaped while Fig. 2 and Fig. 4 show the hazard function and survival function plots respectively. the hazard has increasing, decreasing, and bathtub shapes.

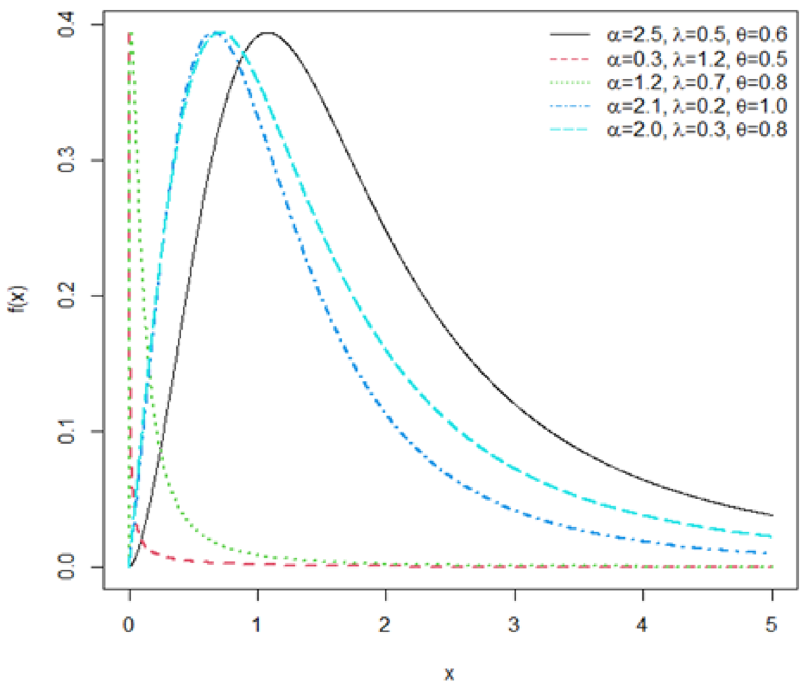

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Figure 1: *The IDD pdf plot*

3. RESULTS AND DISCUSSION

3.1. Simulation

Here, we performed a numerical simulation study to assess the performance of the MLE procedure for estimating IDD parameters based on the Monte Carlo simulation method. A random sample of size $n = 50, 100, 250, 500 \text{and} 1000$ with 10000 replicate was generated from IDD using the quantile function $\tau = \left[\frac{1}{\lambda}(1-u)^{-\theta} - 1\right]^{\frac{1}{\alpha}}$, where u is uniform (0,1). Two sets of values of the parameters $\theta = 0.5$, $\alpha = 4$ $\lambda = 0.1$ and $\theta = 1.5$, $\alpha = 1$ $\lambda = 2$ were considered. In both cases, considered. The emphirical results is presented in table 1 and table 2 which show that for both cases ML estimate are consistant as the root mean square errror (RMSE) and the average bias (BIAS) decreases as the sample size increases. i.e the Ml estimates converge to the true value of the parameter of IDD.

Table 1: *MEANS, BIAS and RMSE of* $\alpha = 4.0$, $\theta = 0.5$ *and* $\lambda = 0.1$

n	α			θ			λ		
	MEANS	BIAS	RMSE	MEANS	BIAS	RMSE	MEANS	BIAS	RMSE
20	4.5416	0.5416	1.5253	0.6184	0.1184	0.4057	0.1054	0.0054	0.1250
50	4.2997	0.2997	1.0773	0.5442	0.0442	0.2325	0.1004	0.0004	0.0549
100	4.1918	0.1918	0.7441	0.5125	0.1497	0.1497	0.0999	-0.0001	0.0375
250	4.0669	0.0669	0.4564	0.5069	0.0069	0.0942	0.0995	-0.0005	0.0224
500	4.0366	0.0366	0.3143	0.5018	0.0018	0.0664	0.1002	0.0002	0.0149
1000	4.0163	0.0163	0.2159	0.5014	0.0014	0.0462	0.1004	0.0004	0.0109

3.2. Applications

In this section, we apply IDD to two-lifetime datasets. We employed several information criteria and the goodness of ft statistics that allowed us to compare the fits of the IDD with seven different

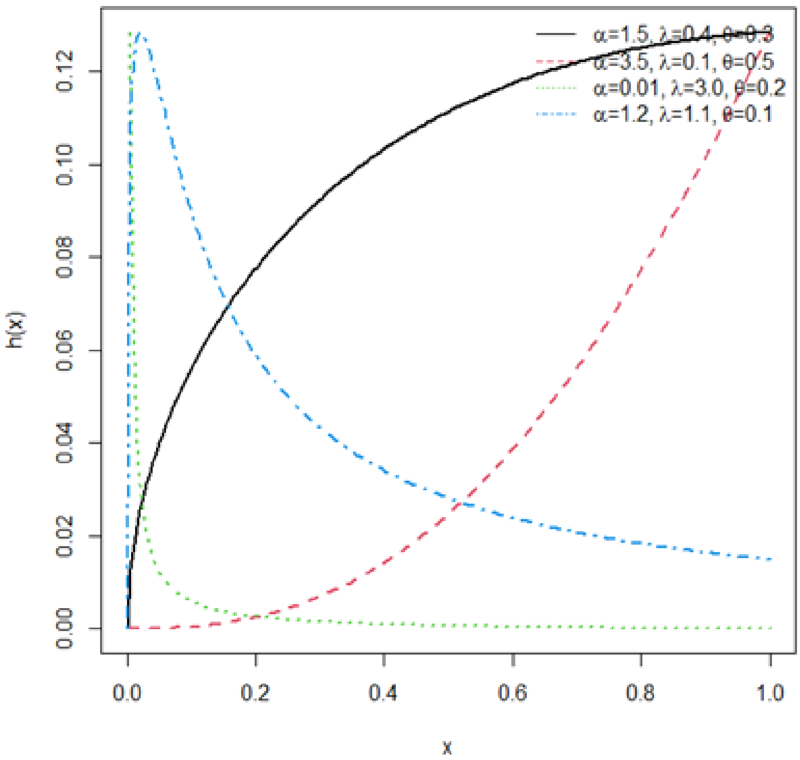

Figure 2: *The IDD hazard function plot*

existing distributions.

The first data set consists of the daily new fatalities caused by COVID-19 in New Jersey, USA, from March 12, 2020, to July 25, 2021, as retrieved from https://www.worldometers.info/coronavirus/usa/new-jersey/. The data are "1, 1, 2, 5, 2, 6, 5, 8, 19, 21, 22, 31, 37, 24, 42, 79, 100, 206, 124, 226, 81, 98, 259, 308, 222, 263, 284, 189, 106, 409, 398, 409, 365, 260, 150, 198, 425, 352, 413, 214, 279, 86, 121, 449, 372, 518, 352, 231, 162, 74, 386, 318, 297, 171, 150, 165, 88, 226, 210, 248, 231, 126, 121, 94, 161, 176, 120, 151, 111, 64, 18, 48, 162, 81, 141, 115, 84, 24, 58, 139, 114, 87, 73, 80, 87, 88, 112, 97, 56, 108, 42, 56, 63, 62, 41, 38, 29, 14, 36, 55, 52, 33, 45, 34, 27, 5, 17, 41, 52, 24, 22, 23, 50, 64, 106, 31, 50, 6, 30, 23, 43, 31, 20, 20, 5, 2, 2, 23, 33, 18, 13, 17, 16, 18, 15, 9, 10, 6, 3, 6, 3, 13, 3, 10, 11, 2, 5, 4, 15, 2, 31, 9, 7, 9, 1, 2, 2, 7, 2, 6, 8, 4, 3, 7, 5, 15, 9, 6, 5, 3, 22, 7, 11, 5, 9, 4, 4, 5, 8, 9, 4, 5, 4, 2, 1, 3, 9, 10, 3, 5, 4, 13, 7, 3, 3, 3, 4, 3, 7, 5, 7, 3, 7, 2, 1, 17, 12, 4, 4, 2, 4, 3, 18, 16, 17, 8, 12, 2, 10, 17, 15, 10, 6, 9, 1, 2, 16, 23, 16, 10, 12, 4, 10, 22, 12, 19, 25, 28, 15, 14, 38, 36, 36, 24, 32, 17, 16, 46, 59, 36, 27, 38, 11, 20, 84, 56, 62, 45, 42, 23, 15, 87, 111, 77, 59, 61, 29, 24, 86, 125, 66, 51, 50, 28, 26, 106, 139, 81, 47, 23, 19, 39, 115, 188, 82, 106, 32, 34, 38, 120, 124, 110, 99, 86, 173, 113, 34, 113, 87, 24, 16, 51, 136, 86, 109, 59, 17, 22, 132, 115, 81, 82, 72, 29, 30, 70, 109, 100, 93, 77, 24, 23, 93, 147, 79, 64, 47, 13, 13, 31, 135, 89, 62, 50, 24, 17, 104, 99, 69, 46, 46, 14, 21, 48, 128, 42, 30, 36, 16, 17, 45, 133, 46, 40, 34, 15, 22, 41, 79, 31, 27, 31, 40, 7, 61, 50, 38, 28, 24, 7, 15, 82, 75, 30, 24, 11, 9, 14, 51, 49, 34, 42, 33, 12, 25, 50, 59, 47, 43, 40, 9, 18, 45, 65, 30, 27, 39, 13, 19, 61, 49, 20, 25, 34, 12, 16, 42, 49, 33, 27, 22, 11, 10, 28, 43, 25, 26, 20, 9, 14, 23, 32, 23, 17, 14, 7, 9, 25, 34, 14, 12, 16, 6, 5, 7, 28, 6, 12, 8, 6, 5, 1, 31, 6, 2, 2, 3, 5, 11, 12, 7, 4, 4, 2, 3, 15, 18, 6, 12, 4, 3, 3, 6, 13, 5, 5, 5, 1, 4, 13, 3, 3, 5, 4, 4, 7, 11, 2, 2, 5, 3, 6, 12, 5, 2, 4, 5, 2, 4, 6, 3, 2, 5, 7, 3, 1, 8, 11, 4, 7, 5, 5, 4, 9, 6, 7, 9".

Table 3 lists ML estimates of the parameters together with the values of information criteria; the Akaike Information Criteria (AIC), the Consistent Akaike Information Criteria, and the Bayesian

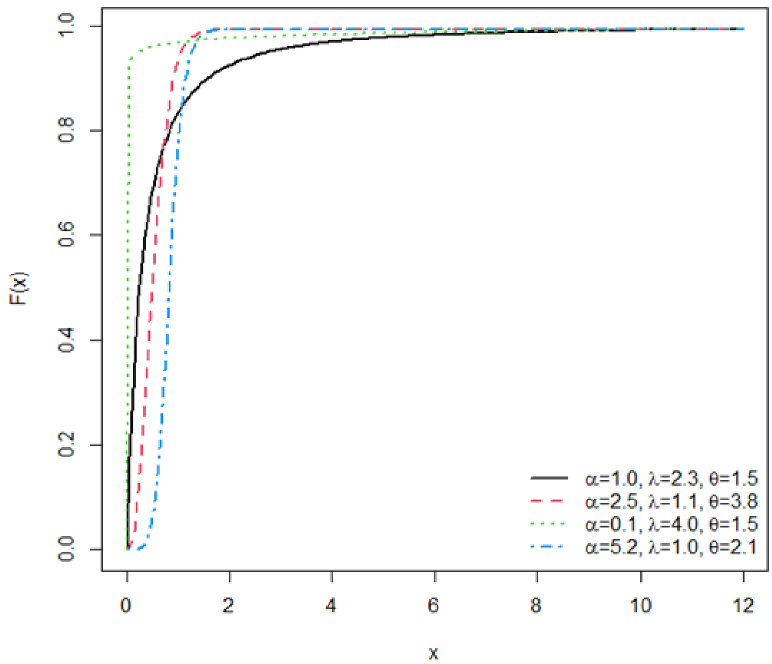

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Figure 3: *The IDD cdf Plot*

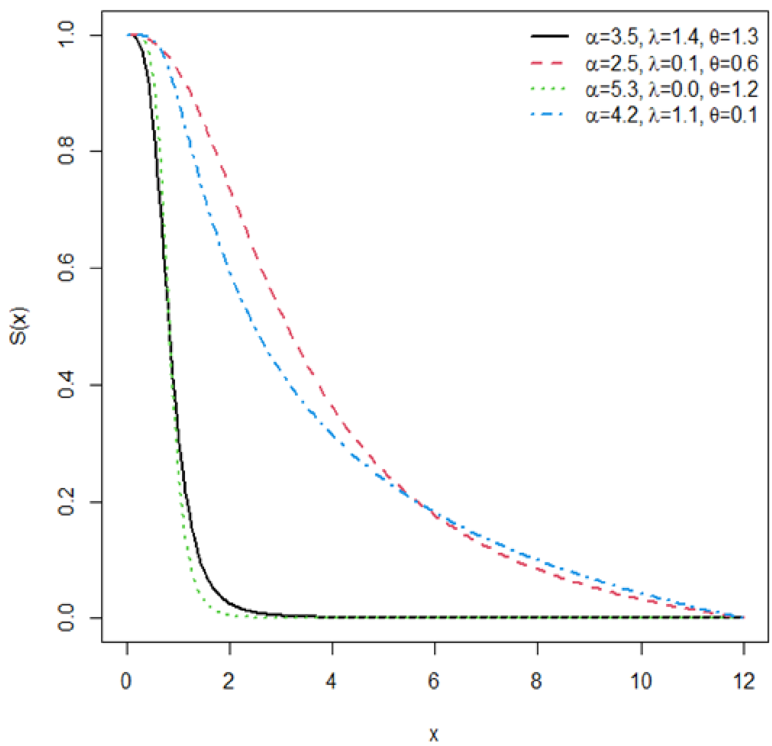

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Figure 4: *The IDD Survival Function plot*

Table 2: *MEANS, BIAS and RMSE of* $\alpha = 1.0$, $\theta = 1.5$ *and* $\lambda = 2.0$

n	α			θ			λ		
	MEANS	BIAS	RMSE	MEANS	BIAS	RMSE	MEANS	BIAS	RMSE
20	1.0820	0.0820	0.2427	1.8485	0.3485	0.9268	2.4032	0.4032	1.8064
50	1.0343	0.0343	0.1533	1.6827	0.1827	0.6381	2.3160	0.3160	1.5318
100	1.0189	0.0189	0.1066	1.5963	0.0963	0.4689	2.1882	0.1882	1.0169
250	1.0073	0.0073	0.0746	1.5592	0.0592	0.3407	2.0970	0.0970	0.7676
500	1.0036	0.0036	0.0553	1.5287	0.0287	0.2552	2.0712	0.0712	0.5735
1000	1.0015	0.0015	0.0397	1.5168	0.0168	0.1811	2.0389	0.0389	0.4127

Information Criteria. It can be observed that the IDD provides a better fit in comparison with seven competitors; the Weibull Weibull Distribution (WW), the Beta Exponentiated Exponential (BEE), the Beta Gamma Distribution (BGA), the Weibull Exponential (WE), the Inverse Weibull (IW), the Inverse Rayleigh (IR) and the Inverse Exponential (IE). Because the IDD has a lower value for all information criteria. Hence, we can conclude that the IDD gives a better fit to the COVID-19 dataset than the other probability distributions considered in the study. Table 4 displays the goodness of fit statistics; the Anderson Darling (A^*), the Cramer-Von Mises (W^*), and Kolmogorov-Smirnov (KS) statistics with its p-value. The proposed distribution appears to be a very competitive model for these data since the values of the investigated metrics are lower and the probability value of Kolmogorov-Smirnov statistics is larger than those of the other models.

Table 3: *the model parameters MLE estimates and information criteria for the dataset two.*

Model	α	θ	λ	β	AIC	CAIC	BIC
IDD	-	1.097	1.393	0.0213	**4935.08**	**4935.12**	**4947.76**
WW	-	3.7360	0.1490	0.3920	4980.01	4980.05	4992.69
BEE	0.2510	0.4350	0.0317	3.1370	4968.67	4968.75	4985.59
BGA	0.0602	3.5560	8.5420	0.0220	50.94.17	5094.25	5111.09
WE	-	-	0.5280	0.0090	5111.31	5119.76	5114.62
IW	-	-	6.0356	0.7537	4964.96	4964.99	4973.42
IR	-	-	24.1579	-	6463.86	6463.87	6468.08
IE	-	-	8.9379	-	5048.32	5048.33	5052.55

Table 4: *The test results for the Goodness-of-fit for dataset two.*

Model	A^*	W^*	KS	KS p-value
IDD	2.404	0.333	0.056	0.082
WW	5.835	0.858	0.081	0.003
BEE	4.811	0.677	0.084	0.002
BGA	11.688	1.761	0.177	<0.0001
WE	13.177	2.017	0.183	<0.0001
IW	5.771	0.936	0.082	0.002
IR	16.032	2.661	0.514	<0.0001
IE	8.041	1.316	0.193	0.081

The second dataset comprises 100 observations of the breaking stress of carbon fibers provided by [21]. the data are "0.920, 0.9280, 0.997, 0.9971, 1.0610, 1.117, 1.1620, 1.183, 1.187, 1.1920, 1.196, 1.2130, 1.215, 1.2199, 1.220, 1.2240, 1.225, 1.2280, 1.237, 1.240, 1.244, 1.259, 1.2610, 1.263, 1.276, 1.310, 1.3210, 1.3290, 1.3310, 1.337, 1.351, 1.359, 1.388, 1.4080, 1.449, 1.4497, 1.450, 1.459, 1.471, 1.475, 1.477, 1.480, 1.489, 1.501, 1.507, 1.515, 1.530, 1.5304, 1.533, 1.544, 1.5443, 1.552, 1.556, 1.5620, 1.566, 1.585, 1.586, 1.599, 1.602, 1.6140, 1.6160, 1.617, 1.6280, 1.6840, 1.7110, 1.7180, 1.733, 1.7380, 1.7430, 1.7590, 1.777, 1.7940, 1.799, 1.806, 1.814, 1.814, 1.8160, 1.8280, 1.830, 1.884, 1.892, 1.944, 1.972, 1.9840, 1.987, 2.02, 2.0304, 2.0290, 2.0350, 2.0370, 2.0430, 2.0460, 2.0590, 2.111, 2.165, 2.686, 2.778, 2.972, 3.504, 3.863, 5.3060".

Table 5 shows the statistics of AIC, CAIC, and BIC for all models investigated. When compared to the values of the other models, the IDD values are lower, demonstrating that this new distribution is a very competitive model for this data. Table 6 reveals that the IDD has the highest KS p-value and the lowest KS, A^*, and W^* values for the carbon fibers dataset. This demonstrates that the IDD distribution is superior at fitting this dataset.

Table 5: *the model parameters MLE estimates and information criteria for the dataset two.*

Model	α	θ	λ	β	AIC	CAIC	BIC
IDD	-	9.9065	0.5073	0.0577	**111.88**	**112.13**	**119.72**
WW	-	5.557	0.332	0.564	196.20	196.45	204.04
BEE	14.951	0.8590	4.0340	10.8910	119.40	119.81	129.86
BGA	10.6136	0.4324	5.6310	7.5055	122.65	123.07	133.11
WE	-	-	1.4008	0.3543	216.18	216.30	221.41
IW	-	-	4.3427	4.3754	112.54	112.66	117.77
IR	-	-	2.2095	-	183.67	183.71	186.29
IE	-	-	1.5317	-	304.42	304.03	307.03

Table 6: *The test results for the Goodness-of-fit for dataset two.*

Model	A^*	W^*	KS	KS p-value
IDD	0.497	0.067	0.070	0.704
WW	6.323	1.031	0.206	0.003
BEE	0.854	0.107	0.086	0.4444
BGA	1.022	0.123	0.097	0.298
WE	7.943	1.348	0.244	<0.0001
IW	0.805	0.116	0.089	0.399
IR	0.713	0.087	0.327	<0.0001
IE	0.986	0.121	0.447	<0.0001

4. Conclusion

In this work, we presented a new univariate continuous distribution called the inverted Dagum distribution by taking the transformation of the reciprocal of the random variable of the Dagum distribution on the ground of the type I Dagum distribution. After introducing the distribution, we obtained its basic properties like r^th moments, mean, moment generating function, quantile function, hazard function, survival function, Renyi entropy, and order statistics. The maximum likelihood method was used to provide the estimates of the model parameters. We also provided the density hazard rate plots of the distribution with some assumed values. Additionally, a simulation study was performed to show the ML estimates are consistence as the number as sample size increases the estimates converge to the actual values of the parameters. Our empirical analysis of two real datasets shows that the proposed distribution outperforms the well-known distribution based on goodness of fit metrics and information criteria.

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