ANALYSIS OF $M^X/G/1$ QUEUE WITH OPTIONAL SECOND SERVICE, FEEDBACK AND BERNOULLI VACATION

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Abstract

In this article the single-server queue situation described with batch arrivals, a mandatory first service and a choice of second service are provided to the customers. A general distribution governs the service times, whereas a compound Poisson distribution follows customer arrivals. Although each new customer requests the first mandatory service, only some of them choose the optional second service. Customers who are dissatisfied with mandatory service are more likely to get the required services later on. After every service is finished, the server might choose to go on Bernoulli vacation. Time dependent probability generating functions are constructed in terms of Laplace transforms using the supplementary variable approach, and explicit results are obtained for the steady state. Additionally, mean waiting time and mean queue length expressions are examined. The graphical and numerical representations improve comprehension of the results even further.

Keywords: Optional second service, Feedback, Bernoulli vacation

1. Introduction

Queueing system is useful in a wide range of scenarios. Wireless networks, supermarkets, restaurants, hospitals, modulation lines, and communication systems all use queueing systems. There will always be lines. People are permitted to wait in line if a service time exceeds the pace of arrivals. Recently, there have been several contributions on taking the $M/G/1$ queue with second optional service into consideration. These kinds of queue situations arise in daily life. An optional second service is part of a single-server queueing system that has been studied by Madan [11]. Madheswari and Suganthi [17] have discussed an $M/G/1$ orbital queue with an optional service and beginning failures. The retrial time distribution is assumed to be exponential. Chowdary and Paul [7] have explained a customers arrival in batches with an optional second service within an N-policy framework. A batch arrival and single server with two service phases (the second of which is optional) and working breakdown has been developed by Somasundaram et al. [22] The system's performance, steady state outcomes, and optimization analysis are examined and also discusses how the concept to be applied to cellular networks and how crucial it is to take server failures into account while providing services.

Queueing systems with feedback is allow customers to return to the same server for re-service with certain probability. Li and Jinting [10] have suggested the implementation of a single server orbital queue featuring numerous optional services and feedback options. Varalakshmi et al. [24]

added to the idea of instant feedback in a unique way. After service completion, if a customer need further service, they will instantly get the another service. A two-phase $M/G/1$ queueing model with instantaneous feedback only for finite number of customers was studied by Kalidass and Kasturi [8]. Moreover, there are several real-world situations where an orbital queue with feedback arises.

Queueing models with vacations have been examined by several researchers. Maragathasundari and Srinivasan [15] investigated the $M/G/1$ queue models with a single vacation in a transient study. Madan [12] examined a single server queue that required mandatory server shutdowns. In the industrial business, Karpagam et al. [9] examined a bulk queueing system with rework that had a single vacation and beginning failure. Thangaraj and Vanitha [23] have studied two stages of service, with single server and single arrival subject to compulsory server vacation and random breakdown.

A batch arrival and single server queueing model with balking and vacation was studied by Charan et al. [21]. Customers may leave the system when server is busy or on vacation. Ayyappan and Deepa [3] discovered batch arrival bulk service queuing system with mandatory and optional repair process. They formulate for the number of performance metrics, including expected queue length, expected waiting time and idle duration. A single server retrial queue with working vacations under multiple vacation policy, vacation interruptions, breakdown and impatient of the customers was examined by Rajadurai et al. [18]. In this study, server breakdown due to the arrival of negative consumers. Shanmugasundaram and Sivaram [20] discussed a single server queue that includes feedback for a client and sever vacation. The steady-state probability and some importance measures were obtained in this investigation. A two-stage batch arrival queue system with reneging during vacation and breakdown times was presented by Baruah et al. [16].

Vacations can be categorized into various types, which are single vacations, multiple vacations, compulsory vacations, modified vacations, J-vacations, Bernoulli vacations, modified Bernoulli vacations, working vacations, multiple working vacations etc., Numerous of studies have used the Bernoulli vacation as a parameter while analyzing queueing models. Arivudainambi and Gowsalya [1] analysed an $M/G/1$ retrial queue with staring failure and vacation scheduled on Bernoulli type. An $M^{[X]}G/1$ retrial queue was studied by Madhu and Kaur [13], who combined Bernoulli feedback, optional service, and the Bernoulli vacation idea for an unreliable server. A retrial queue with batch arrivals was studied by Madhu et al. [14]. Bernoulli vacation was used to provide the server with the opportunity to take a rest during both service phases.

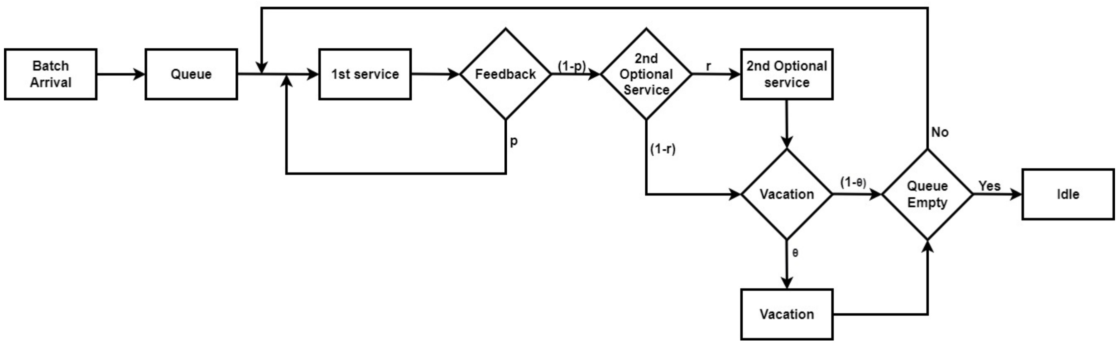
Arivudainambi and Gowsalya [2] developed a Bernoulli vacation schedule and two types of service for a retrial queueing system. The study covered the growing applications in teletraffic theory, client-server communication, etc. A discrete time retrial queue with Bernoulli vacation, preemptive resume priority, general Bernoulli feedback, and retrial periods was analysed by Chen et al. [6]. This study indicates that if the server becomes idle after service, it will either wait for a customer or initiate a single vacation. A repairable queue model with Bernoulli vacation and a two-phase service structure was created by Wang and Li [26]. An $M^{[X]}/G/1$ feedback retrial queue with two-phase service, Bernoulli vacation, delayed repair, and orbit search was analyzed by Chandrasekaran et al. [5].

Ayyappan and Somasundaram [4] analyzed a two-stage retrial G-queue with Bernoulli vacation, working breakdown, and discretionary priority services. During the first stage of this investigation, incoming priority units are free to disrupt service; however, during the second stage, they are not allowed to do so. A modified Bernoulli vacation of batch arrival and retrial queue with balking customers due to beginning failure was communicated by Rajam and Uma [19]. There are several uses for this model in the fields of healthcare, mail, manufacturing, production lines, and communication networks. A retrial queue with feedback, working breakdowns, and Bernoulli vacation was examined by Varalakshmi and Rajadurai [25].

In this paper, we study an $M^{[X]}/G/1$ queue with feedback, optional second service, and Bernoulli vacations where the second service stage is optional and the first service stage is mandatory. Customers might choose to try again if they are not satisfied with the service they received re-service after the first stage. The server can also take a Bernoulli vacation when each service is finished.

The primary goal of this inquiry is to ascertain average customer wait times and queue lengths, which are crucial metrics for assessing the system effectiveness. The study comprehensive outline is given in the parts that follow. The mathematical model description, integrating the essential presumptions, is presented in Section 2. Section 3, explores real-world examples and applications. Section 4, presents a series of equations that define the model governing the system. Section 5, deduces time-dependent solutions, while Section 6 concentrates on figuring out steady-state outcomes. Some important performance metrics are calculated in Section 7. Section 8, follows provides graphical representations and numerical results.

2. Methods



Diagrammatic representation of this model

- Customers use a compound Poisson process to enter the system in batches of different sizes. $\leftthreetimes C_k dt$ (where $k = 1, 2, 3, \dots$) represents the likelihood that a batch of consumers of size k would join the system during a short time interval $(t, t + dt)$. The probabilities are $\sum_{k=1}^{\infty} C_k = 1$ and $0 \le C_k \le n$.
- The single server provides each customer with the first mandatory service. Let $r_1(V)$ stand for the first service time density function and $\Re_1(V)$ for the distribution function.
- Following the fulfillment of the mandatory service, the customer dissatisfaction of service, there is a probability 'p' that they will receive their standard service again. Alternatively, the customer may permanently exit the system with a probability 'q=1-p'.
- Customers might choose to proceed with a second optional service after completing their mandatory service. With the probability 'r', if they desire the optional service they will immediately get the service again or else with probability '1-r' the customer may leave the system.
- Consider $\Re_2(V)$ as the distribution function and let $r_2(V)$ represent the density function specifically associated with the second optional service.
- The likelihood that the i^{th} service will be completed within the time range $[\ell, \ell + d\ell]$, provided that ℓ has elapsed, is expressed by the equation $\mu_i(\ell)d\ell$.

$$\mu_i(\ell) = \frac{r_i(\ell)}{1 - \Re_i(\ell)} \qquad \text{i} = 1, 2...$$

and therefore

$$r_i(v) = \mu_\text{i}(v) e^{-\int_0^v \mu_\text{i}(\ell) d\ell} \qquad \text{i=1,2,....}$$

- Following the completion of each service, either the server may go for Bernoulli vacation with probability 'θ' or stay in the system to serve the next customer with a probability of '1-θ'. If the queue length is > 0, the server will start the next service; else, it remains idle.
- The system follows first-come, first-served queue discipline.
- Let $v(\ell)$ be the vacation density function and $\mathcal{M}(\ell)$ be the distribution function.

$$\gamma(\ell) = \frac{v(\ell)}{1 - \mathcal{M}(\ell)}$$

and therefore

$$v(s) = \gamma(s) e^{-\int_0^\infty \gamma(\ell) d\ell}$$

- All stochastic processes involved are mutually independent.

3. Application of the Model

This model is useful in many different kinds of real-world situations. Take a scenario where a consumer goes to a bank to get money from the cashier. Should the client be dissatisfied with draw the amount, he might request another service. Upon receiving the money, the customer's choice is taken into account while making a passbook entry. Following the end of each customer's service, the cashier may choose to work on other duties (Bernoulli vacation).

4. System-governing definitions and equations.

$\mathcal{H}_n^{(1)}(\ell, \mathtt{t})$ and $\mathcal{H}_n^{(2)}(\ell, \mathtt{t})$ reflects the probability that, at time 't' with elapsed service time 'ℓ', the server is active and occupied with service, excluding the one customer being serviced at the server station and '$\mathfrak{n}$' (≥ 0) customers in the queue.

$\mathcal{M}_n(\ell, t)$ reflects the probability that the server is on vacation with an elapsed vacation time 'ℓ' while there are $n (\geq 0)$ customers in the queue.

$\mathfrak{I}(t)$ reflects the probability that the system has no customers at time 't' and the server is idle but available.

The following differential-difference equations are framed based on the model defined:

$$\frac{\partial}{\partial \ell}\mathcal{H}_n^{(1)}(\ell, t) + \frac{\partial}{\partial \mathtt{t}}\mathcal{H}_n^{(1)}(\ell, \mathtt{t}) = -(\lambda + \mu_1(\ell))\mathcal{H}_n^{(1)}(\ell, t) + \lambda(1 - \delta_{n0}) \sum_{k=1}^{n} c_k \mathcal{H}_{n-k}^{(1)}(\ell, t) \tag{1}$$

$$\frac{\partial}{\partial \ell}\mathcal{H}_n^{(2)}(\ell, t) + \frac{\partial}{\partial t}\mathcal{H}_n^{(2)}(\ell, t) = -(\lambda + \mu_2(\ell))\mathcal{H}_n^{(2)}(\ell, t) + \lambda(1 - \delta_{n0}) \sum_{k=1}^{n} c_k \mathcal{H}_{n-k}^{(2)}(\ell, t) \tag{2}$$

$$\frac{\partial}{\partial \ell}\mathcal{M}_n(\ell, t) + \frac{\partial}{\partial t}\mathcal{M}_n(\ell, t) = -(\lambda + \gamma(\ell))\mathcal{M}_n(\ell, t) + \lambda(1 - \delta_{n0}) \sum_{k=1}^{n} c_k \mathcal{M}_{n-k}(\ell, t) \tag{3}$$

$$\frac{d}{dt}\mathfrak{I}(t) = -\lambda\mathfrak{I}(t) + (1 - r)(1 - \theta)\int_0^\infty \mathcal{H}_0^{(1)}(\ell, t)\mu_1(\ell) d\ell + (1 - \theta)\int_0^\infty \mathcal{H}_0^{(2)}(\ell, t)\mu_2(\ell) d\ell \tag{4}$$

Equations (1) to (4) must be solved under the following boundary conditions:

$$\begin{aligned}\mathcal{H}_n^{(1)}(0, \mathtt{t}) = &\lambda c_{n+1}\mathfrak{I}(\mathtt{t}) + p\int_0^\infty \mathcal{H}_n^{(1)}(\ell, \mathtt{t})\mu_1(\ell) d\ell + q(1 - r)(1 - \theta)\int_0^\infty \mathcal{H}_{n+1}^{(1)}(\ell, \mathtt{t})\mu_1(\ell) d\ell \\ &+ (1 - \theta)\int_0^\infty \mathcal{H}_{n+1}^{(2)}(\ell, \mathtt{t})\mu_2(\ell) d\ell + \int_0^\infty \mathcal{M}_{n+1}(\ell, \mathtt{t})\gamma(\ell) d\ell\end{aligned} \tag{5}$$

$$\mathcal{H}_n^{(2)}(0, \mathtt{t}) = rq\int_0^\infty \mathcal{H}_n^{(1)}(\ell, \mathtt{t})\mu_1(\ell) d\ell \tag{6}$$

$$\mathcal{M}_n(0, \mathtt{t}) = \theta\int_0^\infty \mathcal{H}_n^{(2)}(\ell, \mathtt{t})\mu_2(\ell) d\ell + \theta(1 - r)q\int_0^\infty \mathcal{H}_n^{(1)}(\ell, \mathtt{t})\mu_1(\ell) d\ell \tag{7}$$

The initial conditions are

$$\mathfrak{I}(0) = 1, \mathcal{H}^{(1)}(0) = \mathcal{H}^{(2)}(0) = \mathcal{M}(0) = 0 \tag{8}$$

5. The Time-Dependent Solution for Generating Queue Functions:

We establish the probability generating functions as follows:

$$\mathcal{H}_q^{(1)}(\ell,z,t) = \sum_{n=0}^{\infty} z^n \mathcal{H}_n^{(1)}(\ell,t) \tag{9}$$

$$\mathcal{H}_q^{(2)}(\ell,z,t) = \sum_{n=0}^{\infty} z^n \mathcal{H}_n^{(2)}(\ell,t) \tag{10}$$

$$\mathcal{M}_q(\ell,z,t) = \sum_{n=0}^{\infty} z^n \mathcal{M}_n(\ell,t) \tag{11}$$

$$C(z) = \sum_{k=1}^{\infty} z^k c_k(t) \tag{12}$$

By taking the Laplace transforms of Equations (1) through (7) and applying Equation (8), we derive:

$$\frac{\partial}{\partial \ell}\bar{\mathcal{H}}_n^{(1)}(\ell,\wp)+(\wp+\curlywedge+{}_1(\ell))\bar{\mathcal{H}}_n^{(1)}(\ell,\wp)=\curlywedge(1-\delta_{n0})\sum_{k=1}^{n} c_k\bar{\mathcal{H}}_{n-k}^{(1)}(\ell,\wp) \tag{13}$$

$$\frac{\partial}{\partial \ell}\bar{\mathcal{H}}_n^{(2)}(\ell,\wp)+(\wp+\curlywedge+{}_2(\ell))\bar{\mathcal{H}}_n^{(2)}(\ell,\wp)=\curlywedge(1-\delta_{n0})\sum_{k=1}^{n} c_k\bar{\mathcal{H}}_{n-k}^{(2)}(\ell,\wp) \tag{14}$$

$$\frac{\partial}{\partial \ell}\bar{\mathcal{M}}_n(\ell,\wp)+(\wp+\curlywedge+\gamma(\ell))\bar{\mathcal{M}}_n(\ell,\wp)=\curlywedge(1-\delta_{n0})\sum_{k=1}^{n} c_k\bar{\mathcal{M}}_{n-k}(\ell,\wp) \tag{15}$$

$$\begin{aligned}(\wp+\curlywedge)\bar{\Im}(\wp) =&1+(1-p)(1-r)(1-\theta)\int_0^\infty \bar{\mathcal{H}}_0^{(1)}(\ell,\wp)\mu_1(\ell)d\ell+(1-\theta)\int_0^\infty \bar{\mathcal{H}}_0^{(2)}(\ell,\wp)\mu_2(\ell)d\ell\\ &+\int_0^\infty \bar{\mathcal{M}}_0(\ell,\wp)\gamma(\ell)d\ell\end{aligned} \tag{16}$$

$$\begin{aligned}\bar{\mathcal{H}}_n^{(1)}(0,\wp) =&\curlywedge c_{n+1}\bar{\Im}(\wp)+p\int_0^\infty \bar{\mathcal{H}}_n^{(1)}(\ell,\wp)\mu_1(\ell)d\ell+q(1-r)(1-\theta)\int_0^\infty \bar{\mathcal{H}}_{n+1}^{(1)}(\ell,\wp)\mu_1(\ell)d\ell\\ &+(1-\theta)\int_0^\infty \bar{\mathcal{H}}_{n+1}^{(2)}(\ell,\wp)\mu_2(\ell)d\ell+\int_0^\infty \bar{\mathcal{M}}_{n+1}(\ell,\wp)\gamma(\ell)d\ell\end{aligned} \tag{17}$$

$$\bar{\mathcal{H}}_n^{(2)}(0,\wp) = rq\int_0^\infty \bar{\mathcal{H}}_n^{(1)}(\ell,\wp)\mu_1(\ell)d\ell \tag{18}$$

$$\bar{\mathcal{M}}_n(0,\wp) = \theta\int_0^\infty \bar{\mathcal{H}}_n^{(2)}(\ell,\wp)\mu_2(\ell)d\ell+\theta(1-r)q\int_0^\infty \bar{\mathcal{H}}_n^{(1)}(\ell,\wp)\mu_1(\ell)d\ell \tag{19}$$

Equations (13) multiplied by suitable powers of z, summed over n, and simplified using (9). Following algebraic computations, we obtain:

$$\frac{\partial}{\partial \ell}\bar{\mathcal{H}}_q^{(1)}(\ell,z,\wp)+(\wp+[\curlywedge(1-C(z)]+\mu_1(\ell))\bar{\mathcal{H}}_q^{(1)}(\ell,z,\wp)=0 \tag{20}$$

By using (10) and carrying out similar procedures on (14), we derive:

$$\frac{\partial}{\partial \ell}\bar{\mathcal{H}}_q^{(2)}(\ell,z,\wp)+(\wp+[\curlywedge(1-C(z)]+\mu_2(\ell))\bar{\mathcal{H}}_q^{(2)}(\ell,z,\wp)=0 \tag{21}$$

By using (11) and carrying out similar procedures on (15), we derive:

$$\frac{\partial}{\partial \ell}\bar{\mathcal{M}}_q(\ell,z,\wp)+(\wp+[\curlywedge(1-C(z)]+\gamma(\ell))\bar{\mathcal{M}}_q(\ell,z,\wp)=0 \tag{22}$$

The two sides of equation (17) are then multiplied by z, the sum over n from 0 to ∞, we get

$$\begin{aligned}\bar{\mathcal{H}}_q^{(1)}(0,z,\wp) =& \lambda C(z)\bar{\Im}(\wp) + pz\int_0^\infty \bar{\mathcal{H}}_q^{(1)}(\ell,z,\wp)\mu_1(\ell)d\ell + (1-p)(1-r)(1-\theta)\\ &\left[\int_0^\infty \bar{\mathcal{H}}_q^{(1)}(0,z,\wp)\mu_1(\ell)d\ell - \int_0^\infty \bar{\mathcal{H}}_0^{(1)}(\ell,\wp)\mu_1(\ell)d\ell\right]\\ &+(1-\theta)\left[\int_0^\infty \bar{\mathcal{H}}_q^{(2)}(0,z,\wp)\mu_2(\ell)d\ell - \int_0^\infty \bar{\mathcal{H}}_0^{(2)}(\ell,\wp)\mu_2(\ell)d\ell\right]\\ &+\left[\int_0^\infty \bar{\mathcal{M}}_q(0,z,\wp)\gamma(\ell)d\ell - \int_0^\infty \bar{\mathcal{M}}_0(\ell,\wp)\gamma(\ell)d\ell\right]\end{aligned} \tag{23}$$

By carrying out analogous procedures on equations (18) and (19), we obtain

$$\bar{\mathcal{H}}_q^{(2)}(0,z,s) = r(1-p)\int_0^\infty \bar{\mathcal{H}}_q^{(1)}(\ell,z,s)\mu_1(\ell)d\ell \tag{24}$$

$$\bar{\mathcal{M}}_q(0,z,\wp) = \theta\int_0^\infty \bar{\mathcal{H}}_q^{(2)}(\ell,z,\wp)\mu_2(\ell)d\ell + \theta(1-r)(1-p)\int_0^\infty \bar{\mathcal{H}}_q^{(1)}(\ell,z,\wp)\mu_1(\ell)d\ell \tag{25}$$

Using equation (16) in (23), we get

$$\begin{aligned}z\bar{\mathcal{H}}_q^{(1)}(0,z,\wp) =& 1-(\wp+\lambda(1-C(z))\bar{\Im}(\wp)) + pz\int_0^\infty \bar{\mathcal{H}}_q^{(1)}(\ell,z,\wp)\mu_1(\ell)d\ell\\ &+(1-p)(1-r)(1-\theta)\int_0^\infty \bar{\mathcal{H}}_q^{(1)}(\ell,z,\wp)\mu_1(\ell)d\ell\\ &+(1-\theta)\int_0^\infty \bar{\mathcal{H}}_q^{(2)}(\ell,z,\wp)\mu_2(\ell)d\ell + \int_0^\infty \bar{\mathcal{M}}_q(\ell,z,\wp)\gamma(\ell)d\ell\end{aligned} \tag{26}$$

Integrating equation (20), (21) and (22), from 0 to ℓ yields

$$\bar{\mathcal{H}}_q^{(1)}(\ell,z,\wp) = \bar{\mathcal{H}}_q^{(1)}(0,z,\wp)e^{-(\wp+\lambda(1-C(z))\ell-\int_0^\ell \mu_1(t)dt} \tag{27}$$

$$\bar{\mathcal{H}}_q^{(2)}(\ell,z,\wp) = \bar{\mathcal{H}}_q^{(2)}(0,z,\wp)e^{-(s+\lambda(1-C(z))\ell-\int_0^\ell \mu_2(t)dt} \tag{28}$$

$$\bar{\mathcal{M}}_q(\ell,z,\wp) = \bar{\mathcal{M}}_q(0,z,\wp)e^{-(\wp+\lambda(1-C(z))\ell-\int_0^\ell \gamma(t)dt} \tag{29}$$

Integrating equation (27) to (29) by parts with respect to x yields, we get

$$\bar{\mathcal{H}}_q^{(1)}(z,\wp) = \bar{\mathcal{H}}_q^{(1)}(0,z,\wp)\left[\frac{1-\bar{\Re}_1[f(z,\wp)]}{[f(z,\wp)]}\right] \tag{30}$$

where
$\bar{\Re}_1[f(z,\wp)] = \int_0^\infty e^{-f(z,\wp)\ell}d\Re_1(\ell)$

$$\bar{\mathcal{H}}_q^{(2)}(z,\wp) = \bar{\mathcal{H}}_q^{(2)}(0,z,\wp)\left[\frac{1-\bar{\Re}_2[f(z,\wp)]}{[f(z,\wp)]}\right] \tag{31}$$

where
$\bar{\Re}_2[f(z,\wp)] = \int_0^\infty e^{-f(z,\wp)\ell}d\Re_2(\ell)$

$$\bar{\mathcal{M}}_q(z,\wp) = \bar{\mathcal{M}}_q(0,z,\wp)\left[\frac{1-\bar{\mathcal{M}}[f(z,\wp)]}{[f(z,\wp)]}\right] \tag{32}$$

where
$\bar{\mathcal{M}}[f(z,\wp)] = \int_0^\infty e^{-f(z,\wp)\ell}d\mathcal{H}(\ell)$

where $f(z,\wp) = \wp + \lambda(1-C(z))$

Now, by multiplying $\mu_1(\ell)$ by both sides of equation (27), $\mu_2(\ell)$ by (28), & $\gamma(\ell)$ by (29), & integrating over ℓ, we get

$$\int_0^\infty \bar{\mathcal{H}}_q^{(1)}(\ell, z, \wp)\mu_1(\ell)d\ell = \bar{\mathcal{H}}_q^{(1)}(0, z, \wp)\bar{\mathfrak{R}}_1[f(z, \wp)] \tag{33}$$

$$\int_0^\infty \bar{\mathcal{H}}_q^{(2)}(\ell, z, \wp)\mu_2(\ell)d\ell = \bar{\mathcal{H}}_q^{(2)}(0, z, \wp)\bar{\mathfrak{R}}_2[f(z, \wp)] \tag{34}$$

$$\int_0^\infty \bar{\mathcal{M}}_q(\ell, z, \wp)\gamma(\ell) = \bar{\mathcal{M}}_q(0, z, \wp)\bar{\mathcal{M}}[f(z, \wp)] \tag{35}$$

Substituting equation (34) in (24), we obtain

$$\bar{\mathcal{H}}_q^{(2)}(0, z, \wp) = r(1-p)\bar{\mathcal{H}}_q^{(2)}(0, z, \wp)\bar{\mathfrak{R}}_1[f(z, \wp)] \tag{36}$$

Similarly using equation (33) and (34) in (25), we get

$$\bar{\mathcal{M}}_q(0, z, \wp) = \theta(1-r)q\bar{\mathcal{H}}_q^{(1)}(0, z, \wp)\bar{\mathfrak{R}}_1[f(z, \wp)] + \theta r\bar{\mathcal{H}}_q^{(1)}(0, z, \wp)\bar{\mathfrak{R}}_1[f(z, \wp)]\bar{\mathfrak{R}}_2[f(z, \wp)] \tag{37}$$

Using equation (33), (34) and (35) in (26) and solving $\bar{\mathcal{H}}_q(0, z, \wp)$

$$\bar{\mathcal{H}}_q^{(1)}(0, z, \wp) = \frac{f(z, \wp)\bar{\mathfrak{I}}(\wp)}{\begin{bmatrix} z - \bar{\mathfrak{R}}_1[f(z, \wp)]pz + q(1-r)(1-\theta) + (1-\theta)r(1-p)\bar{\mathfrak{R}}_2[f(z, \wp)] \\ + \theta(1-r)(1-p)\bar{\mathcal{M}}[f(z, \wp)] + r\theta(1-p)\bar{\mathfrak{R}}_2[f(z, \wp)]\bar{\mathcal{M}}[f(z, \wp)] \end{bmatrix}} \tag{38}$$

Equation (30), (31) and (32) becomes

$$\bar{\mathcal{H}}_q^{(1)}(z, \wp) = \bar{\mathcal{H}}_q^{(1)}(0, z, \wp)\left[\frac{1 - \bar{\mathfrak{R}}_1[f(z, \wp)]}{f(z, \wp)}\right] \tag{39}$$

$$\bar{\mathcal{H}}_q^{(2)}(z, \wp) = r(1-p)\bar{\mathcal{H}}_q^{(1)}(0, z, \wp)\bar{\mathfrak{R}}_1[f(z, \wp)]\left[\frac{1 - \bar{\mathfrak{R}}_2[f(z, \wp)]}{f(z, \wp)}\right] \tag{40}$$

$$\begin{aligned}\bar{\mathcal{M}}_q(z, \wp) = &\{\theta(1-r)(1-p)\bar{\mathcal{M}}_q^{(1)}(0, z, \wp)\bar{\mathfrak{R}}_1[f(z, \wp)] \\ &+ \theta r(1-p)\bar{\mathcal{H}}_q^{(1)}(0, z, \wp)\bar{\mathfrak{R}}_1[f(z, \wp)]\}\left[\frac{1 - \bar{\mathcal{M}}[f(z, \wp)]}{f(z, \wp)}\right]\end{aligned} \tag{41}$$

6. The steady-state results

The steady-state probability distribution for our queueing model is what we want to achieve. We exclude the time argument 't' from the time-dependent analysis in order to get the steady-state probability. This can be made easier by applying the well-known Tauberian property.

$$\lim_{\wp \to 0} \wp\bar{f}(\wp) = \lim_{t \to \infty} f(t) \tag{42}$$

We will use the normalizing condition

$$\mathcal{H}_q^{(1)}(1) + \mathcal{H}_q^{(2)}(1) + \mathcal{M}_q(1) + \mathfrak{I} = 1 \tag{43}$$

The probability generating function of the queue size $\mathcal{P}(z)$ irrespective of the state of the system.

$$\begin{aligned}\mathcal{P}(z) &= \mathcal{H}_q^{(1)}(z) + \mathcal{H}_q^{(2)}(z) + \mathcal{M}_q(z)\\ &= \mathcal{H}_q^{(1)}(0,z)\left[\frac{1-\Re_1[f(z)]}{f(z)}\right] + \mathcal{H}_q^{(2)}(0,z)\left[\frac{1-\Re_2[f(z)]}{f(z)}\right] + \mathcal{M}_q(0,z)\left[\frac{1-\mathcal{M}[f(z)]}{(z)}\right]\end{aligned} \tag{44}$$

where $\mathcal{H}_q^{(1)}(z)$, $\mathcal{H}_q^{(2)}(z)$ and $\mathcal{M}_q(z)$ are given by the following equations.

$$\mathcal{H}_q^{(1)}(z) = \mathcal{H}_q^{(1)}(z,0)\left[\frac{1-\Re_1[f(z)]}{f(z)}\right] \tag{45}$$

$$\mathcal{H}_q^{(2)}(z) = r(1-p)\mathcal{H}_q^{(1)}(z,0)\Re_1[f(z)]\left[\frac{1-\Re_2([f(z)])}{f(z)}\right] \tag{46}$$

$$\begin{aligned}\mathcal{M}_q(z) = \mathcal{H}_q^{(1)}(z,0)\Big\{&\theta(1-r)(1-p)\Re_1[f(z)]\left[\frac{1-\mathcal{M}[f(z)]}{f(z)}\right]\\ &+\theta r(1-p)\Re_1[f(z)]\Re_2[f(z)]\left[\frac{1-\mathcal{M}[f(z)]}{f(z)}\right]\Big\}\end{aligned} \tag{47}$$

$$\mathcal{P}(z) = \frac{\begin{bmatrix}\Im\{[1-\Re_1[f(z)]] + \Psi\theta r[\Re_2[f(z)]][1-\mathcal{M}[f(z)]]\\ +\theta(1-r)\Psi[1-\mathcal{M}[f(z)]] + \Psi r[1-\Re_1[f(z)]]\}\end{bmatrix}}{\begin{bmatrix}z - r\Psi(1-\theta)\Re_2[f(z)] - \Psi\theta r\Re_2[f(z)]\mathcal{M}[f(z)]\\ -\Re_1[f(z)]pz - (1-r)\Psi(1-\theta) - \theta\Psi(1-r)\mathcal{M}[f(z)]\end{bmatrix}} \tag{48}$$

where

$$\Psi = (1-p)\Re_1[f(z)]$$

Observing that for z=1, $\mathcal{P}(z)$ takes on an indeterminate form of $0/0$, we apply L'Hopital's rule on equation (44) using the fact $\Re_1(0)=1$, $\Re_2(0)=1$, $\mathcal{M}(0)=1$, $-\mathcal{M}'(0)=E(v)$, $-\Re_i'(0)=E(\Re_i)$, and $\Re_i''(0)=E(\Re_i^2)$. We get,

$$\mathcal{P}(1) = \frac{\Im[\curlywedge[-E(X)]]\{E(\Re_1) + rE(\Re_2)(1-p) + (1-p)E(V)\theta\}}{-\curlywedge[E(X)] + (1-p)\{E(\Re_1)p + (1-p)E(\Re_1) + r(1-p)E(\Re_2) + \theta(1-p)E(V)\}} \tag{49}$$

Therefore adding $\Im$ to equation (49), we get

$$\Im = \frac{-\curlywedge[E(X)] + (1-p)\{E(\Re_1) + r(1-p)E(\Re_2) + \theta(1-p)E(V)\}}{-2\curlywedge[E(X)] + (1-p)\{E(\Re_1) + r(1-p)E(\Re_2) + \theta(1-p)E(V)\}} \tag{50}$$

Consequently, The system's utilization factor is established by

$$\rho = 1 - \Im$$

where $\rho < 1$ is the stability condition under which steady state exists, for the model.

7. Performance metrics

Let L_q be the mean number of customers in the queue. Following this,

$$L_q = \frac{d}{dt}\mathcal{P}(z)\bigg|_{z=1}$$

$$L_q = \lim_{z\to 1}\frac{d}{dt}\mathcal{P}(z)$$

$$= \lim_{z\to 1} \frac{[\Re'(z)\aleph''(z) - \aleph'(z)\Re''(z)]}{2[\Re'(z)]^2}$$

$$= \frac{[\Re'(1)\aleph''(1) - \aleph'(1)\Re''(1)]}{2[\Re'(1)]^2} \tag{51}$$

where,

$$\aleph'(1) = \mathrm{I}[\leftthreetimes[-E(X)][E(\Re_1) + rE(\Re_2)(1-p) + \theta(1-p)E(V)]$$

$$\begin{aligned}\aleph''(1) = &-\mathrm{I}[[\leftthreetimes^2[E(X)]^2[E(\Re_1)^2 + r(1-p)[2E(\Re_1)E(\Re_2) + E(\Re_2)^2]\\ &+ \theta(1-p)[2E(\Re_1)E(V) + E(V)^2] + \theta r(1-p)[2E(\Re_2)E(V)]]\\ &- \leftthreetimes[E(X^2)][E(\Re_1) + r(1-p)E(\Re_2) + \theta(1-p)E(V)]]\end{aligned}$$

$$\Re'(1) = -\leftthreetimes[E(X)] + (1-p)[E(\Re_1) + r(1-p)E(\Re_2) + \theta(1-p)E(V)]$$

$$\begin{aligned}\Re''(1) = &\leftthreetimes[E(X)][2E(\Re_1)p - \leftthreetimes^2[E(X)]^2[E(\Re_1)^2 + r(1-p)[E(\Re_2)^2 + 2E(\Re_1)E(\Re_2)]\\ &+ \theta(1-p)[E(v)^2 + 2E(\Re_1)E(\Re_2)]] + \theta r(1-p)[2E(\Re_2)E(V)]]\\ &- \leftthreetimes[E(X^2)][E(\Re_1) + r(1-p)[E(\Re_2)] + \theta(1-p)E(V)]\end{aligned}$$

Let $\mathfrak{W}_q$ be the mean time while customers in the line have been waiting.
By using Little's formula

$$\mathfrak{W}_q = \frac{\mathfrak{L}_q}{\leftthreetimes} \tag{52}$$

8. The numerical results

In this section, we shows various factors affect system performance metrics using MATLAB. We assume that vacation and service time are exponentially distributed. All the parameter values are selected to satisfy the stability condition.

Table 1: *Service rate effectiveness*

μ_1	$\mathfrak{I}$	$\mathfrak{L}_q$	$\mathfrak{W}_q$
11	0.5118	0.0174	0.0348
12	0.5508	0.0167	0.0335
13	0.5838	0.0159	0.0318
14	0.6121	0.0150	0.0300
15	0.6367	0.0141	0.0282
16	0.6581	0.0132	0.0265

From table 1, shows that idle time increase, mean waiting time and mean queue length decreases while service rate increase.

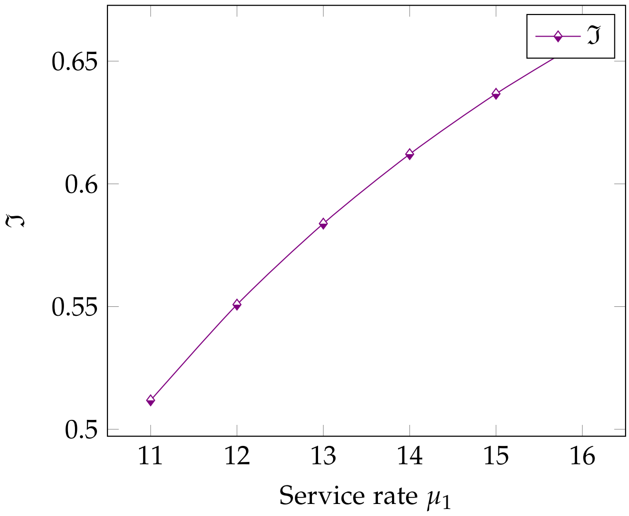


Figure 1: *Service rate vs Idle*

Figure 1 clearly demonstrates that as the mandatory service times increase, the idle time also increases.

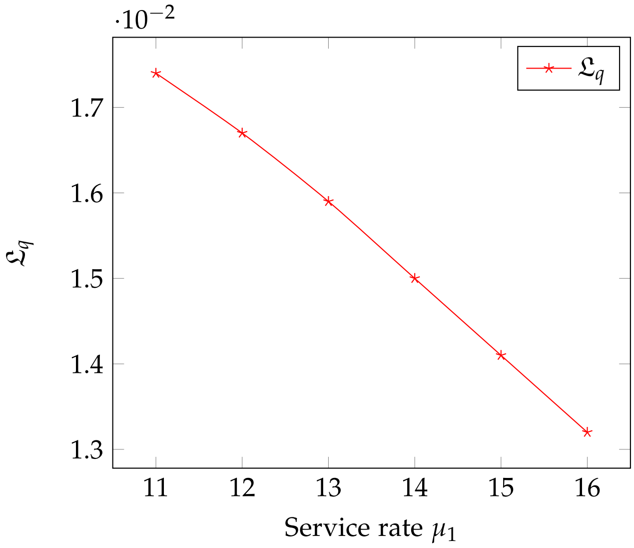


Figure 2: *Service rate vs Queue length*

Figure 2 clearly demonstrates that as the mandatory service times increase, the mean queue length decreases.

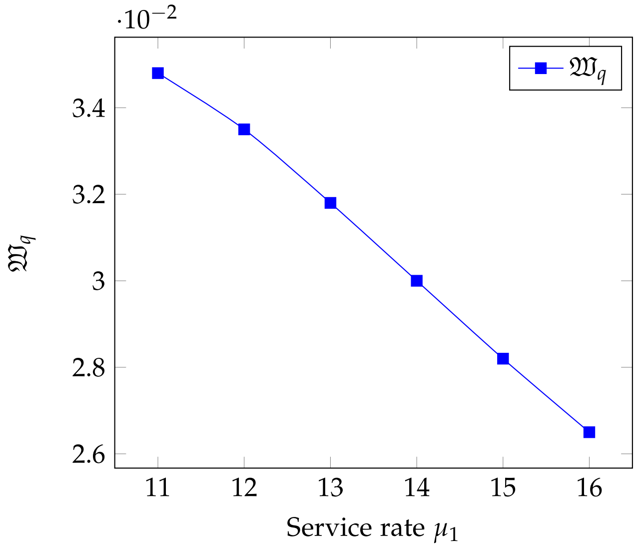


Figure 3: *Service rate vs Waiting time*

Figure 3 indicates that as the mandatory service times increase, the mean waiting time of customers in the queue decreases.

9. CONCLUSION

We analyzed an $M^X/G/1$ queueing system with optional second service, feedback, and Bernoulli vacation. Key performance indicators are obtained by applying the supplementary variable approach. Numerical outcomes validate using analytical results. Both the mean waiting time and mean queue length decrease while increase in service rate. For instance, if service rates rise in the banking industry, this model helps reduce customer mean waiting times and mean queue length.

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