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ESTIMATION OF PARAMETERS FOR KUMARASWAMY EXPONENTIAL DISTRIBUTION BASED ON PROGRESSIVE TYPE-I INTERVAL CENSORED SAMPLE

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Abstract

In this paper, we consider the problem of estimation of parameters of the Kumaraswamy exponential distribution using progressive type-I interval censored data. The maximum likelihood estimators (MLEs) of the parameters are obtained. As it is observed that there is no closed-form solutions for the MLEs, we implement the Expectation-Maximization (EM) algorithm for the computation of MLEs. Bayes estimators are also obtained using different loss functions such as the squared error loss function and the LINEX loss function. For the Bayesian estimation, Lindley's approximation method has been applied. To evaluate the performance of the various estimators developed, we conduct an extensive simulation study. The different estimators and censoring schemes are compared based on average bias and mean squared error. A real data set is also taken into consideration for illustration.

Keywords: Maximum likelihood estimate, EM algorithm, Bayesian inference, Lindley's approximation

1. Introduction

In life testing experiment and survival analysis, the test units may leave the experiment before failure due to restriction of time, budget cost or accidental breakage. A censored sample refers to data that was gathered from such cases but may not be complete. Over the last few decades, a number of censoring methodologies have been developed for the analysis of such situations. In the exiting literature, two commonly used traditional censoring schemes are type-I and type-II, in which experiment is terminated after a prescribed time point and number of failures, respectively. However, neither of these two censoring strategies permit the experimenter to remove live units from the experiment prior to its termination time. To remove the units in between the experiments, the idea of progressive censoring was developed by [7]. It is further observed that, in many practical situations it is not possible for the experimenter to continuously observe the life test units to observe the precise failure lifetimes. For example, in medical and clinical trials, specific information regarding the patient survival lifetime for those diagnosed with a particular treatment may not be available. In such cases, the failure lifetimes are often observed in the intervals, known as interval censoring. However, this censoring does not allow to remove the units in between the experiments. The concept of progressive type-I interval censoring, incorporating the principles of type-I, progressive, and interval censoring schemes, was introduced by [2]. In this type of censoring, items can be withdrawn between two successive time points that have been prescheduled.

The progressive type-I interval censored sample is gathered in the following manner. Assume that n units are placed on a life test at the time $t_0 = 0$. Units are inspected at m predefined times $t_1, t_2, ..., t_m$, with t_m being the experiment's scheduled finish time. At the ith inspection time $t_i, i = 1, ..., m$, the number X_i , of failures within $(t_{i-1}, t_i]$ is recorded and R_i surviving units are randomly removed from the life test. The number of surviving units at time $t_1, ..., t_m$ is a random variable, hence the number of removals $R_1, ..., R_m$ can be estimated as a percentage of the remaining surviving units. Specifically, $\lfloor q_i \times$ (number of surviving units at time t_i)$\rfloor$ remaining surviving units are eliminated from the life test with pre-specified values of $q_1, ..., q_{m-1}$ and $q_m = 1$, where $\lfloor w \rfloor$ = the largest integer less than or equal to w. Alternatively, $R_1, R_2, ..., R_m$ can be pre-specified non-negative integers, with $R_i^{obs} = \min(R_i$, number of surviving units at time $t_i)$, $i = 1, 2, ..., m-1$, and R_m^{obs} = number of surviving units at time t_m. Data observed under this censoring scheme can be represented as $(X_i, R_i, t_i)_{i=1}^m$. If $F(x, \theta)$ is the cumulative distribution function (cdf) of the population from which the progressive type-I censored sample is taken, then the likelihood function of θ can be constructed as follows (see, [2])

$$L(\theta) \propto \prod_{i=1}^{m} \left[F(t_i, \theta) - F(t_{i-1}, \theta)\right]^{X_i} \left[1 - F(t_i, \theta)\right]^{R_i} \ , \tag{1}$$

where $t_0 = 0$.

In the recent past, several authors studied progressive type-I interval censored sampling schemes under various circumstances. The maximum likelihood estimates of the parameters of the exponentiated Weibull family and their asymptotic variances were obtained by [4]. Optimally spaced inspection times for the log-normal distribution were determined by [12], while different estimation methods based on progressive type-I interval censoring were considered for the Weibull distribution by [17] and for the Generalized exponential distribution by [6]. The statistical inference under this censoring for Inverse Weibull distribution was further discussed by [19]. Bayesian inference under this censoring has been discussed by [3] for Dagum distribution.
In this paper, we consider progressive type-I interval censored sample taken from a Kumaraswamy exponential (KE) distribution with probability density function (pdf) given by

$$f(x) = \beta\lambda e^{-x}(1 - e^{-x})^{\beta-1}(1 - (1 - e^{-x})^{\beta})^{\lambda-1}, x > 0 \ . \tag{2}$$

The cdf corresponding to the above pdf is given by

$$F(x) = 1 - (1 - (1 - e^{-x})^{\beta})^{\lambda}, x > 0 \ , \tag{3}$$

where $\beta > 0, \lambda > 0$ are two shape parameters. Through out the paper, we use the notation KE(β, λ) to denote Kumaraswamy exponential distribution with shape parameters β and λ. The KE distribution is a generalisation of the exponential distribution that was created as a model for issues in environmental studies and survival analysis. Several studies on Kumaraswamy distribution and its generalisations have been published in recent years. An exponentiated Kumaraswamy distribution and its properties were considered and discussed by [11].The Kumaraswamy linear exponential distribution with four parameters was introduced by [9], who also derived some of its mathematical properties. The maximum likelihood estimation of the unknown parameters for the Kumaraswamy exponential distribution was considered by [1]. The exponentiated Kumaraswamy exponential distribution and its characterization properties were introduced by [18]. The estimation of parameters for the Kumaraswamy exponential distribution under a progressive type-II censored scheme was considered by [5].

The structure of this paper is outlined as follows. The maximum likelihood estimators of KE(β, λ) parameters are obtained in Section 2. In this section, estimators are also obtained using EM algorithm. In Section 3, Bayes estimates for β and λ are obtained for different loss functions such as squared error and LINEX. Here, Lindley's approximation method is used to evaluate these Bayes estimates. In Section 4, a simulation study is carried out for analysing the properties of various estimators developed in this paper. In Section 5, a real data is considered for illustration. Finally, in Section 6, we present some concluding remarks.

2. Maximum Likelihood Estimation

Let (X_i, R_i, t_i) , $i = 1, 2, \cdots n$ be a progressively type-I interval censored sample taken from the KE(β, λ) distribution defined in (2), then by using (1), the likelihood function is given by

$$L(\beta,\lambda) \propto \sum_{i=1}^{m} \left[\left[1-(1-e^{-t_{i-1}})^{\beta}\right]^{\lambda} - \left[1-(1-e^{-t_i})^{\beta}\right]^{\lambda} \right]^{X_i} \left[\left[1-(1-e^{-t_i})^{\beta}\right]^{\lambda} \right]^{R_i}. \quad (4)$$

Then the log-likelihood function is given by

$$\begin{aligned} l(\beta,\lambda) = ln\ L(\beta,\lambda) = & \sum_{i=1}^{m} X_i\ ln\ \left[[1-(1-e^{-t_{i-1}})^{\beta}\right]^{\lambda} - \left[1-(1-e^{-t_i})^{\beta}]^{\lambda}\right] + \\ & \sum_{i=1}^{m} R_i\ ln\ \left[1-(1-e^{-t_i})^{\beta}\right]^{\lambda}. \end{aligned} \quad (5)$$

The MLEs of β and λ are the solutions to the following normal equations

$$\sum_{i=1}^{m} \frac{R_i\ \lambda[1-Z_i^{\beta}]^{\lambda} Z_i^{\beta}\ ln\ Z_i}{[1-Z_i^{\beta}]^{\lambda}} = -\sum_{i=1}^{m} \frac{X_i \left[\lambda[1-Z_i^{\beta}]^{\lambda-1} Z_i^{\beta}\ ln\ Z_i - \lambda[1-Z_{i-1}^{\beta}]^{\lambda-1} Z_{i-1}^{\beta}\ ln\ Z_{i-1}\right]}{[(1-Z_i^{\beta})^{\lambda} - (1-Z_{i-1}^{\beta})^{\lambda}]} \quad (6)$$

and

$$\sum_{i=1}^{m} \frac{R_i\ \lambda[1-Z_i^{\beta}]^{\lambda} ln(1-Z_i^{\beta})}{[1-Z_i^{\beta}]^{\lambda}} = -\sum_{i=1}^{m} \frac{X_i \left[(1-Z_i^{\beta})^{\lambda} ln(1-Z_i^{\beta}) - (1-Z_{i-1}^{\beta})^{\lambda} \ln(1-Z_i^{\beta})\right]}{[(1-Z_i^{\beta})^{\lambda} - (1-Z_{i-1}^{\beta})^{\lambda}]}, \quad (7)$$

where $Z_i^{\beta} = (1-e^{-t_i})$.
As the above equations have no closed form solutions, the MLEs can be obtained through an iterative numerical methods such as Newton-Raphson method. Since the MLEs are obtained using numerical method, in the following subsection, the EM algorithm is used to find the MLEs of β and λ.

2.1. EM Algorithm

The Expectation-Maximization (EM) algorithm is a broadly applicable method of iterative computing of maximum likelihood estimates and useful in a variety of incomplete-data scenarios where methods like the Newton-Raphson method may prove to be more difficult. The expectation step, also known as the E-step, and the maximisation step, often known as the M-step, are two steps that comprise each iteration of the EM algorithm. Therefore, the algorithm is known as the EM algorithm, and its detailed development can be found in [8]. The EM algorithm for finding MLEs of the parameter of the two-parameter Kumaraswamy exponential distribution is as follows.
Let $\psi_{i,j}, j = 1, 2, ...X_i$, be the survival times of the units failed within subinterval $(t_{i-1} t_i]$ and $\psi^*_{i,j}, j = 1, 2, ...R_i$ be the durations of survival for those units withdrawn at t_i for $i = 1, 2, 3, ...m$, then the log likelihood function, $ln(L^c)$, based on the lifetimes of all n items (complete sample) from the two-parameter KE(β, λ) distribution is given by

$$ln(L^c) = \sum_{i=1}^{m} \left[\sum_{j=1}^{X_i} log(f(\psi_{i,j}, \theta)) + \sum_{j=1}^{R_i} log(f(\psi^*_{i,j}, \theta)) \right],$$

$$
\begin{aligned}
ln(L^c) = [ln(\beta)+ln(\lambda)]\sum_{i=1}^{m}[X_i+R_i] - \sum_{i=1}^{m}\left[\sum_{j=1}^{X_i}\psi_{i,j}+\sum_{j=1}^{R_i}\psi^*_{i,j}\right] + \\
(\beta-1)\sum_{i=1}^{m}\left[\sum_{j=1}^{X_i} ln(1-e^{-\psi_{i,j}}) + \sum_{j=1}^{R_i} ln(1-e^{-\psi^*_{i,j}})\right] + \\
(\lambda-1)\sum_{i=1}^{m}\left[\sum_{j=1}^{X_i} ln\left[1-(1-e^{-\psi_{i,j}})^{\beta}\right] + \sum_{j=1}^{R_i} ln\left[1-(1-e^{-\psi^*_{i,j}})^{\beta}\right]\right],
\end{aligned} \tag{8}
$$

where $\sum_{i=1}^{m}(X_i+R_i)=n$
Taking the derivatives of (8) with respect to β and λ, respectively, the following normal equations are obtained:

$$
\begin{aligned}
\frac{n}{\beta} =& (\lambda-1)\sum_{i=1}^{m}\left[\sum_{j=1}^{X_i}\frac{(1-e^{-\psi_{i,j}})^{\beta} ln(1-e^{-\psi_{i,j}})}{\left[1-(1-e^{-\psi_{i,j}})^{\beta}\right]}\right] + \sum_{i=1}^{m}\left[\sum_{j=1}^{X_i}\frac{(1-e^{-\psi^*_{i,j}})^{\beta} ln(1-e^{-\psi^*_{i,j}})}{\left[1-(1-e^{-\psi^*_{i,j}})^{\beta}\right]}\right] \\
& -\sum_{i=1}^{m}\left[\sum_{j=1}^{X_i} ln(1-e^{-\psi_{i,j}}) + \sum_{j=1}^{R_i} ln(1-e^{-\psi^*_{i,j}})\right]
\end{aligned} \tag{9}
$$

and

$$
\frac{n}{\lambda} = -\sum_{i=1}^{m}\left[\sum_{j=1}^{X_i} ln\left[1-(1-e^{-\psi_{i,j}})^{\beta}\right] + \sum_{j=1}^{R_i} ln\left[1-(1-e^{-\psi^*_{i,j}})^{\beta}\right]\right]. \tag{10}
$$

The lifetimes of X_i failures in the i^{th} interval $(t_{i-1}, t_i]$ are independent and follow a doubly truncated Kumaraswamy exponential distribution from left at t_{i-1} and right at t_i, while the lifetimes of the R_i censored items at the time t_i are independent and follow a truncated Kumaraswamy exponential distribution from the left at $t_i, i = 1, 2, ...m$.

For the EM algorithm, the following expected values of a doubly truncated Kumaraswamy exponential random variable Y, from a on the left and b on the right with $0 < a < b \le \infty$ are needed.

$$
E_{\beta,\lambda}\left[ln(1-e^{-Y})|Y\in[a,b)\right] = \int_a^b \frac{\ln(1-e^{-y})f(y;\beta,\lambda)dy}{F(b:\beta,\lambda)-F(a;\beta,\lambda)} \;,
$$

$$
E_{\beta,\lambda}\left[ln\left[1-(1-e^{-Y})^{\beta}\right]|Y\in[a,b)\right] = \int_a^b \frac{\ln\left[1-(1-e^{-y})^{\beta}\right]f(y;\beta,\lambda)dy}{F(b:\beta,\lambda)-F(a;\beta,\lambda)}
$$

and

$$
E_{\beta,\lambda}\left[\frac{(1-e^{-Y})^{\beta} ln(1-e^{-Y})}{\left[1-(1-e^{-Y})^{\beta}\right]}|Y\in[a,b)\right] = \int_a^b \frac{\frac{(1-e^{-Y})^{\beta} ln(1-e^{-Y})}{\left[1-(1-e^{-Y})^{\beta}\right]}f(y;\beta,\lambda)dy}{F(b:\beta,\lambda)-F(a;\beta,\lambda)} \;.
$$

The iterative process that results in the EM algorithm is as follows:

Step 1: Given starting values of β and λ, say $\beta^{(0)}$ and $\lambda^{(0)}$ and set k=0.

Step 2: In the $(k+1)^{th}$ iteration, the following conditional expectations are computed by the E-step. For $i = 1, 2, \cdots, m$

$$E_{1i} = E_{\hat{\beta^{(k)}},\hat{\lambda^{(k)}}} \left[ln(1-e^{-Y})|Y \in [t_{i-1}, t_i) \right] ,$$
$$E_{2i} = E_{\hat{\beta^{(k)}},\hat{\lambda^{(k)}}} \left[ln(1-e^{-Y})|Y \in [t_i, \infty) \right] ,$$
$$E_{3i} = E_{\hat{\beta^{(k)}},\hat{\lambda^{(k)}}} \left[ln[1-(1-e^{-Y})^{\hat{\beta}^{(k)}}]|Y \in [t_{i-1}, t_i) \right] ,$$
$$E_{4i} = E_{\hat{\beta^{(k)}},\hat{\lambda^{(k)}}} \left[ln[1-(1-e^{-Y})^{\hat{\beta}^{(k)}}]|Y \in [t_i, \infty) \right] ,$$
$$E_{5i} = E_{\hat{\beta^{(k)}},\hat{\lambda^{(k)}}} \left[\frac{(1-e^{-Y})^{\hat{\beta}^{(k)}} ln(1-e^{-Y})}{[1-(1-e^{-Y})^{\hat{\beta}^{(k)}}]} |Y \in [t_{i-1}, t_i) \right]$$

and

$$E_{6i} = E_{\hat{\beta^{(k)}},\hat{\lambda^{(k)}}} \left[\frac{(1-e^{-Y})^{\hat{\beta}^{(k)}} ln(1-e^{-Y})}{[1-(1-e^{-Y})^{\hat{\beta}^{(k)}}]} |Y \in [t_i, \infty) \right] .$$

Then, the likelihood equations (9) and (10) are respectively given by

$$\frac{n}{\beta} = (\lambda - 1) \sum_{i=1}^{m} [X_i E_{5i} + R_i E_{6i}] - \sum_{i=1}^{m} [X_i E_{1i} + R_i E_{2i}] \tag{11}$$

and

$$\frac{n}{\lambda} = - \sum_{i=1}^{m} [X_i E_{3i} + R_i E_{4i}] \quad . \tag{12}$$

Step 3: The M-step requires to solve the equations (11) and (12) and obtains the next values, $\beta^{(\hat{k+1})}$ and $\lambda^{(\hat{k+1})}$, of β and λ, respectively, as follows:

$$\beta^{(\hat{k+1})} = \frac{n}{(\hat{\lambda}^{(k+1)} - 1) \sum_{i=1}^{m} [X_i E_{5i} + R_i E_{6i}] - \sum_{i=1}^{m} [X_i E_{1i} + R_i E_{2i}]}$$

and

$$\lambda^{(\hat{k+1})} = - \frac{n}{\sum_{i=1}^{m} [X_i E_{3i} + R_i E_{4i}]} \quad .$$

Step 4: Checking for convergence; if convergence happens, then the current $\beta^{(\hat{k+1})}$ and $\lambda^{(\hat{k+1})}$ are the approximated maximum likelihood estimates of β and λ via EM algorithm. If the convergence doesn't happens, then set $k = k+1$ and go to step 2.

3. Bayesian Estimation

In this section, Bayesian estimation of parameters of $KE(\beta, \lambda)$ are obtained under both symmetric and assymetric loss functions.

The squared error is a symmetric loss function and is defined as

$$L_1(\delta, \hat{\delta}) = (\hat{\delta} - \delta)^2,$$

where $\hat{\delta}$ is the estimate of parameter δ.

An asymmetric loss function is the LINEX loss function, defined as

$$L_2(\delta, \hat{\delta}) \propto e^{h(\hat{\delta}-\delta)} - h(\hat{\delta} - \delta) - 1, \quad h \neq 0 \ .$$

We assume that the prior distributions for β and λ follow independent gamma distributions given by

$$\pi_1(\beta|a,b) \propto \beta^{a-1}e^{-b\beta}, \qquad \beta > 0, a > 0, b > 0,$$

and

$$\pi_2(\lambda|c,d) \propto \lambda^{c-1}e^{-d\lambda}, \qquad \lambda > 0, c > 0, d > 0\ .$$

In addition, the hyper-parameters $a, b, c,$ and d represent the prior knowledge of the unknown parameters.

The joint prior distribution of β and λ is of the form

$$\pi(\beta,\lambda) \propto \beta^{a-1}e^{-b\beta}\lambda^{c-1}e^{-d\lambda}, \qquad \beta > 0, \lambda > 0\ . \tag{13}$$

Then, the posterior density of (β, λ) is given by

$$\pi^*(\beta,\lambda|\ \underline{x}) = \frac{L(\beta,\lambda|\ \underline{x})\pi(\beta,\lambda)}{\int_0^\infty \int_0^\infty L(\beta,\lambda)\pi(\beta,\lambda|\ \underline{x})d\lambda\ d\beta}\ . \tag{14}$$

The Bayes estimates of β and λ against the loss function L_1 are respectively obtained as

$$\hat{\beta}_{SB} = E(\beta|\underline{x}) = \frac{\int_\beta \int_\lambda \beta\ \ l(\beta,\lambda)\pi(\beta,\lambda)d\lambda d\beta}{\int_\beta \int_\lambda l(\beta,\lambda)\pi(\beta,\lambda)d\lambda d\beta} \tag{15}$$

and

$$\hat{\lambda}_{SB} = E(\lambda|\underline{x}) = \frac{\int_\beta \int_\lambda \lambda\ \ l(\beta,\lambda)\pi(\beta,\lambda)d\lambda d\beta}{\int_\beta \int_\lambda l(\beta,\lambda)\pi(\beta,\lambda)d\lambda d\beta}\ . \tag{16}$$

The Bayes estimates of β under the loss function L_2 is obtained as

$$\hat{\beta}_{LB} = -\frac{1}{h}\ log\ E(e^{-h\beta}|\underline{x}), \quad h \neq 0,$$

where

$$E(e^{-h\beta}|\underline{x}) = \frac{\int_\beta \int_\lambda e^{-h\beta} l(\beta,\lambda)\pi(\beta,\lambda)d\lambda d\beta}{\int_\beta \int_\lambda l(\beta,\lambda)\pi(\beta,\lambda)d\lambda d\beta}\ . \tag{17}$$

The Bayes estimates of λ under the loss function L_2 is obtained as

$$\hat{\lambda}_{LB} = -\frac{1}{h}\ log\ E(e^{-h\lambda}|\underline{x}), \quad h \neq 0,$$

where

$$E(e^{-h\lambda}|\underline{x}) = \frac{\int_\beta \int_\lambda e^{-h\lambda} l(\beta,\lambda)\pi(\beta,\lambda)d\lambda d\beta}{\int_\beta \int_\lambda l(\beta,\lambda)\pi(\beta,\lambda)d\lambda d\beta}\ . \tag{18}$$

The ratios of integrals given in equations (15), (16), (17) and (18) cannot be obtained in a closed form. Thus, [13] approximation method for evaluating the ratio of two integrals have been used. This has been adopted by several researchers, such as [10], [5], to obtain the approximate Bayes estimates.

3.1. Lindley approximation method

Since all estimates have the forms of ratios of two integrals, to obtain these estimates numerically, we use the Lindley's approximation method. Since Bayes estimates of β and λ depend on the ratio of two integrals, we define,

$$I(\underline{x}) = \frac{\int_0^\infty \int_0^\infty u(\beta,\lambda) e^{l(\beta,\lambda|\underline{x})+\rho(\beta,\lambda)} d\beta d\lambda}{\int_0^\infty \int_0^\infty e^{l(\beta,\lambda|\underline{x})+\rho(\beta,\lambda)} d\beta d\lambda}, \tag{19}$$

where $u(\beta,\lambda)$ is function of β and λ only and $l(\beta,\lambda|\underline{x})$ is the same as $logL(\beta,\lambda|\underline{x})$ and $\rho(\beta,\lambda) = log\ \pi(\beta,\lambda)$. Then by Lindley's method, $I(\underline{x})$ can be approximated as

$$\begin{aligned}
\hat{I}(\underline{x}) =& u(\hat{\beta},\hat{\lambda}) + \frac{1}{2}\left[\left(\hat{u_{\beta\beta}} + 2\hat{u_\beta}\hat{\rho_\beta}\right)\hat{\sigma_{\beta\beta}} + \left(\hat{u_{\lambda\beta}} + 2\hat{u_\lambda}\hat{\rho_\beta}\right)\hat{\sigma_{\lambda\beta}} + \right.\\
&\left.\left(\hat{u_{\beta\lambda}} + 2\hat{u_\beta}\hat{\rho_\lambda}\right)\hat{\sigma_{\beta\lambda}} + \left(\hat{u_{\lambda\lambda}} + 2\hat{u_\lambda}\hat{\rho_\lambda}\right)\hat{\sigma_{\lambda\lambda}}\right]\\
&+ \frac{1}{2}\left[\left(\hat{u_\beta}\hat{\sigma_{\beta\beta}} + \hat{u_\lambda}\hat{\sigma_{\beta\lambda}}\right)\left(\hat{l_{\beta\beta\beta}}\hat{\sigma_{\beta\beta}} + \hat{l_{\beta\lambda\beta}}\hat{\sigma_{\beta\lambda}} + \hat{l_{\lambda\beta\beta}}\hat{\sigma_{\lambda\beta}} + \hat{l_{\lambda\lambda\beta}}\hat{\sigma_{\lambda\lambda}}\right)\right.\\
&\left.+ \left(\hat{u_\beta}\hat{\sigma_{\lambda\beta}} + \hat{u_\lambda}\hat{\sigma_{\lambda\lambda}}\right)\left(\hat{l_{\lambda\beta\beta}}\hat{\sigma_{\beta\beta}} + \hat{l_{\beta\lambda\lambda}}\hat{\sigma_{\beta\lambda}} + \hat{l_{\lambda\beta\lambda}}\hat{\sigma_{\lambda\beta}} + \hat{l_{\lambda\lambda\lambda}}\hat{\sigma_{\lambda\beta}}\right)\right],
\end{aligned} \tag{20}$$

where $\hat{\beta}$ and $\hat{\lambda}$ are the ML estimators of β and λ, respectively. Also, $u_{\beta\beta}$ is the second derivative of the function $u(\beta,\lambda)$ with respect to β and $u_{\hat{\beta}\hat{\beta}}$ is the same expression evaluated at $(\hat{\beta},\hat{\lambda})$. Other expressions are given by

$$\begin{aligned}
l_{\beta\beta} &= \frac{\partial^2 l(\beta,\lambda)}{\partial\beta^2}\\
&= \sum_{i=1}^{m}\left[Xi\left(\frac{\left(\frac{\partial^2 F_i}{\partial\beta^2} - \frac{\partial^2 F_{i-1}}{\partial\beta^2}\right)}{(F_i - F_{i-1})} - \frac{\left(\frac{\partial F_i}{\partial\beta} - \frac{\partial F_{i-1}}{\partial\beta}\right)^2}{(F_i - F_{i-1})^2}\right) + R_i\left(\frac{-\frac{\partial^3 F_i}{\partial\beta^3}}{(1-F_i)} - \frac{\left(\frac{\partial F_i}{\partial\beta}\right)^2}{(1-F_i)^2}\right)\right]
\end{aligned}$$

$$\begin{aligned}
l_{\lambda\lambda} &= \frac{\partial^2 l(\beta,\lambda)}{\partial\lambda^2}\\
&= \sum_{i=1}^{m}\left[Xi\left(\frac{\left(\frac{\partial^2 F_i}{\partial\lambda^2} - \frac{\partial^2 F_{i-1}}{\partial\lambda^2}\right)}{(F_i - F_{i-1})} - \frac{\left(\frac{\partial F_i}{\partial\lambda} - \frac{\partial F_{i-1}}{\partial\lambda}\right)^2}{(F_i - F_{i-1})^2}\right) + R_i\left(\frac{-\frac{\partial^3 F_i}{\partial\lambda^3}}{(1-F_i)} - \frac{\left(\frac{\partial F_i}{\partial\lambda}\right)^2}{(1-F_i)^2}\right)\right]
\end{aligned}$$

$$\begin{aligned}
l_{\beta\lambda} =& \frac{\partial^2 l(\beta,\lambda)}{\partial\lambda^2}\\
=& \sum_{i=1}^{m}\left[Xi\left(\frac{\left(\frac{\partial^2 F_i}{\partial\beta\partial\lambda} - \frac{\partial^2 F_{i-1}}{\partial\beta\partial\lambda}\right)}{(F_i - F_{i-1})} - \frac{\left(\frac{\partial F_i}{\partial\lambda} - \frac{\partial F_{i-1}}{\partial\lambda}\right)\left(\frac{\partial F_i}{\partial\beta} - \frac{\partial F_{i-1}}{\partial\beta}\right)}{(F_i - F_{i-1})^2}\right) + \right.\\
&\left. R_i\left(\frac{-\frac{\partial^2 F_i}{\partial\beta\partial\lambda}}{(1-F_i)} - \frac{\left(\frac{\partial F_i}{\partial\beta}\right)\left(\frac{\partial F_i}{\partial\lambda}\right)}{(1-F_i)^2}\right)\right]
\end{aligned} \tag{21}$$

$$l_{\lambda\beta} = \frac{\partial^2 l(\beta,\lambda)}{\partial\lambda^2}$$
$$= \sum_{i=1}^{m}\left[Xi\left(\frac{\left(\frac{\partial^2 F_i}{\partial\beta\partial\lambda}-\frac{\partial^2 F_{i-1}}{\partial\beta\partial\lambda}\right)}{(F_i-F_{i-1})}-\frac{\left(\frac{\partial F_i}{\partial\lambda}-\frac{\partial F_{i-1}}{\partial\lambda}\right)\left(\frac{\partial F_i}{\partial\beta}-\frac{\partial F_{i-1}}{\partial\beta}\right)}{(F_i-F_{i-1})^2}\right)+\right.$$
$$\left. R_i\left(\frac{-\frac{\partial^2 F_i}{\partial\beta\partial\lambda}}{(1-F_i)}-\frac{\left(\frac{\partial F_i}{\partial\beta}\right)\left(\frac{\partial F_i}{\partial\lambda}\right)}{(1-F_i)^2}\right)\right]. \quad (22)$$

From equations (21) and (22), we have

$$l_{\beta\beta\beta} = \frac{\partial^3 l(\beta,\lambda)}{\partial\beta^3}$$
$$= \sum_{i=1}^{m}\left[Xi\left(\frac{\frac{\partial^3 F_i}{\partial\beta^3}-\frac{\partial^3 F_{i-1}}{\partial\beta^3}}{F_i-F_{i-1}}-\frac{3\left(\frac{\partial^2 F_i}{\partial\beta^2}-\frac{\partial^2 F_{i-1}}{\partial\beta^2}\right)\left(\frac{\partial F_i}{\partial\beta}-\frac{\partial F_{i-1}}{\partial\beta}\right)}{(F_i-F_{i-1})^2}+\frac{2\left(\frac{\partial F_i}{\partial\beta}-\frac{\partial F_{i-1}}{\partial\beta}\right)^3}{(F_i-F_{i-1})^3}\right)\right.$$
$$\left.-R_i\left(\frac{\frac{\partial^3 F_i}{\partial\beta^3}}{1-F_i}+\frac{3\left(\frac{\partial^2 F_i}{\partial\beta^2}\right)\left(\frac{\partial F_i}{\partial\beta}\right)}{(1-F_i)^2}+\frac{2\left(\frac{\partial F_i}{\partial\beta}\right)^3}{(1-F_i)^3}\right)\right],$$

$$l_{\lambda\lambda\lambda} = \frac{\partial^3 l(\beta,\lambda)}{\partial\lambda^3}$$
$$= \sum_{i=1}^{m}\left[Xi\left(\frac{\frac{\partial^3 F_i}{\partial\lambda^3}-\frac{\partial^3 F_{i-1}}{\partial\lambda^3}}{F_i-F_{i-1}}-\frac{3\left(\frac{\partial^2 F_i}{\partial\lambda^2}-\frac{\partial^2 F_{i-1}}{\partial\lambda^2}\right)\left(\frac{\partial F_i}{\partial\lambda}-\frac{\partial F_{i-1}}{\partial\lambda}\right)}{(F_i-F_{i-1})^2}+\frac{2\left(\frac{\partial F_i}{\partial\lambda}-\frac{\partial F_{i-1}}{\partial\lambda}\right)^3}{(F_i-F_{i-1})^3}\right)\right.$$
$$\left.-R_i\left(\frac{\frac{\partial^3 F_i}{\partial\lambda^3}}{1-F_i}+\frac{3\left(\frac{\partial^2 F_i}{\partial\lambda^2}\right)\left(\frac{\partial F_i}{\partial\lambda}\right)}{(1-F_i)^2}+\frac{2\left(\frac{\partial F_i}{\partial\lambda}\right)^3}{(1-F_i)^3}\right)\right],$$

$$l_{\lambda\beta\beta} = \frac{\partial^3 l(\beta,\lambda)}{\partial\lambda\partial\beta^2}$$
$$= \sum_{i=1}^{m}\left[Xi\left(\frac{\frac{\partial^3 F_i}{\partial\lambda\partial\beta^2}-\frac{\partial^3 F_{i-1}}{\partial\lambda\partial\beta^2}}{F_i-F_{i-1}}-\frac{\left(\frac{\partial^2 F_i}{\partial\beta^2}-\frac{\partial^2 F_{i-1}}{\partial\beta^2}\right)\left(\frac{\partial F_i}{\partial\lambda}-\frac{\partial F_{i-1}}{\partial\lambda}\right)}{(F_i-F_{i-1})^2}\right.\right.$$
$$\left.-\frac{2\left(\frac{\partial F_i}{\partial\beta}-\frac{\partial F_{i-1}}{\partial\beta}\right)\left(\frac{\partial^2 F_i}{\partial\lambda\partial\beta}-\frac{\partial^2 F_{i-1}}{\partial\lambda\partial\beta}\right)}{(F_i-F_{i-1})^2}+\frac{2\left(\frac{\partial F_i}{\partial\beta}-\frac{\partial F_{i-1}}{\partial\beta}\right)^2\left(\frac{\partial F_i}{\partial\lambda}-\frac{\partial F_{i-1}}{\partial\lambda}\right)}{(F_i-F_{i-1})^3}\right)$$
$$\left.-R_i\left(\frac{\frac{\partial^3 F_i}{\partial\lambda\partial\beta^2}}{1-F_i}+\frac{\left(\frac{\partial^2 F_i}{\partial\beta^2}\right)\left(\frac{\partial F_i}{\partial\lambda}\right)}{(1-F_i)^2}+2\frac{\left(\frac{\partial F_i}{\partial\beta}\right)\left(\frac{\partial^2 F_i}{\partial\lambda\partial\beta}\right)}{(1-F_i)^3}+\frac{2\left(\frac{\partial F_i}{\partial\beta}\right)^2\left(\frac{\partial F_i}{\partial\lambda}\right)}{(1-F_i)^3}\right)\right],$$

$$
\begin{aligned}
l_{\beta\lambda\beta} &= \frac{\partial^3 l(\beta,\lambda)}{\partial\lambda\partial\beta^2} \\
&= \sum_{i=1}^{m}\left[Xi\left(\frac{\frac{\partial^3 F_i}{\partial\beta\partial\lambda\partial\beta}-\frac{\partial^3 F_{i-1}}{\partial\beta\partial\lambda\partial\beta}}{F_i-F_{i-1}}-\frac{\left(\frac{\partial^2 F_i}{\partial\beta\partial\lambda}-\frac{\partial^2 F_{i-1}}{\partial\beta\partial\lambda}\right)\left(\frac{\partial F_i}{\partial\beta}-\frac{\partial F_{i-1}}{\partial\beta}\right)}{(F_i-F_{i-1})^2}\right.\right. \\
&- \left(\frac{\left(\frac{\partial F_i}{\partial\lambda}-\frac{\partial F_{i-1}}{\partial\lambda}\right)\left(\frac{\partial^2 F_i}{\partial\beta^2}-\frac{\partial^2 F_{i-1}}{\partial\beta}\right)+\left(\frac{\partial^2 F_i}{\partial\lambda\partial\beta}-\frac{\partial^2 F_{i-1}}{\partial\lambda\partial\beta}\right)\left(\frac{\partial F_i}{\partial\beta}-\frac{\partial F_{i-1}}{\partial\beta}\right)}{(F_i-F_{i-1})^2}\right) \\
&\left.-\frac{2\left(\frac{\partial F_i}{\partial\lambda}-\frac{\partial F_{i-1}}{\partial\lambda}\right)\left(\frac{\partial F_i}{\partial\beta}-\frac{\partial F_{i-1}}{\partial\beta}\right)^2}{(F_i-F_{i-1})^2}\right) \\
&+ R_i\left(\left(-\frac{\frac{\partial^3 F_i}{\partial\beta\partial\lambda\partial\beta^2}}{1-F_i}+\frac{\left(\frac{\partial^2 F_i}{\partial\beta\partial\lambda}\right)\left(\frac{\partial F_i}{\partial\beta}\right)}{(1-F_i)^2}\right)-\left(\frac{\left(\frac{\partial F_i}{\partial\beta}\right)\left(\frac{\partial^2 F_i}{\partial\lambda\partial\beta}\right)}{(1-F_i)^3}+\frac{\left(\frac{\partial F_i}{\partial\beta}\right)\left(\frac{\partial F_i}{\partial\lambda}\right)}{(1-F_i)^2}\right)+\right. \\
&\left.\left.\frac{2\left(\frac{\partial F_i}{\partial\lambda}\right)\left(\frac{\partial F_i}{\partial\lambda}\right)^2}{(1-F_i^3)}\right)\right].
\end{aligned}
$$

Let $u_i = 1-e^{-t_i}$. Then

$$
\begin{aligned}
\frac{\partial F_i}{\partial\beta} &= \lambda\left[1-u_i^{\beta}\right]^{\lambda-1}u_i^{\beta}ln\ u_i \\
\frac{\partial^2 F_i}{\partial\beta^2} &= \lambda\left[1-u_i^{\beta}\right]^{\lambda-1}u_i^{\beta}\left(ln\ u_i\right)^2-\lambda\left(\lambda-1\right)\left[1-u_i^{\beta}\right]^{\lambda-2}\left(u_i^{\beta}ln\ u_i\right)^2 \\
\frac{\partial^3 F_i}{\partial\beta^3} &= \lambda\left[1-u_i^{\beta}\right]^{\lambda-1}u_i^{\beta}\left(ln\ u_i\right)^3+u_i^{\beta}\left(ln\ u_i\right)^2\lambda\left(\lambda-1\right)\left[1-u_i^{\beta}\right]^{\lambda-2} \\
\frac{\partial F_i}{\partial\lambda} &= -\lambda\left[1-u_i^{\beta}\right]^{\lambda-1} \\
\frac{\partial^2 F_i}{\partial\lambda^2} &= -\lambda(\lambda-1)\left[1-u_i^{\beta}\right]^{\lambda-2} \\
\frac{\partial^3 F_i}{\partial\lambda^3} &= -\lambda(\lambda-1)(\lambda-2)\left[1-u_i^{\beta}\right]^{\lambda-3} \\
&-2\lambda\left(\lambda-1\right)\left[1-u_i^{\beta}\right]^{\lambda-2}u_i^{\beta}\left(ln\ u_i\right)^3+\lambda\left(\lambda-1\right)\left(\lambda-2\right)\left[1-u_i^{\beta}\right]^{\lambda-3}\left(u_i^{\beta}ln\ u_i\right)^3 \\
\frac{\partial^2 F_i}{\partial\beta\partial\lambda} &= \lambda\left(\lambda-1\right)\left[1-u_i^{\beta}\right]^{\lambda-2}u_i^{\beta}ln\ u_i \\
\frac{\partial^3 F_i}{\partial\beta^2\partial\lambda} &= \lambda\left(\lambda-1\right)\left[1-u_i^{\beta}\right]^{\lambda-2}u_i^{\beta}\left(ln\ u_i\right)^2-\lambda\left(\lambda-1\right)\left[1-u_i^{\beta}\right]^{\lambda-3}\left(u_i^{\beta}ln\ u_i\right)^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3 F_i}{\partial\beta\partial\lambda^2} &= \lambda\left(\lambda-1\right)\left(\lambda-2\right)\left[1-u_i^{\beta}\right]^{\lambda-3}u_i^{\beta}ln\ u_i \\
\frac{\partial^2 F_i}{\partial\lambda\partial\beta} &= \lambda\left(\lambda-1\right)\left[1-u_i^{\beta}\right]^{\lambda-2}u_i^{\beta}ln\ u_i \\
\frac{\partial^3 F_i}{\partial\beta\partial\lambda\partial\beta} &= \lambda\left(\lambda-1\right)\left[1-u_i^{\beta}\right]^{\lambda-2}u_i^{\beta}\left(ln\ u_i\right)^2-\lambda\left(\lambda-1\right)\left(\lambda-2\right)\left[1-u_i^{\beta}\right]^{\lambda-3}u_i^{\beta}ln\ u_i \\
\frac{\partial^3 F_i}{\partial\lambda\partial\beta^2} &= \lambda\left(\lambda-1\right)\left[1-u_i^{\beta}\right]^{\lambda-2}u_i^{\beta}\left(ln\ u_i\right)^2-\lambda\left(\lambda-1\right)\left(\lambda-2\right)\left[1-u_i^{\beta}\right]^{\lambda-3}\left(u_i^{\beta}ln\ u_i\right)^2
\end{aligned}
$$

Also,

$$\rho(\beta,\lambda) \propto (c-1)log\beta - d\beta + (b-1)log\lambda - a\lambda \; . \tag{23}$$

Thus,

$$\hat{\rho}_\beta = \frac{c-1}{\hat{\beta}} - d$$

and

$$\hat{\rho}_\lambda = \frac{b-1}{\hat{\lambda}} - a \; .$$

Here

$$\begin{pmatrix} \hat{\sigma}_{\beta\beta} & \hat{\sigma}_{\beta\lambda} \\ \hat{\sigma}_{\lambda\beta} & \hat{\sigma}_{\lambda\lambda} \end{pmatrix} = - \begin{pmatrix} \hat{l}_{\beta\beta} & \hat{l}_{\beta\lambda} \\ \hat{l}_{\lambda\beta} & \hat{l}_{\lambda\lambda} \end{pmatrix}^{-1}$$

We now determine the approximate Bayes estimates of β and λ under various loss functions using the above-mentioned equations. First, we derive the Bayes estimates for β and λ under the squared error loss function L_1. For estimating β, we take $u(\beta,\lambda) = \beta$. Therefore $u_\beta = 1$ and $u_{\beta\beta} = u_\lambda = u_{\lambda\lambda} = u_{\beta\lambda} = u_{\lambda\beta} = 0$. Then the Bayes estimate of β under the loss function L_1 is obtained as

$$\begin{aligned}\hat{\beta}_{SB} =& \hat{\beta} + 0.5[2\hat{\beta}_\beta\hat{\sigma}_{\beta\beta} + 2\hat{\beta}_\lambda\hat{\sigma}_{\beta\lambda} + \hat{\sigma}_{\beta\beta}\hat{l}_{\beta\beta\beta} + \\ & \hat{\sigma}_{\beta\beta}\hat{\sigma}_{\beta\lambda}\hat{l}_{\beta\lambda\beta} + 2\hat{\sigma}_{\beta\beta}\hat{\sigma}_{\lambda\beta}\hat{l}_{\lambda\beta\beta} + \hat{\sigma}_{\lambda\beta}\hat{\sigma}_{\lambda\lambda}\hat{l}_{\lambda\lambda\lambda}] \; .\end{aligned}$$

To estimate λ, we take $u(\beta,\lambda) = \lambda$. Thus $u_\lambda = 1$ and $u_\beta = u_{\beta\beta} = u_{\lambda\lambda} = u_{\lambda\beta} = u_{\beta\lambda} = 0$. Then the Bayes estimate of λ under the loss function L_1 can be determined as

$$\begin{aligned}\hat{\lambda}_{SB} =& \hat{\lambda} + 0.5[2\hat{\beta}_\beta\hat{\sigma}_{\lambda\beta} + 2\hat{\beta}_\lambda\hat{\sigma}_{\lambda\lambda} + \hat{\sigma}_{\beta\beta}\hat{\sigma}_{\beta\lambda}\hat{l}_{\beta\beta\beta} + \\ & \hat{\sigma}^2_{\beta\lambda}\hat{l}_{\beta\lambda\beta} + \hat{\sigma}_{\beta\lambda}\hat{\sigma}_{\lambda\beta}\hat{l}_{\lambda\beta\beta} + \hat{\sigma}_{\beta\beta}\hat{\sigma}_{\lambda\lambda}\hat{l}_{\lambda\beta\beta} + \hat{\sigma}^2_{\lambda\lambda}\hat{l}_{\lambda\lambda\lambda}] \; .\end{aligned}$$

Now, we obtain the Bayes estimates of β and λ under LINEX loss function L_2. For estimating β, we take $u\,(\beta,\lambda) = e^{-h\beta}$. Thus $u_\beta = -he^{-h\beta}, u_{\beta\beta} = h^2e^{-h\beta}$ and $u_\lambda = u_{\lambda\lambda} = u_{\lambda\beta} = u_{\beta\lambda} = 0$. Therefore, Bayes estimate of β under the loss function L_2 is obtained as

$$\hat{\beta}_{LB} = -\frac{1}{h}log[E(e^{-h\beta}|\underline{x})] \; , \tag{24}$$

where

$$\begin{aligned}E(e^{-h\beta}|x) =& e^{-h\hat{\beta}} + 0.5[\hat{u}_{\beta\beta}\hat{\sigma}_{\beta\beta} + \hat{u}_\lambda(2\hat{\beta}_\lambda\hat{\sigma}_{\beta\beta} + 2\hat{\beta}_\lambda\hat{\sigma}_\lambda + \hat{\sigma}^2_{\beta\beta}\hat{l}_{\beta\beta\beta} + \\ & \hat{\sigma}_{\beta\beta}\hat{\sigma}_{\beta\lambda}\hat{l}_{\beta\lambda\beta} + 2\hat{\sigma}_{\beta\beta\beta}\hat{\sigma}_{\lambda\beta}\hat{l}_{\lambda\beta\beta}] \; ,\end{aligned} \tag{25}$$

To estimate λ, we take $u(\beta,\lambda) = e^{-h\lambda}$. Thus, $u_\lambda = -he^{-h\lambda}, u_{\lambda\lambda} = h^2e^{-h\lambda}$ and $u_\beta = u_{\beta\beta} = u_{\lambda\beta} = u_{\beta\lambda} = 0$. Therefore, the Bayes estimate of λ under the loss function L_2 is obtained as

$$\hat{\lambda}_{LB} = -\frac{1}{h}log[E(e^{-h\lambda}|\underline{x})] \; , \tag{26}$$

where

$$\begin{aligned}E(e^{-h\lambda}|x) =& e^{-h\hat{\lambda}} + 0.5\left[\hat{u}_{\lambda\lambda}\hat{\sigma}_{\lambda\lambda} + \hat{u}_\lambda\left(2\hat{\beta}_\beta\hat{\sigma}_{\lambda\beta} + 2\hat{\beta}_\lambda\hat{\sigma}_{\lambda\lambda} + \hat{\sigma}_{\beta\beta}\hat{\sigma}_{\beta\lambda}\hat{l}_{\beta\beta\beta} + \right.\right. \\ & \left.\left.\hat{\sigma}^2_{\beta\lambda}\hat{\mathrm{l}}_{\beta\lambda\beta} + \hat{\sigma}_{\beta\lambda}\hat{\sigma}_{\lambda\beta}\hat{\mathrm{l}}_{\lambda\beta\beta} + \hat{\sigma}_{\lambda\lambda}\hat{\sigma}_{\beta\beta}\hat{l}_{\lambda\beta\beta} + \hat{\sigma}^2_{\lambda\lambda}\hat{l}_{\lambda\lambda\lambda}\right)\right] \; .\end{aligned} \tag{27}$$

4. Simulation Study

A simulation study is carried out in this section to investigate the behaviours of the proposed methods of estimation for the KE distribution. Five different censoring schemes are suggested to generate progressive type-I interval censored data from the KE distribution, and a comparison of all the estimating techniques mentioned above will be addressed. The simulation is run in R programming. The different censoring schemes used to compare the performance of the estimation procedures is given in the following table.

Scheme i	**n**	**m**	$q_{(i)}$
1	75	10	$(0.25, 0.25, 0.25, 0.25, 0.5, 0.5, 0.5, 0.5, 0.5, 1)$
2	75	12	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$
3	100	15	$(0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.25, 0.25, 0.25, 0.25$ $0.25, 0.25, 0.25, 1)$
4	100	20	$(0.25, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$
5	100	25	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$

Here, for the censoring scheme 1, the first few intervals are lighter, and the remaining intervals are heavier. Schemes 2 and 5 represent conventional interval censoring, in which no removals are made prior to the end of the experiment, scheme 3 is the reverse of scheme 1, and censoring in scheme 4 occurs only at the beginning and end. For various combinations of n, m, and various censoring schemes, the performance of each estimators is numerically compared in terms of their bias and mean square error (MSE) values. The bias and MSE of the MLE's and the estimates obtained using the EM algorithm are given in Table 1.

For Bayes estimation, we considered both informative and non-informative priors for the unknown parameters. For informative prior, we consider two priors; Prior 1 and Prior 2. The hyper-parameters for Prior 1 and Prior 2 are chosen in such a way that mean of the prior distribution is equal to the parameter value and variance of the prior distribution is high (Prior 1) and low (Prior 2). The values of hyper-parameters we considered for different choices of the parameters β and λ are given below.

Parameter β/λ	Prior	Hyper parameters a/c	b/d
1.25	Prior 1	1.25	1
	Prior 2	2.5	2
1.5	Prior 1	1.5	1
	Prior 2	3	2
1.75	Prior 1	1.75	1
	Prior 2	3.5	2
2	Prior 1	1	0.5
	Prior 2	4	2

We use $h = 1$ to evaluate Bayes estimators under the LINEX loss function L_2. In each case, we have assessed bias and MSE based on 500 iterations. We repeat the simulation study for various values of β and λ also. The bias and MSE for the estimate of β for both informative and non-informative priors are given in Table 2. Table 3 provides the bias and MSE for the estimate of λ for both informative and non-informative priors.

The tabulated values shows that all of the estimates do improve with a higher value of n. From Table 1, regarding MSE and Bias, we found that the estimates based on the censoring schemes 2 and 5 give the better estimates of β and λ, followed by the scheme 4. In case of maximum likelihood estimation given in table 1, as n increases the MSE of estimates decrease as expected. Also, we can see that bias and MSEs of the estimates of β and λ via EM algorithm are smaller than bias and MSEs of the corresponding MLEs. Also the Bayes estimators based on informative prior perform much better than the MLEs in terms of biases and MSEs. From the tables 1, 2 and 3, it is clear that the bias and MSE of Bayes estimators under informative prior are smaller than those of MLE's.

As expected, the Bayes estimators based on informative prior perform much better than the Bayes estimators based on non-informative prior in terms of biases and MSEs. From Tables 2 and 3, one can see that for β and λ, estimators based on informative priors perform better to those of non-informative priors in terms of bias and MSE. Also, among the Bayes estimators of β, the estimator under the LINEX loss function performs better. Again, when compared to squared error loss functions, estimators of λ under the LINEX loss function have the least bias and MSE.

Table 1: *Bias and MSE of parameters under different censoring schemes for different values of β and λ*

				$\hat{\beta}$				$\hat{\lambda}$			
(β, λ)	n	m	c.s	MLE		EM		MLE		EM	
				Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
		10	1	-0.3509	0.1807	-0.3055	0.0956	-0.4086	0.3751	-0.3131	0.3318
		12	2	-0.2609	0.1073	-0.1063	0.0268	-0.2298	0.1260	-0.1192	0.1057
(1.25,1.5)	75	15	3	-0.3948	0.2090	-0.3129	0.0985	-0.6539	0.7903	-0.2564	0.7338
		20	4	-0.2811	0.1276	-0.2259	0.0631	-0.2734	0.2140	-0.1871	0.2175
		25	5	-0.2725	0.1174	-0.2195	0.0495	-0.2567	0.1782	-0.1691	0.1555
		10	1	-0.3487	0.1360	-0.3160	0.0772	-0.1784	0.2405	-0.1686	0.1903
		12	2	-0.0542	0.0394	-0.0528	0.0901	-0.1275	0.1113	-0.1136	0.0511
(1.75,2)	100	15	3	-0.3526	0.1757	-0.3016	0.0912	-0.3358	0.3049	-0.2373	0.2762
		20	4	-0.1386	0.0538	-0.1029	0.0558	-0.1628	0.1496	-0.1421	0.1040
		25	5	-0.0982	0.0501	-0.0898	0.0242	-0.1422	0.1386	-0.1315	0.0757

Table 2: *Bias and MSE of the Bayes estimates of β under squared error loss function, $\hat{\beta_{SB}}$, and LINEX loss function, $\hat{\beta_{LB}}$, for different values of β and λ*

					Informative Prior 1				Informative Prior 2				Non-Informative Prior			
β	λ	n	m	c.s	$\hat{\beta_{SB}}$		$\hat{\beta_{LB}}$		$\hat{\beta_{SB}}$		$\hat{\beta_{LB}}$		$\hat{\beta_{SB}}$		$\hat{\beta_{LB}}$	
					Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
1.25	1.5	75	10	1	-0.3408	0.1736	-0.3378	0.1717	-0.3324	0.1602	-0.3282	0.1580	-0.5498	0.3766	-0.5468	0.3733
			12	2	-0.2562	0.1043	-0.2540	0.1033	-0.2159	0.0878	-0.2141	0.0872	-0.2804	0.1246	-0.2776	0.1183
		100	15	3	-0.3821	0.1986	-0.3786	0.1961	-0.1147	0.1000	-0.1151	0.0802	-0.4722	0.2432	-0.4592	0.2268
			20	4	-0.2477	0.0860	-0.2468	0.0856	-0.2338	0.0770	-0.2322	0.0764	-0.2708	0.1103	-0.2696	0.1032
			25	5	-0.2670	0.0943	-0.2664	0.0940	-0.1117	0.0375	-0.1109	0.0373	-0.2694	0.0980	-0.2682	0.0975
1.75	2	75	10	1	-0.2720	0.1067	-0.2711	0.1063	-0.2338	0.0991	-0.2318	0.9014	-0.3475	0.1689	-0.3473	0.1688
			12	2	-0.0533	0.0368	-0.0530	0.0361	-0.1740	0.1046	-0.1733	0.1043	-0.0656	0.0306	-0.0654	0.0304
		100	15	3	-0.3469	0.1683	-0.3459	0.1677	-0.1537	0.0748	-0.1542	0.0747	-0.3824	0.1997	-0.3790	0.1974
			20	4	-0.0995	0.0537	-0.0986	0.0536	-0.1106	0.0389	-0.1163	0.0389	-0.2372	0.0930	-0.2363	0.0927
			25	5	-0.0929	0.0499	-0.0921	0.0498	-0.0894	0.0541	-0.0886	0.0540	-0.1559	0.0635	-0.1552	0.0634

Table 3: *Bias and MSE of the Bayes estimates of λ under squared error loss function, $\hat{\lambda_{SB}}$, and LINEX loss function, $\hat{\lambda_{LB}}$, for different values of β and λ*

					Informative Prior 1				Informative Prior 2				Non-Informative Prior			
β	λ	n	m	c.s	$\hat{\lambda_{SB}}$		$\hat{\lambda_{LB}}$		$\hat{\lambda_{SB}}$		$\hat{\lambda_{LB}}$		$\hat{\lambda_{SB}}$		$\hat{\lambda_{LB}}$	
					Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
1.25	1.5	75	10	1	-0.4077	0.3745	-0.4076	0.3744	-0.4548	0.4265	-0.4546	0.4264	-0.6067	0.6583	-0.6066	0.6581
			12	2	-0.2519	0.1573	-0.2518	0.1573	-0.1998	0.1438	-0.1998	0.1438	-0.3475	0.3341	-0.3473	0.3340
		100	15	3	-0.6532	0.7865	-0.6529	0.7862	-0.2819	0.3809	-0.2818	0.3809	-1.4426	2.6977	-1.4425	2.6973
			20	4	-0.2291	0.1237	-0.2291	0.1237	-0.2376	0.1202	-0.2376	0.1202	-0.2761	0.1893	-0.2760	0.1892
			25	5	-0.2283	0.1258	-0.2283	0.1258	-0.1312	0.0791	-0.1312	0.0790	-0.2436	0.1420	-0.2435	0.1419
1.75	2	75	10	1	-0.1742	0.1449	-0.1742	0.1449	-0.1293	0.2154	-0.1293	0.2154	-0.3583	0.2185	-0.3583	0.2185
			12	2	-0.1348	0.1299	-0.1346	0.1297	-0.1774	0.2478	-0.1771	0.2477	-0.2544	0.1782	-0.2544	0.1782
		100	15	3	-0.3182	0.2985	-0.3182	0.2985	0.3172	0.1586	0.3172	0.1586	-0.4004	0.3473	-0.4004	0.3473
			20	4	-0.1550	0.1466	-0.1550	0.1466	-0.1297	0.1313	-0.1297	0.1313	-0.2658	0.1912	-0.2657	0.1912
			25	5	-0.1440	0.1479	-0.1440	0.1479	-0.1242	0.1357	-0.1242	0.1357	-0.2098	0.1581	-0.2098	0.1581

5. Illustrations using real data

In this section, a real-life data is utilised to demonstrate the inference methods proposed in this paper. The data was previously studied by [14] and [15].
The data shows the running and failure times for a sample of devices from the larger system's eld-tracking research. The failure times are:
2.75, 0.13, 1.47, 0.23, 1.81, 0.30, 0.65, 0.10, 3.00, 1.73, 1.06, 3.00, 3.00, 2.12, 3.00, 3.00, 3.00, 0.02, 2.61, 2.93, 0.88, 2.47, 0.28, 1.43, 3.00, 0.23, 3.00, 0.80, 2.45, 2.66.
The data was also previously considered by [16] and fitted for KE(β, λ) distribution. For evaluating the goodness of fit, they used the Anderson-Darling test. The Anderson-Darling test statistic has a value of 2.00757 and the related P-value is 0.0913729. Based on the aforementioned estimation procedures, we have obtained the estimates of β and λ, which is included in Table 4.

Table 4: *Estimates of β and λ for the real data*

						Bayes	
n	m	Censoring scheme		MLE	EM	SE	$LINEX$
30	5	$q=(0.25,0.25,0.5,0.5,1)$	β	1.2857	1.5756	1.5875	1.5173
		$X_i=(8,3,4,1,4)$	λ	0.5192	0.5739	0.5324	0.5338
		$R_i=(2,1,3,4,0)$					
30	7	$q=(0.5,0,0,0,0,0,1)$	β	1.1000	1.3050	0.9374	0.9214
		$X_i=(7,3,3,2,3,5,0)$	λ	0.5355	0.5784	0.5875	0.5866
		$R_i=(3,0,0,0,0,4,0)$					
30	12	$q=(0,0,0,0,0,0,0,0,0,0,0,1)$	β	0.8453	1.1082	0.7944	0.7866
		$X_i=(5,2,1,2,1,2,1,1,1,2,3,4)$	λ	0.4445	0.4961	0.4869	0.4346
		$R_i=(0,0,0,0,0,0,0,0,0,0,0,5)$					

6. Conclusion

In this paper, we considered the problem of estimation of parameters of Kumaraswamy-exponential distribution based on progressive type-I interval censored sample. The maximum likelihood estimators of the parameters β and λ were obtained. Since the MLEs of the unknown parameters of the distribution does not admit closed form, we employed the EM algorithm approach. The Bayes estimators were also obtained using different loss functions such as squared error loss function and LINEX loss function. To evaluate the Bayes estimators, Lindley's approximation method was applied. Based on simulation study, we have the following conclusions. We observed that the performance of EM algorithm was quite satisfactory. In addition, it was found that for both β and λ, the bias and MSE of the Bayes estimators under an informative prior are smaller than those of MLEs. The performance of Informative prior was better than the Non-informative prior both β and λ in terms of bias and MSE values. For both β and λ, Bayes estimators under LINEX loss function perform better with regard to bias and MSE. The estimation methods employed in this paper were also illustrated using real data sets.

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