RT&A, No 3 (79)
Volume 19, September 2024

THE EXPONENTIATED SKEW LAPLACE DISTRIBUTION: PROPERTIES AND APPLICATIONS

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Abstract

In this paper, a 4-parameter Exponentiated Skew Laplace distribution is defined and studied. Various statistical properties including its moment generating function, characteristics function, hazard function, and reliability function of the proposed ESLD were derived. The estimation of its parameters was carried out using the maximum likelihood method of estimation. The performance of the proposed ESLD compared with other similar distributions was demonstrated empirically with daily returns of S & P 500 between 2/02/24 and 28/03/2024 and daily returns of Bitcoin between 2/02/24 and 1/04/24 as obtained from Yahoo Finance. The fitness performance of the proposed distribution was evaluated based on log-likelihood, AIC, and BIC. Results obtained show that the proposed ESLD reported the highest log likelihood as well as the lowest AIC and BIC in the two data sets. This study therefore underscores the superiority of the proposed distribution over the some of the similar existing distributions.

Keywords: Skew Laplace, Exponentiated, Distribution, Reliability function, Maximum Likelihood

I. Introduction

The Laplace distribution also commonly known as the double exponential distribution is named after Pierre Simon Laplace. The growing popularity of the Laplace-based models as described by Lakshmi and Sebastian [1] is due to the properties of the sharp peak at the mode, heavier than normal distribution making it an asymmetric distribution Laplace distribution as described by Nadarajah and Kotz [2] is a tractable lifetime model with applications in various areas including telecommunication, biological sciences, engineering, and life testing among other areas of human endaviours. Despite the significance of this distribution in probabilistic modelling, the Laplace distribution lacks a skewness parameter and hence not be able to account for skewness in data. To overcome this challenge, the Laplace distribution as noted by Kotz *et al.*, [3] and Safavinejad *et al.*,[4]) was extended by adding a skewness parameter. This extension therefore gave rise to the skew Laplace distribution.

The use Skew-Laplace distribution in Economics, Finance and Engineering has been emphasized by Puig and Stephens [5] while Julia and Vives-Rego [6] used Skew Laplace distribution to analyze bacterial sizes in axenic cultures. The skew Laplace distribution has been applied to different areas of research. This is due to its flexibility to model real data that exhibit skewness. Skew Laplace distribution has also been used in probabilistic modelling of financial data. For instance, Jing *et al*. [7] applied asymmetric Laplace to currency exchange rates in Australia, Canada, European and United Kingdom. Similarly, Shi *et al*. [8] used the asymmetric Laplace distribution in portfolio selection. The use of Skew Laplace distribution can also be extended to cryptocurrencies and stock data because of its ability to capture skewed data as cryptocurrencies and stocks have some stylized properties which include departure from normality, price jumps, and high volatility patterns.

In this study, a new form of skew Laplace distribution of Safavinejad *et al*. [4] is proposed, and some of the statistical properties are proposed with application to real data. The fitness performance of the proposed ESLD compared to some existing distributions will also be investigated in this study. This study adopts the method of exponentiation introduced by Gupta *et al*. [9] in generating the proposed Exponentiated Skew Laplace Distribution (ESLD). In the exponentiation method, only one shape parameters is introduced into the parent distribution. This method has been used by Agboola *et al*., [10], Oguntunde *et al*., [11], Nadarajah and Bakar [12], Datta and Datta [13], Andrade *et al*., [14], Adubisi *et al*., [15] among others to generate more flexible distribution.

II. Methods

Exponentiated Skew Laplace Distribution (ESLD)

The probability density function (pdf) and the Cumulative Density Function (CDF) of the Skewed Laplace Distribution (SLD) are defined in (1) and (2) as follows:

$$f(x;\theta,\varepsilon,\delta)=\begin{cases}\frac{1}{\theta+\varepsilon}\exp\left(\frac{x-\delta}{\theta}\right), x<\delta \\ \frac{1}{\theta+\varepsilon}\exp\left(-\frac{x-\delta}{\varepsilon}\right), x\geq\delta\end{cases} \tag{1}$$

Safavinejad *et al*. [4]

$$F(x;\theta,\varepsilon,\delta)=\begin{cases}\frac{\theta}{\theta+\varepsilon}\exp\left(\frac{x-\delta}{\theta}\right), x<\delta \\ 1-\frac{\varepsilon}{\theta+\varepsilon}\exp\left(-\frac{x-\delta}{\varepsilon}\right), x\geq\delta\end{cases} \tag{2}$$

$$-\infty<x<\infty,\ \lambda>0,\ \theta,\varepsilon,>0,\ -\infty<\delta<\infty$$

Safavinejad *et al*. [4]

where, θ is a scale parameter, δ location parameter and skewness parameter.

The probability density function of an Exponentiated generation distribution is given as:

$$g(x)=\lambda F(x)^{\lambda-1}f(x),\ \lambda>0 \tag{3}$$

The Cumulative Density Function is given as:

$$G(x)=\left[F(x)\right]^{\lambda} \tag{4}$$

Where, $f(x)$ is the density function of the Skewed Laplace Distribution (SLD) and $F(x)$ is its corresponding cumulative density function.

Substituting for $f(x)$ in (3) and $F(x)$ in (4) give the pdf and the CDF of the proposed Exponentiated Skew Laplace Distribution.

$$g(x)=\begin{cases}\frac{\lambda}{\theta+\varepsilon}\exp\left(\frac{x-\delta}{\theta}\right)\left[\frac{\theta}{\theta+\varepsilon}\exp\left(\frac{x-\delta}{\theta}\right)\right]^{\lambda-1}, x<\delta\\ \frac{\lambda}{\theta+\varepsilon}\exp\left(-\frac{x-\delta}{\varepsilon}\right)\left[1-\frac{\varepsilon}{\theta+\varepsilon}\exp\left(-\frac{x-\delta}{\varepsilon}\right)\right]^{\lambda-1}, x\geq\delta\end{cases} \tag{5}$$

$$\theta,\varepsilon>0,\ \lambda>0,\ -\infty<x<\infty,$$

$$G(x)=\begin{cases}\left[\frac{\theta}{\theta+\varepsilon}\exp\left(\frac{x-\delta}{\theta}\right)\right]^{\lambda-1}, x<\delta\\ \left[1-\frac{\varepsilon}{\theta+\varepsilon}\exp\left(-\frac{x-\delta}{\varepsilon}\right)\right]^{\lambda-1}, x\geq\delta\end{cases} \tag{6}$$

Theorem 1

A random variable X is said to follow an Exponentiated Skew Laplace Distribution (ESLD), if its probability density function can be expressed as:

$$g(x)=\begin{cases}\frac{\lambda}{\theta+\varepsilon}\exp\left(\frac{x-\delta}{\theta}\right)\left[\frac{\theta}{\theta+\varepsilon}\exp\left(\frac{x-\delta}{\theta}\right)\right]^{\lambda-1}, x<\delta\\ \frac{\lambda}{\theta+\varepsilon}\exp\left(-\frac{x-\delta}{\varepsilon}\right)\left[1-\frac{\varepsilon}{\theta+\varepsilon}\exp\left(-\frac{x-\delta}{\varepsilon}\right)\right]^{\lambda-1}, x\geq\delta\end{cases} \tag{7}$$

$$\theta,\varepsilon>0,\ \lambda>0,\ -\infty<x<\infty,$$

Proof

The purpose of this proof is to show that the proposed ESLD is a probability density function.
Since, it a continuous distribution function, then

$$\int_{-\infty}^{\infty} g(x)dx=1 \tag{8}$$

$$\int_{-\infty}^{\delta}\frac{\lambda}{\theta+\varepsilon}\exp\left(\frac{x-\delta}{\theta}\right)\left[\frac{\theta}{\theta+\varepsilon}\exp\left(\frac{x-\delta}{\theta}\right)\right]^{\lambda-1}dx+\int_{\delta}^{\infty}\left(\frac{\lambda}{\theta+\varepsilon}\right)\exp\left(-\frac{x-\delta}{\varepsilon}\right)\left[1-\frac{\varepsilon}{\theta+\varepsilon}\exp\left(-\frac{x-\delta}{\theta}\right)\right]^{\lambda-1}dx \tag{9}$$

Splitting the expression in equation (9) into two parts as follows:

$$\text{Let } g_1=\int_{-\infty}^{\delta}\frac{\lambda}{\theta+\varepsilon}\exp\left(\frac{x-\delta}{\theta}\right)\left[\frac{\theta}{\theta+\varepsilon}\exp\left(\frac{x-\delta}{\theta}\right)\right]^{\lambda-1}dx \tag{10}$$

$$g_2=\frac{\lambda}{\theta+\varepsilon}\int_{\delta}^{\infty}\exp\left\{-\frac{x-\delta}{\varepsilon}\right\}\left\{1-\frac{\varepsilon}{\theta+\varepsilon}\exp\left(-\frac{x-\delta}{\theta}\right)\right\}^{\lambda-1}dx \tag{11}$$

Solving for g_1.

$$\text{Let } z=\frac{\theta}{\theta+\varepsilon}e^{\frac{x-\delta}{\theta}}\ \Rightarrow\frac{dz}{dx}=\frac{1}{\theta+\varepsilon}e^{\frac{x-\delta}{\theta}} \tag{12}$$

when, $x=-\infty$, $z=0$ and when, $x=\delta$, $z=\dfrac{\theta}{\theta+\varepsilon}$ (13)

Then

$$= \; g_1 = \left(\frac{\theta}{\theta+\varepsilon}\right)^{\lambda} - 0 = \left(\frac{\theta}{\theta+\varepsilon}\right)^{\lambda} \tag{14}$$

Similarly,

$$g_2 = \frac{\lambda}{\theta+\varepsilon}\int_{\delta}^{\infty}\exp\left\{-\frac{x-\delta}{\varepsilon}\right\}\left\{1-\frac{\varepsilon}{\theta+\varepsilon}\exp\left(-\frac{x-\delta}{\theta}\right)\right\}^{\lambda-1}dx \;=\; 1-\left(\frac{\theta}{\theta+\varepsilon}\right)^{\lambda} \tag{15}$$

Therefore,

$$\int_{-\infty}^{\infty} g(x)dx = \left(\frac{\theta}{\theta+\varepsilon}\right)^{\lambda} + 1 - \left(\frac{\theta}{\theta+\varepsilon}\right)^{\lambda} = 1\,. \tag{16}$$

This implies that the proposed ESLD is a probability density function.
When $\lambda=1$, the ESLD in (7) reduces to Skew Laplace distribution
When $\lambda=\theta=1$, the ESLD in (5) reduces to Laplace distribution.

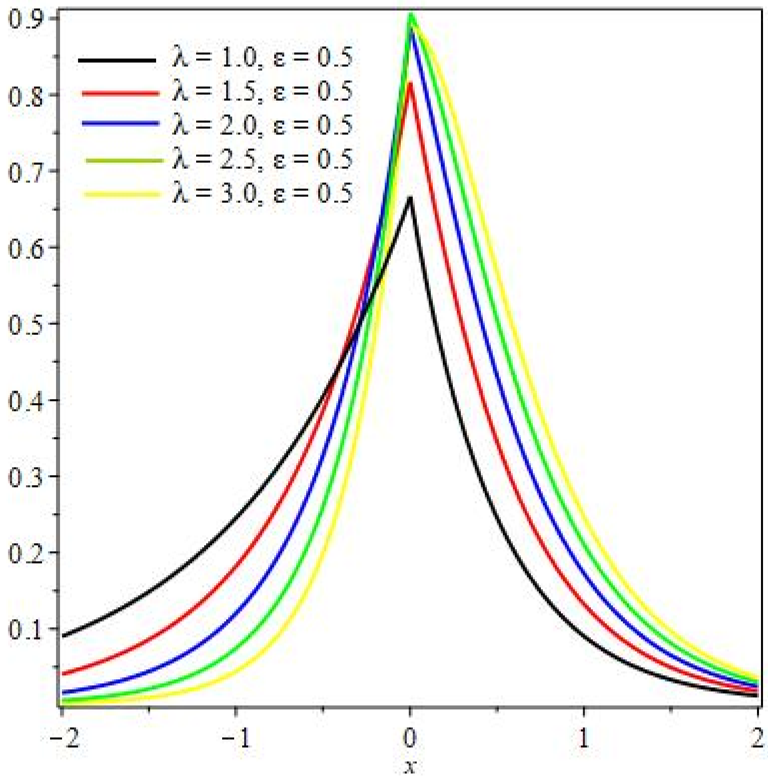

Figure 1: *Plot of pdf of the proposed Exponentiated Skewed Laplace distribution (ESLD).*

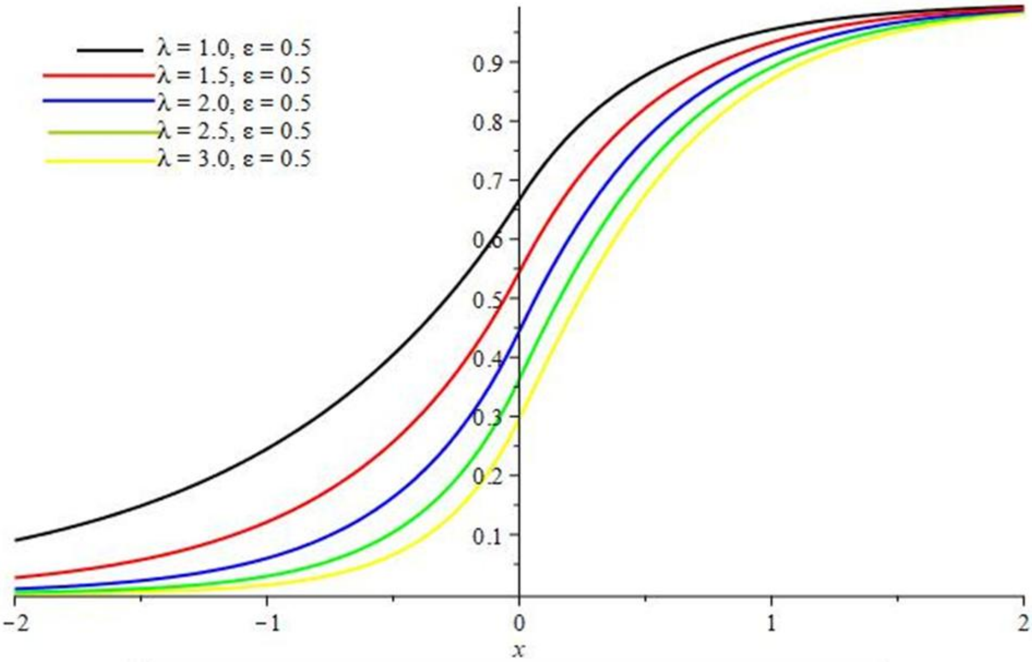

Figure 2: *Plot of the CDF of the proposed distribution ESLD.*

Asymptotic Behaviour of the Proposed ESLD

Theorem 2

If the random variable X ~ESLD$(\theta,\delta,\varepsilon,\lambda)$, $\lim_{x\to-\infty} g(x)=0$ and $\lim_{x\to\infty} g(x)=0$.

Proof

$$g(x)=\begin{cases}\dfrac{\lambda}{\theta+\varepsilon}\exp\left(\dfrac{x-\delta}{\theta}\right)\left[\dfrac{\theta}{\theta+\varepsilon}\exp\left(\dfrac{x-\delta}{\theta}\right)\right]^{\lambda-1}, x<\delta\\ \dfrac{\lambda}{\theta+\varepsilon}\exp\left(-\dfrac{x-\delta}{\varepsilon}\right)\left[1-\dfrac{\varepsilon}{\theta+\varepsilon}\exp\left(-\dfrac{x-\delta}{\varepsilon}\right)\right]^{\lambda-1}, x\ge\delta\end{cases}$$

For, $$\lim_{x\to-\infty} g(x) = \left(\frac{\lambda}{\theta+\varepsilon}\right)\exp\left(\frac{-\infty-\delta}{\theta}\right)\left[\frac{\theta}{\theta+\varepsilon}\exp\left(\frac{-\infty-\delta}{\theta}\right)\right]^{\lambda-1}=0 \quad (17)$$

Similarly,

$$\lim_{x\to\infty} g(x) = \left(\frac{\lambda}{\theta+\varepsilon}\right)\exp\left\{-\frac{\infty-\delta}{\varepsilon}\right\}\left\{1-\frac{\varepsilon}{\theta+\varepsilon}\exp\left(-\frac{\infty-\delta}{\varepsilon}\right)\right\}^{\lambda-1}=0 \quad (18)$$

Hence, $\lim_{x\to-\infty} g(x)=0$ and $\lim_{x\to\infty} g(x)=0$

Theorem 4

If, X ~ESLD$(\theta,\delta,\varepsilon,\lambda)$, $\lim_{x\to-\infty} G(x)=0$ and $\lim_{x\to\infty} G(x)=1$.

Proof

$$\lim_{x\to-\infty} G(x)=\lim_{x\to-\infty}\left[\frac{\theta}{\theta+\varepsilon}\exp\left(\frac{x-\delta}{\theta}\right)\right]^{\lambda} \quad (19)$$

$$= \left[\frac{\theta}{\theta+\varepsilon}\exp\left(\frac{-\infty-\delta}{\theta}\right)\right]^{\lambda} = 0 \tag{20}$$

Similarly, $\lim_{x\to\infty} G(x) = \lim_{x\to\infty}\left[1-\frac{\varepsilon}{\theta+\varepsilon}\exp\left(-\frac{x-\delta}{\varepsilon}\right)\right]^{\lambda}$

$$\left[1-\frac{\varepsilon}{\theta+\varepsilon}\exp\left(-\frac{\infty-\delta}{\varepsilon}\right)\right]^{\lambda} = 1-0 = 1 \tag{21}$$

Hence, $\lim_{x\to-\infty} G(x) = 0$ and $\lim_{x\to\infty} G(x) = 1$

The proof of the Theorems 3 and 4 indicate that the proposed distribution satisfied the property of the limiting property a probability density function and cumulative density function respectively.

Statistical Properties of the proposed ESLD

Moment Generating Function of the Proposed ESLD

Suppose that X is a random variable that follows ESLD with parameters $(\theta,\delta,\varepsilon,\lambda)$, the moment generation function is given as:

$$M_x(t) = \frac{\lambda}{\theta+\varepsilon}\exp(t\delta)\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\left(\frac{\varepsilon}{\theta+\varepsilon}\right)^j\left[\frac{\theta}{t\theta+1+j} - \frac{\varepsilon}{-t\varepsilon+1+j}\right] \tag{12}$$

Proof

By definition of the moment generating function:

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} g(x)dx \tag{22}$$

$$\frac{\lambda}{\theta+\varepsilon}\left[\begin{array}{l}\int_{-\infty}^{\delta}\exp(tx)\exp\left(\frac{x-\delta}{\theta}\right)\left[\frac{\theta}{\theta+\varepsilon}\exp\left(\frac{x-\delta}{\theta}\right)\right]^{\lambda-1}dx + \\ \int_{\delta}^{\infty}\exp(tx)\exp\left(-\frac{x-\delta}{\varepsilon}\right)\left[1-\frac{\varepsilon}{\theta+\varepsilon}\exp\left(-\frac{x-\delta}{\varepsilon}\right)\right]^{\lambda-1}dx\end{array}\right] \tag{23}$$

Let, $P_1 = \int_{-\infty}^{\delta}\frac{\lambda}{\theta+\varepsilon}\exp(tx)\exp\left(\frac{x-\delta}{\theta}\right)\left[\frac{\theta}{\theta+\varepsilon}\exp\left(\frac{x-\delta}{\theta}\right)\right]^{\lambda-1}dx$ and

$$P_2 = \int_{-\infty}^{\delta}\frac{\lambda}{\theta+\varepsilon}\exp(tx)\exp\left(-\frac{x-\delta}{\varepsilon}\right)\left[1-\frac{\varepsilon}{\theta+\varepsilon}\exp\left(-\frac{x-\delta}{\varepsilon}\right)\right]^{\lambda-1} \tag{24}$$

Let $w = x-\delta$, $\Rightarrow x = \delta + w$, if $x = -\infty$, $w = -\infty - \infty = -\infty$

$x = \delta$, $w = \delta - \delta = 0$

$$P_1 = \int_{-\infty}^{0}\exp(tw+t\delta)\exp\left(\frac{w}{\theta}\right)\left\{\frac{\theta}{\theta+\varepsilon}\exp\left(\frac{w}{\theta}\right)\right\}^{\lambda-1}dw \tag{25}$$

$$P_1 = \int_{-\infty}^{0}\exp(tw+t\delta)\exp\left(\frac{w}{\theta}\right)\left\{1-\left(1-\frac{\theta}{\theta+\varepsilon}\right)\exp\left(\frac{w}{\theta}\right)\right\}^{\lambda-1}dw \tag{26}$$

$$P_1 = \int_{-\infty}^{0}\exp(tw+t\delta)\exp\left(\frac{w}{\theta}\right)\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\left(1-\frac{\theta}{\theta+\varepsilon}\right)^j\exp\left[\frac{jw}{\theta}\right]dw \tag{27}$$

$$P_1=\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\left(1-\frac{\theta}{\theta+\varepsilon}\right)^j\int_{-\infty}^{0}\exp(tw+t\delta)\exp\left(\frac{w}{\theta}\right)\exp\left[\frac{jw}{\theta}\right]dw \tag{28}$$

$$P_1=\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\left(1-\frac{\theta}{\theta+\varepsilon}\right)^j\int_{-\infty}^{0}\exp\left(tw+t\delta+\frac{w}{\theta}+\frac{jw}{\theta}\right)dw \tag{29}$$

$$P_1=\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\left(1-\frac{\theta}{\theta+\varepsilon}\right)^j\exp(t\delta)\int_{-\infty}^{0}\exp\left(tw+\frac{w}{\theta}+\frac{jw}{\theta}\right)dw \tag{30}$$

$$P_1=\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\left(1-\frac{\theta}{\theta+\varepsilon}\right)^j\exp(t\delta)\int_{-\infty}^{0}\exp\left[w\left(t+\frac{1}{\theta}+\frac{j}{\theta}\right)\right]dw \tag{31}$$

$$P_1=\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\left(1-\frac{\theta}{\theta+\varepsilon}\right)^j\exp(t\delta)\frac{1}{t+\frac{1}{\theta}+\frac{j}{\theta}}\exp\left[w\left(t+\frac{1}{\theta}+\frac{j}{\theta}\right)\right]_{-\infty}^{0} \tag{32}$$

$$P_1=\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\left(1-\frac{\theta}{\theta+\varepsilon}\right)^j\frac{\exp(t\delta)}{\frac{t\theta+1+j}{\theta}}[1-0] \tag{33}$$

$$P_1=\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\left(1-\frac{\theta}{\theta+\varepsilon}\right)^j\frac{\exp(t\delta)}{\frac{t\theta+1+j}{\theta}} \tag{34}$$

Similarly,

$$P_2=\int_{-\infty}^{\delta}\frac{\lambda}{\theta+\varepsilon}\exp(tx)\exp\left(-\frac{x-\delta}{\varepsilon}\right)\left[1-\frac{\varepsilon}{\theta+\varepsilon}\exp\left(-\frac{x-\delta}{\varepsilon}\right)\right]^{\lambda-1} \tag{35}$$

When $x=\delta,$ since $w=x-\delta\ \ w=0$, When $x=\infty, u=\infty$

$$P_2=\int_{0}^{\infty}\exp(tw+t\delta)\exp\left(-\frac{w}{\varepsilon}\right)\left[1-\frac{\varepsilon}{\theta+\varepsilon}\exp\left(-\frac{w}{\varepsilon}\right)\right]^{\lambda-1}dw \tag{36}$$

$$P_2=\int_{0}^{\infty}\exp(tw+t\delta)\exp\left(-\frac{w}{\varepsilon}\right)\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\left(\frac{\varepsilon}{\theta+\varepsilon}\right)^j\exp\left(-\frac{jw}{\varepsilon}\right)dw \tag{37}$$

$$P_2=\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\left(\frac{\varepsilon}{\theta+\varepsilon}\right)^j\int_{0}^{\infty}\exp(tw+t\delta)\exp\left(-\frac{w}{\varepsilon}-\frac{jw}{\varepsilon}\right)dw \tag{38}$$

$$P_2=\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\left(\frac{\varepsilon}{\theta+\varepsilon}\right)^j\exp(t\delta)\int_{0}^{\infty}\exp\left(tw-\frac{w}{\varepsilon}-\frac{jw}{\varepsilon}\right)dw \tag{39}$$

$$P_2=\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\left(\frac{\varepsilon}{\theta+\varepsilon}\right)^j\exp(t\delta)\int_{0}^{\infty}\exp\left[w\left(t-\frac{1}{\varepsilon}-\frac{j}{\varepsilon}\right)\right]dw \tag{40}$$

$$P_2=\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\left(\frac{\varepsilon}{\theta+\varepsilon}\right)^j\exp(t\delta)\int_{0}^{\infty}\exp\left[-w\left(-t+\frac{1}{\varepsilon}+\frac{j}{\varepsilon}\right)\right]dw \tag{41}$$

$$P_2=\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\left(\frac{\varepsilon}{\theta+\varepsilon}\right)^j\exp(t\delta)\frac{1}{-t+\frac{1}{\varepsilon}+\frac{j}{\varepsilon}}\exp\left[-w\left(-t+\frac{1}{\varepsilon}+\frac{j}{\varepsilon}\right)\right]_0^{\infty} \tag{42}$$

$$P_2=\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\left(\frac{\varepsilon}{\theta+\varepsilon}\right)^j\frac{\exp(t\delta)\varepsilon}{-t\varepsilon+1+j}[0-1] \tag{43}$$

$$P_2=-\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\left(\frac{\varepsilon}{\theta+\varepsilon}\right)^j\frac{\varepsilon\exp(t\delta)}{-t\varepsilon+1+j} \tag{44}$$

$$M_x(t)=\frac{\lambda}{\theta+\varepsilon}(P_1+P_2) \tag{45}$$

$$M_x(t)=\frac{\lambda}{\theta+\varepsilon}\left[\begin{array}{l}\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\left(1-\frac{\theta}{\theta+\varepsilon}\right)^j\frac{\theta\exp(t\delta)}{t\theta+1+j}\\-\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\left(\frac{\varepsilon}{\theta+\varepsilon}\right)^j\frac{\varepsilon\exp(t\delta)}{-t\varepsilon+1+j}\end{array}\right] \tag{46}$$

Note that, $1-\frac{\theta}{\theta+\varepsilon}=\frac{\theta+\varepsilon-\theta}{\theta+\varepsilon}=\frac{\varepsilon}{\theta+\varepsilon}$ (47)

$$M_{z_t}(t)=\frac{\lambda}{\theta+\varepsilon}\exp(t\delta)\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\left(\frac{\varepsilon}{\theta+\varepsilon}\right)^j\left[\frac{\theta}{t\theta+1+j}-\frac{\varepsilon}{-t\varepsilon+1+j}\right] \tag{48}$$

Characteristic Function of the Proposed ESLD

Suppose that X is a random variable that follows ESLD with parameters $(\theta,\delta,\varepsilon,a)$, the characteristic function, $\varphi_x(t)$ is given by:

$$\varphi_x(t)=\frac{\lambda}{\theta+\varepsilon}\exp(it\delta)\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\left(\frac{\varepsilon}{\theta+\varepsilon}\right)^j\left[\frac{\theta}{it\theta+1+j}-\frac{\varepsilon}{-it\varepsilon+1+j}\right] \tag{49}$$

Proof

By definition,

$$\varphi_x(t)=E\left[\exp(itx)\right]=\int_{-\infty}^{\infty}\exp(itx)g(x)dx \tag{50}$$

Since the mgf of the ESLD has been obtained and presented in (40), then, it is easy to obtain its characteristic function and it is given by:

$$\varphi_x(t)=\frac{\lambda}{\theta+\varepsilon}\exp(it\delta)\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j\left(\frac{\varepsilon}{\theta+\varepsilon}\right)^j\left[\frac{\theta}{it\theta+1+j}-\frac{\varepsilon}{-it\varepsilon+1+j}\right] \tag{51}$$

Hazard Function of ESLD

The Hazard function by definition is given as:

$$H(x)=\frac{g(x)}{1-G(x)} \tag{52}$$

Where, H(x) is the hazard function while g(x) and G(x) are the probability density function and cumulative density function respectively.
Hence, the Hazard function of the proposed ESLD can be expressed as:

$$H(x)=\begin{cases}\dfrac{\dfrac{\lambda}{\theta+\varepsilon}\exp\left(\dfrac{x-\delta}{\theta}\right)\left[\dfrac{\theta}{\theta+\varepsilon}\exp\left(\dfrac{x-\delta}{\theta}\right)\right]^{\lambda-1}}{\left[\dfrac{\theta}{\theta+\varepsilon}\exp\left(\dfrac{x-\delta}{\theta}\right)\right]^{\lambda-1}}, x<\delta \\ \dfrac{\dfrac{\lambda}{\theta+\varepsilon}\exp\left(-\dfrac{x-\delta}{\varepsilon}\right)\left[1-\dfrac{\varepsilon}{\theta+\varepsilon}\exp\left(-\dfrac{x-\delta}{\varepsilon}\right)\right]^{\lambda-1}}{\left[1-\dfrac{\varepsilon}{\theta+\varepsilon}\exp\left(-\dfrac{x-\delta}{\varepsilon}\right)\right]^{\lambda-1}}, x\geq\delta\end{cases} \tag{53}$$

Reliability Function

The reliability function of the ESLD is given by:

$$R(x)=1-G(x) \tag{54}$$

Hence,

$$R(x)=1-\begin{cases}\left[\dfrac{\theta}{\theta+\varepsilon}\exp\left(\dfrac{x-\delta}{\theta}\right)\right]^{\lambda-1}, x<\delta \\ \left[1-\dfrac{\varepsilon}{\theta+\varepsilon}\exp\left(-\dfrac{x-\delta}{\varepsilon}\right)\right]^{\lambda-1}, x\geq\delta\end{cases} \tag{55}$$

Parameter Estimation for the ESLD.

The estimation of the parameter of the ESLD defined in (5) will be estimated using the Method of Maximum Likelihood derived below:
Let X_1, X_2,..., X_n be a random sample of size *n* from Exponentiated Skew Laplace Distribution (ESLD). Then the likelihood function is given by:
For, $x<\delta$

$$L(x)=\prod_{i=1}^{n}\left(\frac{\lambda}{\theta+\varepsilon}\right)\exp\left(\frac{x_i-\delta}{\theta}\right)\left\{\frac{\theta}{\theta+\varepsilon}\exp\left(\frac{x_i-\delta}{\theta}\right)\right\}^{\lambda-1} \tag{56}$$

$$L(x)=\left(\frac{\lambda}{\theta+\varepsilon}\right)^{n}\prod_{i=1}^{n}\exp\left(\frac{x_i-\delta}{\theta}\right)\left\{\frac{\theta}{\theta+\varepsilon}\exp\left(\frac{x_i-\delta}{\theta}\right)\right\}^{\lambda-1} \tag{57}$$

Taking the natural log of both sides.

$$\log L(x)=n\log\left(\frac{\lambda}{\theta+\varepsilon}\right)+(\lambda-1)\log\left[\prod_{i=1}^{n}\exp\left(\frac{x_i-\delta}{\theta}\right)\left\{\frac{\theta}{\theta+\varepsilon}\exp\left(\frac{x_i-\delta}{\theta}\right)\right\}\right] \tag{58}$$

$$\log L(x)=n\log\left(\frac{\lambda}{\theta+\varepsilon}\right)+(\lambda-1)\left[\log\prod_{i=1}^{n}\frac{\theta}{\theta+\varepsilon}\exp\left(\frac{2x_i-2\delta}{\theta}\right)\right] \tag{59}$$

$$\log L(x) = n\log\left(\frac{\lambda}{\theta+\varepsilon}\right) + (\lambda-1)\log\left[\left(\frac{\theta}{\theta+\varepsilon}\right)^n \exp\left(\frac{2\sum_{i=1}^{n}(x_i-\delta)}{\theta}\right)\right] \tag{60}$$

$$\log L(x) = n\log\left(\frac{\lambda}{\theta+\varepsilon}\right) + (\lambda-1)\left\{n\log\left(\frac{\theta}{\theta+\varepsilon}\right) + \frac{2\sum_{i=1}^{n}(x_i-\delta)}{\theta}\right\} \tag{61}$$

$x \geq \delta$

$$L(x) = \prod_{i=1}^{n}\left(\frac{\lambda}{\theta+\varepsilon}\right)\exp\left(-\frac{x_i-\delta}{\varepsilon}\right)\left\{1-\frac{\varepsilon}{\theta+\varepsilon}\exp\left(-\frac{x_i-\delta}{\varepsilon}\right)\right\}^{\lambda-1} \tag{62}$$

$$L(x) = \left(\frac{\lambda}{\theta+\varepsilon}\right)^n \prod_{i=1}^{n}\exp\left(-\frac{x_i-\delta}{\varepsilon}\right)\left\{1-\frac{\varepsilon}{\theta+\varepsilon}\exp\left(-\frac{x_i-\delta}{\varepsilon}\right)\right\}^{\lambda-1} \tag{63}$$

Taking the natural log of both sides.

$$\log L(x) = n\log\left(\frac{\lambda}{\theta+\varepsilon}\right) + (\lambda-1)\log\left[\prod_{i=1}^{n}\exp\left(-\frac{x_i-\delta}{\varepsilon}\right)\left\{1-\frac{\varepsilon}{\theta+\varepsilon}\exp\left(-\frac{x_i-\delta}{\varepsilon}\right)\right\}\right] \tag{64}$$

$$\log L(x) = n\log\left(\frac{\lambda}{\theta+\varepsilon}\right) + (\lambda-1)\log\left[e^{-\frac{1}{\varepsilon}\sum_{i=1}^{n}(x_i-\delta)} - \left(\frac{\varepsilon}{\theta+\varepsilon}\right)^n e^{\frac{-2}{\varepsilon}\sum_{i=1}^{n}(x_i-\delta)}\right] \tag{65}$$

$$\log L(x) = n\log\left(\frac{\lambda}{\theta+\varepsilon}\right) + (\lambda-1)\left[-\frac{1}{\varepsilon}\sum_{i=1}^{n}(x_i-\delta) - n\log\left(\frac{\varepsilon}{\theta+\varepsilon}\right) - \frac{2}{\varepsilon}\sum_{i=1}^{n}(x_i-\delta)\right] \tag{66}$$

Hence, the Log Likelihood of the proposed Exponentiated Skew Laplace Distribution (ESLD) is given as:

$$\log L(x) = \begin{cases} n\log\left(\frac{\lambda}{\theta+\varepsilon}\right) + (\lambda-1)\left[n\log\left(\frac{\theta}{\theta+\varepsilon}\right) + \frac{2}{\theta}\sum_{i=1}^{n}(x_i-\delta)\right], x < \delta \\ n\log\left(\frac{\lambda}{\theta+\varepsilon}\right) + (\lambda-1)\left[-\frac{1}{\varepsilon}\sum_{i=1}^{n}(x_i-\delta) - n\log\left(\frac{\varepsilon}{\theta+\varepsilon}\right) - \frac{2}{\varepsilon}\sum_{i=1}^{n}(x_i-\delta)\right], x \geq \delta \end{cases} \tag{67}$$

Differentiating log L(x) in (57) with respect to each of the parameter in the model and setting them to zero give the estimate of each of the parameter. This was done in R software using the appropriate optimization function.

III. Results

Applications to Real Data

The empirical application of the proposed ESLD was carried out using two sets of real life data. The first data is the daily returns of S& P 500 between 2/02/2024 and 28/03/2024 as obtained from Yahoo finance. The second data set is on the daily returns of Bitcoin between 2/02/2024 and 01/04/2024 which was also obtained from the Yahoo finance website (www.yahoofinance. com). Daily closing returns series were computed from daily closing prices using the formula below:

$$DCR_t = \log\left(\frac{DCP_t}{DCP_{t-1}}\right)\times 100 \quad (68)$$

Where, DCRt is the closing returns at the day t while DCP_t and DCP_{t-1} are the closing prices at the present day and previous day respectively.
The fitness performance of the proposed ESLD was compared with that of similar distributions such as the skew Laplace distribution and Laplace distribution using Akaike Information Criteria (AIC) and Bayesian Information Criteria (BIC) as defined below:

$$AIC = 2k - 2\ln(LL) \quad (69)$$

$$BIC = -2LL + k\log(n) \quad (70)$$

Where, n is the number of observations, k is the number of parameters and LL is the log-likelihood.

Data set 1

0.46	-0.14	0.10	0.36	0.02	0.25	-0.04	-0.60	0.41	0.25
-0.21	-0.26	0.05	0.91	0.02	-0.16	0.07	-0.07	0.23	0.35
-0.05	-0.44	0.22	0.45	-0.28	-0.05	0.48	-0.08	-0.12	-0.28
0.27	0.24	0.38	0.14	-0.06	-0.13	-0.12	0.37	0.05	

Data set 2

0.11	-0.20	-0.42	0.08	0.43	1.23	0.95	1.73	0.57	0.47
1.47	-0.19	1.78	0.09	0.19	-0.42	0.39	-0.29	0.42	-0.37
-0.45	-0.49	0.71	0.14	2.28	2.00	3.94	-0.92	0.87	-0.29
0.79	3.41	-2.98	1.54	0.54	0.88	0.13	0.33	1.91	-0.39
0.96	-1.01	-1.23	-2.64	2.00	-0.54	-3.78	4.02	-1.58	-1.15
0.19	2.10	1.73	0.02	-0.33	0.80	-0.53	-0.15	1.04	-1.01

Table 1: *Summary statistics for the data set*

Data set	n	Min.	Max.	Mean	SD	Skewness
Data set 1 (S&P 500)	39	-0.60	0.91	.0763	.29451	.268
Data set II (Bitcoin)	60	-3.78	4.02	.3484	1.42987	.013

SD- Standard deviation.

Table 2: *Results of the estimated parameters as well as the goodness fit for the proposed distribution and other similar distribution*

Dataset	Distributions	δ	θ	ε	λ	LL	AIC	BIC
I	ESLD	-0.060	0.134	0.368	1.052	**-5.431**	**18.862**	**17.975**
	SLD	-0.061	0.102	0.348	-	-7.938	21.876	20.649
	LD	*0.048*	*0.233*	-	-	-9.247	22.494	21.677
II	ESLD	0.103	0.431	0.462	1.203	**-98.346**	204.691	**203.804**
	SLD	0.111	0.505	0.442	-	-103.041	212.082	211.417
	LD	0.189	1.034	-		-103.620	211.240	210.796

ESLD- Exponentiated skew Laplace Distribution, SLD- Skew Laplace Distribution, LD- Laplace Distribution, SGED- Skew Generalized Error Distribution.

Result presented in Table 3 show the parameter estimates of the proposed ESLD and compared to other related probability distribution (Skew Laplace distribution and Laplace distribution) as well as their fitness performance. Result of the LogLikelihood shows that among these distributions, the proposed ESLD reported the highest LogLikelihood in the two real data compared with other competing distributions. Similarly, the AIC and BIC reported by the proposed ESLD is lower than that of Skew Laplace distribution and Laplace distribution for both the two data set. These result show better fitness performance of the proposed ESLD than both Skew Laplace distribution and Laplace distribution in modelling financial data.

IV. Discussion

This study improve on the robustness of the Skew Laplace distribution introduced by Safavinejad *et al*. [4] by introducing additional shape parameter using the method of exponentiation. We derived some of the statistical properties of the proposed Skew Laplace Distribution (ESLD) after ensuring that the proposed ESLD satisfied the properties of a statistical distribution. Some of the statistical properties of the proposed ESLD derived include: moment generating function, characteristic function, hazard function and reliability function. The estimation of its parameters was carried out using the maximum likelihood. The performance of the proposed ESLD compared with other similar distributions were demonstrated empirically with returns from S & P 500 (Dataset 1) and returns from Bitcoin (Dataset 2). The findings suggest that the new distribution outperforms the existing models considered, indicating its better representation and flexibility when compared with some existing models. This findings show that the ELSD outperformed Skew Laplace Distribution (SLD) which indicates that the use of the method of exponentiation improve the performance of distribution. This is corroborated by that of other studies, Agboola *et al*., [10], Oguntunde *et al.,* [11], Nadarajah and Bakar [12], Datta and Datta [13], Andrade *et al*., [14] and Adubisi *et al.,* [15] which also found that exponentiated distribution performed better than their parent distributions.

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