ON MODELING OF BIOMEDICAL DATA WITH EXPONENTIATED GOMPERTZ INVERSE RAYLEIGH DISTRIBUTION

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Abstract

This paper introduces and thoroughly examines the Exponentiated Gompertz Inverse Rayleigh (EtGoIr) Distribution, a four-parameter extension of the Gompertz Inverse Rayleigh distribution. The primary focus is on its application to biomedical datasets, shedding light on its mathematical and statistical properties. Some properties of the distribution that were derived include the quantile function, median, moments, incomplete moments, Rényi entropy, and probability weighted moments. The model parameters were estimated using the method of maximum likelihood. A simulation study was conducted to investigate the consistency of the proposed model. The outcome of the investigation revealed that the model demonstrates consistency, as evidenced by the reduction in both root mean square error (RMSE) and bias as sample sizes increase. To showcase the practical relevance of the EtGoIr distribution, the paper applies the model to three distinct biomedical datasets. The results highlight its enhanced flexibility, demonstrating superior fit compared to its counterpart.

Keywords: Exponentiated G, MLE, Moment, Renyi Enropy, Biomedical

I. Introduction

Statistical theory continually evolves to meet the demands of modeling complex natural phenomena effectively. Traditional probability distributions have long served as foundational tools, yet the complexities of modern biomedical datasets often necessitate the development of novel models to extract deeper insights. This necessity is particularly pronounced in biomedical research, where conventional distributions struggle to capture the intricacies of physiological measurements, disease outcomes, and survival times across various medical conditions. Recent advancements in distribution theory have underscored the importance of innovative models for accommodating the skewness prevalent in the aforementioned datasets. This skewness poses a significant challenge to conventional distributions, prompting researchers to explore extensions of established models to better capture these complexities. Notable among these extensions are the works of [1] – [9].

In this study, we focus on extending the Gompertz inverse Rayleigh (GoIR) distribution, introduced by [10], to create a more adaptable model. We investigate the exponentiated (Et) family of distributions, as proposed by [11], to achieve this extension. By combining the GoIR distribution with the Et family, our aim is to develop a versatile model capable of accurately fitting real-world datasets, particularly in biomedical science applications.
The cumulative distribution function (cdf) and probability density function (pdf) of the Et family are given respectively as:

$$F(x) = [G(x)]^{\theta} \;; \tag{1}$$

$$f(x) = \theta g(x)[G(x)]^{\theta-1} \quad : \theta > 0 \tag{2}$$

where $G(x)$and $g(x)$ are the cdf and pdf of the baseline distribution.
The cdf and pdf of GoIR distribution taken as baseline are given as:

$$G(x) = 1 - e^{\left(\frac{\beta}{\sigma}\right)\left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right\}} \tag{3}$$

and

$$g(x) = 2\beta\gamma^2 x^{-3} e^{-\left(\frac{\gamma}{x}\right)^2}\left[1-e^{-\left(\frac{\gamma}{x}\right)^2}\right]^{-\sigma-1} e^{\left(\frac{\beta}{\sigma}\right)\left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right\}} \;; \theta > 0, \beta > 0, \gamma > 0, \sigma > 0 \tag{4}$$

The motivation for this research arises from the recognition that traditional distributions often fall short in accommodating the complexities of biomedical datasets, especially those exhibiting skewness. By extending the GoIR distribution, we seek to contribute to the development of hybrid distributions that better reflect the intricacies of real-world data.

II. Methods

2.1 Derivation of Exponentiated Gompertz Inverse Rayleigh (EtGoIR) Distribution

This section introduces a new model called the EtGoIR distribution. The cdf of the EtGoIR distribution is derived by substituting equation (3) into equation (1), as follows:

$$F(x) = \left\{1 - e^{\left(\frac{\beta}{\sigma}\right)\left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right\}}\right\}^{\theta} \tag{5}$$

On differentiating equation (5) with respect to x, we obtain pdf of EtGoIR distribution given as:

$$f(x) = 2\theta\beta\gamma^2 x^{-3} e^{-\left(\frac{\gamma}{x}\right)^2}\left[1-e^{-\left(\frac{\gamma}{x}\right)^2}\right]^{-\sigma-1} e^{\left(\frac{\beta}{\sigma}\right)\left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right\}} \left\{1 - e^{\left(\frac{\beta}{\sigma}\right)\left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right\}}\right\}^{\theta-1} \tag{6}$$

The pdf plot of the EtGoIR distribution is given in Figure 1 below.

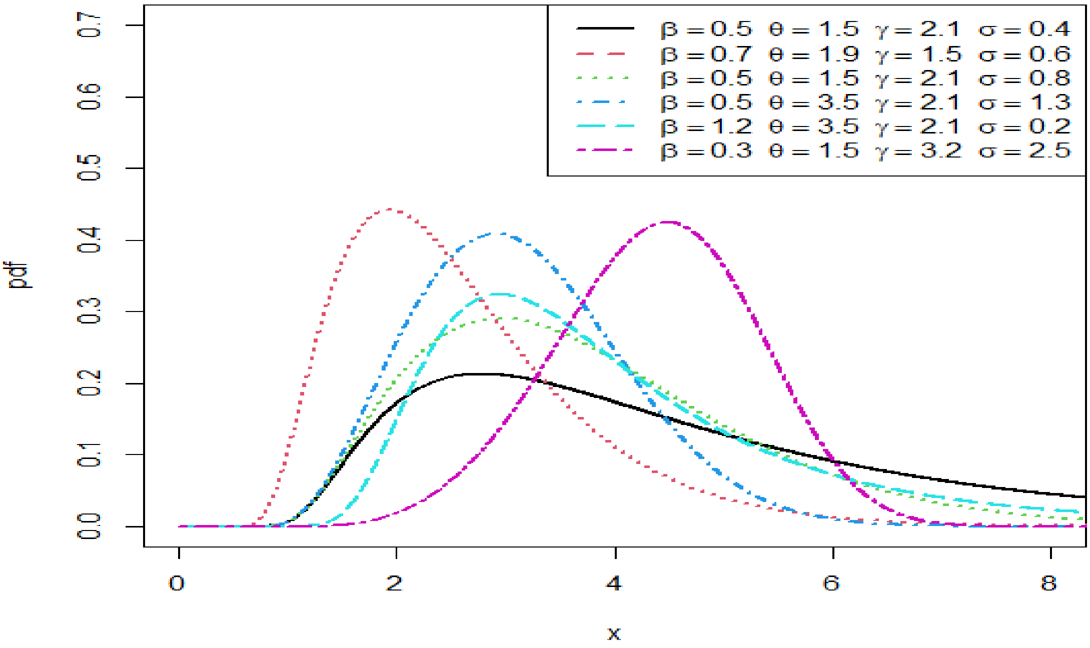

Figure 1: *pdf plot of EtGoIR distribution*

2.2 Expansion of Density

Using the generalized binomial expansion given as

$$(1-y)^{\rho-1} = \sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(\rho)}{i!\,\Gamma(\rho-i)}y^i \tag{7}$$

Applying equation (7) on the last term in equation (6), we have

$$\left[1 - e^{\left(\frac{\beta}{\sigma}\right)\left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right\}}\right]^{\theta-1} = \sum_{i=0}^{\infty}\frac{(-1)^i\Gamma(\rho)}{i!\,\Gamma(\rho-i)}\left[e^{\left(\frac{\beta}{\sigma}\right)\left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right\}}\right]^{i}$$

Where

$$e^{\{-vy^n\}} = \sum_{j=0}^{\infty}\frac{(-1)^j v^j}{j!}y^{vj} \tag{8}$$

and

$$e^{\{-vy^n\}} = \sum_{j=0}^{\infty}\frac{v^j}{j!}y^{vj} \tag{9}$$

Therefore,

$$e^{\left\{\frac{\beta}{\sigma}y^i\right\}} = \sum_{j=0}^{\infty}\frac{\left(\frac{\beta}{\sigma}\right)^j}{j!}y^j$$

where

$$y^j = \left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right\}^j = \sum_{k=0}^{\infty}\binom{j}{k}(-1)^k\left[1-e^{-\left(\frac{\gamma}{x}\right)^2}\right]^{-\sigma k}$$

Substituting back all the expansions into equation (6), we have

$$f(x) = 2\beta\theta\gamma^2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{i+k+l}\left(\frac{\beta}{\sigma}\right)^j \binom{j+1}{k} \Gamma\big(-\sigma(K+1)\big)\Gamma(\theta)}{i!\,j!\,l!\,\Gamma(-\sigma(K+1)-m)\Gamma(\theta-i)} x^{-3} \tag{10}$$

$$f(x) = \psi x^{-3}\left[e^{-\left(\frac{\gamma}{x}\right)^2}\right]^{m+1}$$

where

$$\psi = 2\beta\theta\gamma^2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{i+k+l}\left(\frac{\beta}{\sigma}\right)^j \binom{j+1}{k} \Gamma\big(-\sigma(K+1)\big)\Gamma(\theta)}{i!\,j!\,l!\,\Gamma(-\sigma(K+1)-m)\Gamma(\theta-i)}$$

2.3 Properties of EtGoIR Distribution

This section derives some statistical properties of the EGILx distribution including moments, survival function, hazard function, quantile functions, and order statistics.

2.3.1 Quantile function

The quantile function is the inverse of the cdf of a distribution and is used in simulation studies. It is also applied as a measure of the spread of a distribution. The quantile function is obtained using:

$$Q(u) = F^{-1}(u) \tag{11}$$

Applying equation (13) to the cdf of the new model, we have the quantile function given as

$$x = \gamma\left\{-log\left[1-\left(1-\frac{\sigma log\left(1-u^{\frac{1}{\theta}}\right)}{\beta}\right)^{-\left(\frac{1}{\sigma}\right)}\right]\right\}^{-\left(\frac{1}{2}\right)} \tag{12}$$

2.3.2 Median

Median of EtGoIR distribution is obtained by setting u =0.5 in equation (12) and it is given as

$$x_{median} = \gamma\left\{-log\left[1-\left(1-\frac{\sigma log\left(1-0.5^{\frac{1}{\theta}}\right)}{\beta}\right)^{-\left(\frac{1}{\sigma}\right)}\right]\right\}^{-\left(\frac{1}{2}\right)} \tag{13}$$

2.3.3 Moments

$$E(X^r) = \int_0^{\infty} x^r f(y)dx \tag{14}$$

$$E(X^r) = \psi\int_0^{\infty} x^{r-3}\left[e^{-\left(\frac{\gamma}{x}\right)^2}\right]dx \tag{15}$$

On solving the integral part in equation (15), we have

$$E(X^r) = \psi\left[\frac{\gamma^r\Gamma\left(1-\frac{r}{2}\right)}{(k+1)^{1-\frac{r}{2}}}\right] \tag{16}$$

When r=1 in equation (16), we have mean of EtGoIR distribution

2.3.4 Incomplete Moments

The r^{th} (r > 0) incomplete moments for the EtGoIR distributions follow from equation (10) as

$$\dot{\mu}_r(u) = \int_0^u \psi x^{r-3} e^{-(m+1)\left(\frac{\gamma}{x}\right)^2} dx$$

Let $t = (m+1)\left(\frac{\gamma}{x}\right)^2 \Rightarrow x = \left(\frac{(m+1)\gamma^2}{t}\right)^{\frac{1}{2}}$

When $x = 0 \Rightarrow \mathrm{t} = 0$, $and\ if\ x = u \Rightarrow t = (m+1)\left(\frac{\gamma}{u}\right)^2$

Then

$$\dot{\mu}_r(u) = \frac{\psi}{2\left((m+1)\gamma^2\right)^{1-\frac{r}{2}}} \gamma\left(1-\frac{r}{2}, (m+1)\gamma^2\right) \tag{17}$$

2.3.5 Rényi Entropy

Define the Rényi entropy of the EtGoIR distributions with the following formula [12]

$$T_R(\eta) = \frac{1}{1-\eta} log \int_0^\infty f^{\eta}(x)dx \quad , \quad \eta > 0, \eta \neq 1$$

By equation (6) we find $f(x)^{\eta}$:

$$f(x)^{\eta} = 2\theta^{\eta}\beta^{\eta}\gamma^{2\eta}x^{-3\eta} e^{-\left(\eta\left(\frac{\gamma}{x}\right)^2\right)} \left(1-e^{-\left(\left(\frac{\gamma}{x}\right)^2\right)}\right)^{-(\sigma+1)\eta} e^{\eta\frac{\beta}{\sigma}\left(1-\left(1-e^{-\left(\left(\frac{\gamma}{x}\right)^2\right)}\right)^{-\sigma}\right)} \left\{1-e^{\frac{\beta}{\sigma}\left(1-\left(1-e^{-\left(\left(\frac{\gamma}{x}\right)^2\right)}\right)^{-\sigma}\right)}\right\}^{\eta(\theta-1)}$$

By generalized binomial series and exponential expansion, we get

$$\left(1-e^{\frac{\beta}{\sigma}\left(1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right)}\right)^{\eta(\theta-1)} = \sum_{i=0}^{\infty} \frac{\Gamma(\eta(\theta-1)+i)}{i!\,\Gamma\eta(\theta-1))} e^{\frac{\beta i}{\sigma}\left(1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right)}$$

And

$$e^{\frac{\beta}{\sigma}\left(1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right)(\eta+i)} = \sum_{z=0}^{\infty} \frac{j^z\beta^z(\eta+i)^z}{z!\,\sigma^z}\left(1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right)^z$$

Then

$$f(x)^{\eta} = \sum_{i=z=0}^{\infty} \frac{\Gamma(\eta(\theta-1)+i)j^z\beta^z(\eta+i)^z}{i!\,\Gamma\eta(\theta-1))z!\,\sigma^z}\left(1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right)^z$$

Again using a generalized binomial, we get

$$f(x)^{\eta} = ₩x^{-3\eta}e^{-\left(\frac{\gamma}{x}\right)^2(\eta+q)} \tag{18}$$

Where $₩ = \sum_{i=0}^{\infty} \frac{2\theta^{\eta}\beta^{\eta}\gamma^{2\eta}\Gamma(\eta(\theta-1)+i)j^z\beta^z(\eta+i)^z(-1)^p\Gamma\left((\sigma+1)\eta+\sigma p+q\right)}{i!\,\Gamma\eta(\theta-1))z!\,\sigma^z q!\,\Gamma((\sigma+1)\eta+\sigma p)}\binom{z}{p}$

By substituting equation (18) into the equation above, we get:

$$T_R(\eta) = \frac{1}{1-\eta} log \int_0^\infty ₩x^{-3\eta}e^{-\left(\frac{\gamma}{x}\right)^2(\eta+q)}dx$$

The last integral, we get

$$T_R(\eta)=\frac{1}{1-\eta}\log\left(\frac{\text{₩}\Gamma\left(\frac{3}{2}(\eta-1)+1\right)}{2\left((\eta+q\,)\gamma^2\right)^{\frac{1}{2}(3\eta+1)}}\right) \tag{19}$$

2.3.6 Probability Weighted Moments

The probabilistic weighted moment $(\varphi,\eta)^{th}$ for EtGoIR distributions can be expressed as follows:

$$\rho_{(\varphi,\eta)}=E\left(X^{\varphi}\left(F^{\eta}(X)\right)\right)=\int_{-\infty}^{\infty}x^{\varphi}\,F^{\eta}(x)f(x)dx$$

By equation (5), we can find $F^{\eta}(x)$:

$$F(x)^{\eta}=\left(1-e^{\frac{\beta}{\sigma}\left(1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right)}\right)^{\theta\eta}$$

By generalized binomial series:

$$\left(1-e^{\frac{\beta}{\sigma}\left(1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right)}\right)^{\theta\eta}=\sum_{j=0}^{\infty}(-1)^j\binom{\theta\eta}{j}e^{\frac{j\beta}{\sigma}\left(1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right)}$$

And using exponential expansion

$$e^{\frac{j\beta}{\sigma}\left(1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right)}=\sum_{r=0}^{\infty}\frac{j^r\beta^r}{r!\,\sigma^r}\left(1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right)^r$$

Then

$$F(x)^{\eta}=\sum_{j=r=0}^{\infty}\frac{(-1)^j j^r\beta^r}{r!\,\sigma^r}\binom{\theta\eta}{j}\left(1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right)^r$$

And using generalized binomial

$$\left(1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right)^r=\sum_{s=0}^{\infty}(-1)^s\binom{r}{s}\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-r\sigma}$$

And $\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-r\sigma}=\sum_{w=0}^{\infty}\frac{\Gamma(r\sigma+w)}{w!\,\Gamma(r\sigma)}e^{-\left(\frac{\gamma}{x}\right)^2 w}$

Then

$$F(x)^{\eta}=\text{Ꝁ}e^{-\left(\frac{\gamma}{x}\right)^2 w} \tag{20}$$

Where $\text{Ꝁ}=\sum_{j=r=s=w=0}^{\infty}\frac{(-1)^{j+s}j^r\beta^r\Gamma(r\sigma+w)}{r!\,\sigma^r w!\,\Gamma(r\sigma)}\binom{\theta\eta}{j}\binom{r}{s}$

By substituting equation (20) into the equation above, we get:

$$\rho_{(\varphi,\eta)}=\text{Ꝁ}\psi\int_{-\infty}^{\infty}x^{\varphi-3}\,e^{-(w+m+1\,)\left(\frac{\gamma}{x}\right)^2}dx$$

Let $u=(w+m+1\,)\left(\frac{\gamma}{x}\right)^2$ then

$$\rho_{(\varphi,\eta)}=\frac{\text{Ꝁ}\psi\Gamma\left(1-\frac{\varphi}{2}\right)}{2\left((w+m+1\,)\gamma^2\right)^{1-\frac{\varphi}{2}}} \tag{21}$$

2.3.7 Survival function

$$S(x) = 1 - F(x) \tag{22}$$

$$S(x) = 1 - \left(1 - e^{\left(\frac{\beta}{\sigma}\right)\left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right\}}\right)^{\theta} \tag{23}$$

2.3.8 Hazard function

$$H(x) = \frac{f(x)}{S(x)} \tag{24}$$

$$H(x) = \frac{2\theta\beta\gamma^2 x^{-3} e^{-\left(\frac{\gamma}{x}\right)^2}\left[1-e^{-\left(\frac{\gamma}{x}\right)^2}\right]^{-\sigma-1} e^{\left(\frac{\beta}{\sigma}\right)\left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right\}}\left(1-e^{\left(\frac{\beta}{\sigma}\right)\left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right\}}\right)^{\theta-1}}{1-\left(1-e^{\left(\frac{\beta}{\sigma}\right)\left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right\}}\right)^{\theta}} \tag{25}$$

The hazard plot of the EtGoIR distribution is given in Figure 2 below.

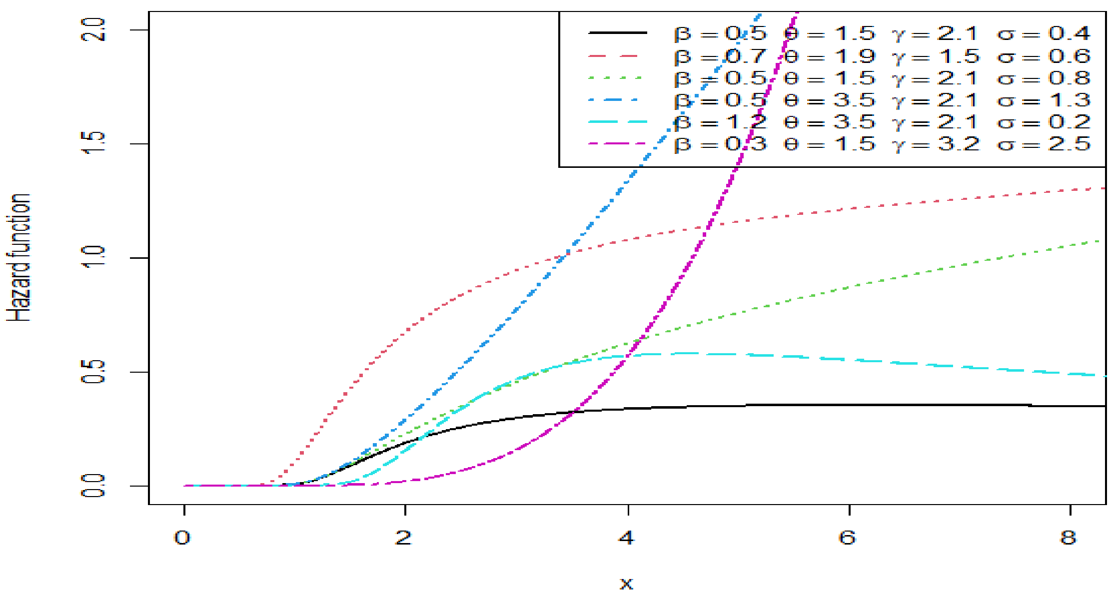

Figure 2: *plot of hazard function of EtGoIR distribution*

2.3.9 Cumulative hazard function

$$C(x) = -\log[S(x)] \tag{26}$$

$$C(x) = -log\left[1 - \left(1 - e^{\left(\frac{\beta}{\sigma}\right)\left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^2}\right)^{-\sigma}\right\}}\right)^{\theta}\right] \tag{27}$$

2.3.10 Reverse hazard function

$$R(x) = \frac{f(x)}{F(x)} \tag{28}$$

$$R(x) = \frac{2\theta\beta\gamma^{2}x^{-3}e^{-\left(\frac{\gamma}{x}\right)^{2}}\left[1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right]^{-\sigma-1} e^{\left(\frac{\beta}{\sigma}\right)\left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right)^{-\sigma}\right\}}\left\{1-e^{\left(\frac{\beta}{\sigma}\right)\left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right)^{-\sigma}\right\}}\right\}^{\theta-1}}{\left\{1-e^{\left(\frac{\beta}{\sigma}\right)\left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right)^{-\sigma}\right\}}\right\}^{\theta}} \tag{29}$$

2.3.11 Order Statistics

The pdf of the r^{th} order statistics of $X_{r:n}$ is given as:

$$f_{r:n}(x) = \frac{1}{B(r,n-r+1)}\sum_{i=0}^{n-r}(-1)^{i}[F(x)]^{r+i-1}f(x) \tag{30}$$

Inserting equation (5) and equation (6) into equation (30), we have

$$f_{r:n}(x) = \frac{1}{B(r,n-r+1)}\sum_{i=0}^{n-r}(-1)^{i}\left[\left\{1-e^{\left(\frac{\beta}{\sigma}\right)\left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right)^{-\sigma}\right\}}\right\}^{\theta}\right]^{r+i-1} 2\theta\beta\gamma^{2}x^{-3}e^{-\left(\frac{\gamma}{x}\right)^{2}}\left[1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right]^{-\sigma-1} e^{\left(\frac{\beta}{\sigma}\right)\left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right)^{-\sigma}\right\}}\left\{1-e^{\left(\frac{\beta}{\sigma}\right)\left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right)^{-\sigma}\right\}}\right\}^{\theta-1}$$

On bringing the like terms together, we have

$$f_{r:n}(x) = \frac{2\theta\beta\gamma^{2}}{B(r,n-r+1)}\sum_{i=0}^{n-r}(-1)^{i}x^{-3}e^{-\left(\frac{\gamma}{x}\right)^{2}}\left[1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right]^{-\sigma-1} e^{\frac{\beta}{\sigma}\left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right)^{-\sigma}\right\}}\left\{1-e^{\frac{\beta}{\sigma}\left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right)^{-\sigma}\right\}}\right\}^{(\theta(r+i)-1)} \tag{31}$$

Using the generalized binomial expansion on the last term in equation (31), we have

$$\left\{1-e^{\left(\frac{\beta}{\sigma}\right)\left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right)^{-\sigma}\right\}}\right\}^{\theta(r+i)-1} = \sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(\theta(r+i))}{j!\,\Gamma(\theta(r+i)-j)}\left[e^{\left(\frac{\beta}{\sigma}\right)\left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right)^{-\sigma}\right\}}\right]^{j}$$

Substituting back into equation (31), we have

$$f_{r:n}(x) = \frac{2\theta\beta\gamma^{2}}{B(r,n-r+1)}\sum_{i=0}^{n-r}\sum_{j=0}^{\infty}\frac{(-1)^{i+1}\Gamma(\theta(r+i))}{j!\,\Gamma(\theta(r+i)-j)}x^{-3}e^{-\left(\frac{\gamma}{x}\right)^{2}}\left[1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right]^{-\sigma-1}\left[e^{\left(\frac{\beta}{\sigma}\right)\left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right)^{-\sigma}\right\}}\right]^{j+1} \tag{32}$$

Also, expanding the last term in equation (32), we have

$$\left[e^{\left(\frac{\beta}{\sigma}\right)\left\{1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right)^{-\sigma}\right\}}\right]^{j+1} = \sum_{k=0}^{\infty}(-1)^{k}\binom{j+1}{k}1-\left(1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right)^{-\sigma k}$$

$$f_{r:n}(x) = \frac{2\theta\beta\gamma^2}{B(r,n-r+1)} \sum_{i=0}^{n-r}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{i+j+1}\binom{j+1}{k}\Gamma\big(\theta(r+i)\big)}{j!\,\Gamma(\theta(r+i)-j)} x^{-3} e^{-\left(\frac{\gamma}{x}\right)^2} \left[1-e^{-\left(\frac{\gamma}{x}\right)^2}\right]^{-\sigma(k+1)-1} \quad (33)$$

$$\left[1-e^{-\left(\frac{\gamma}{x}\right)^2}\right]^{-\sigma(k+1)-1} = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma\big(-\sigma(k+1)\big)}{l!\,\Gamma(-\sigma(k+1)-l)} \left[e^{-\left(\frac{\gamma}{x}\right)^2}\right]^i$$

Putting all the expansions together, we have the r^{th} order statistics of EtGoIR distribution given as:

$$f_{r:n}(x) = \frac{2\theta\beta\gamma^2}{B(r,n-r+1)} \sum_{i=0}^{n-r}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{i+j+k+1}\binom{j+1}{k}\Gamma(-\sigma(k+1))\Gamma(\theta(r+i))}{j!\,l!\,\Gamma(\theta(r+i)-j)\Gamma(-\sigma(k+1)-l)} x^{-3} \left[e^{-\left(\frac{\gamma}{x}\right)^2}\right]^{i+1} \quad (34)$$

To obtain minimum order statistics for EtGoIR distribution, we set r=1 in equation (34) to get

$$f_{1:n}(x) = 2n\theta\beta\gamma^2 \sum_{i=0}^{n-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{i+j+k+1}\binom{j+1}{k}\Gamma(-\sigma(k+1))\Gamma(\theta(1+i))}{j!\,l!\,\Gamma(\theta(1+i)-j)\Gamma(-\sigma(k+1)-l)} x^{-3} \left[e^{-\left(\frac{\gamma}{x}\right)^2}\right]^{i+1} \quad (35)$$

To obtain maximum order statistics for EtGoIR distribution, we set r=n in equation (34) to get

$$f_{n:n}(x) = 2n\theta\beta\gamma^2 \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{j+k+i}\binom{j+1}{k}\Gamma(-\sigma(k+1))\Gamma(\theta(n))}{j!\,l!\,\Gamma(\theta(n)-j)\Gamma(-\sigma(k+1)-l)} x^{-3} \left[e^{-\left(\frac{\gamma}{x}\right)^2}\right]^{i+1} \quad (36)$$

2.4 Maximum Likelihood Estimation (MLE)

Given some observed data, a method known as maximum likelihood estimation (MLE) can be used to estimate a probability distribution's parameters. This is accomplished by maximizing a likelihood function to make the observed data as probable as possible given the assumed statistical model. The log-likelihood function of EtGoIR is given as

$$logL = nlog(2) + nlog(\theta) + nlog(\beta) + 2nlog(\gamma) - 3\sum_{i=1}^{n}\log(x) - \gamma^2\sum_{i=1}^{n}\left(\frac{1}{x}\right)^2 - (\sigma+1)\sum_{i=1}^{n} log\left[1-e^{-\left(\frac{\gamma}{x}\right)^2}\right] + \frac{\beta}{\sigma}\sum_{i=1}^{n}\left[1-\left[1-e^{-\left(\frac{\gamma}{x}\right)^2}\right]^{-\sigma}\right] + (\theta-1)\sum_{i=1}^{n}\left[1-e^{\left(\frac{\beta}{\sigma}\right)\left[1-\left[1-e^{-\left(\frac{\gamma}{x}\right)^2}\right]^{-\sigma}\right]}\right] \quad (37)$$

The maximum likelihood estimate is the location in the parameter space where the likelihood function is maximized. The maximum likelihood estimates of θ, β, γ and σ are the values that maximize the likelihood function. We can find these values by taking the partial derivatives of the likelihood function with respect to θ, β, γ, σ and setting them equal to zero. This gives us the following equations:

$$\frac{\partial logl}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} log\left[1-e^{-\left(\frac{\beta}{\sigma}\right)\left[1-\left[1-e^{-\left(\frac{\gamma}{x}\right)^2}\right]^{-\sigma}\right]}\right] = 0 \quad (38)$$

$$\frac{\partial logl}{\partial\beta} = \frac{n}{\beta} + \frac{1}{\sigma}\sum_{i=1}^{n}\left[1-\left[1-e^{-\left(\frac{\gamma}{x}\right)^2}\right]^{-\sigma}\right] - (\theta-1)\sum_{i=1}^{n}\frac{e^{\left(\frac{\beta}{\sigma}\right)\left[1-\left[1-e^{-\left(\frac{\gamma}{x}\right)^2}\right]^{-\sigma}\right]}}{\left[1-e^{\left(\frac{\beta}{\sigma}\right)\left[1-\left[1-e^{-\left(\frac{\gamma}{x}\right)^2}\right]^{-\sigma}\right]}\right]} = 0 \quad (39)$$

$$\frac{\partial logl}{\partial\gamma} = \frac{2n}{\gamma} + 2\gamma\sum_{i=1}^{n}\left(\frac{1}{x}\right)^{2} + (\sigma+1)\sum_{i=1}^{n}\frac{\gamma e^{-\left(\frac{\gamma}{x}\right)^{2}}}{1-e^{-\left(\frac{\gamma}{x}\right)^{2}}} + \frac{\beta}{\sigma}\sum_{i=1}^{n}2\sigma\gamma e^{-\left(\frac{\gamma}{x}\right)^{2}}\left[1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right]^{-\sigma-1} + (\theta-1)\sum_{i=1}^{n}\frac{e^{\left(\frac{\beta}{\sigma}\right)}2\sigma\gamma e^{-\left(\frac{\gamma}{x}\right)^{2}}e^{\left(\frac{\beta}{\sigma}\right)\left[1-\left[1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right]^{-\sigma}\right]}\left[1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right]^{-\sigma-1}}{1-e^{\left(\frac{\beta}{\sigma}\right)\left[1-\left[1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right]^{-\sigma}\right]}} = 0 \qquad (40)$$

$$\frac{\partial logl}{\partial\sigma} = \frac{\beta}{\sigma}\sum_{i=1}^{n}\left[1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right]^{-\sigma}log\left[1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right] + \frac{\beta}{\sigma^{2}}\sum_{i=1}^{n}\left[1-\left[1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right]^{-\sigma}\right]\sum_{i=1}^{n}log\left[1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right] + (\theta-1)\sum_{i=1}^{n}\frac{e^{-\left(\frac{\beta}{\sigma}\right)\left[1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right]^{-\sigma}}log\left[1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right]-e^{-\left(\frac{\beta}{\sigma^{2}}\right)\left[1-\left[1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right]^{-\sigma}\right]}}{1-e^{\left(\frac{\beta}{\sigma}\right)\left[1-\left[1-e^{-\left(\frac{\gamma}{x}\right)^{2}}\right]^{-\sigma}\right]}} = 0 \qquad (41)$$

Since equations (38), (39), (40) and (41) are non-linear in parameters, techniques such as Newton-Raphson method in R-software can be used to accomplish the task of estimating the parameters from equations (38), (39), (40) and (41).

III. Results

3.1 Simulation

In this section, we conduct a simulation study to assess the performance of the Maximum Likelihood Estimation (MLE) for the EtGoIR distribution. We generate random numbers using the quantile function (qf) of the distribution. Specifically, if U is a uniform random variable on the interval (0, 1), then x follows the EtGoIR distribution. We generated a total of n = 10000 samples, with each sample having sizes n=20, 50, 100, 250, 500, and 1000. These samples were drawn from the EtGoIR distribution using its quantile function. Subsequently, we calculated the empirical means, biases, and root mean squared errors (RMSE) of the MLE.

Table.1 *MLEs, biases and RMSE for some values of parameters*

		(0.5,0.1,0.1,0.5)			(2,1,3,2.5)		
n	Parameters	Estimated Values	Bias	RMSE	Estimated Values	Bias	RMSE
20	θ	0.4548	-0.0452	0.1484	2.2647	0.2647	0.9692
	β	0.1266	0.0266	0.0976	1.0825	0.0825	0.5743
	γ	0.1262	0.0262	0.0579	3.0253	0.0253	0.2659
	σ	0.5770	0.0770	0.1908	2.7354	0.2354	0.9156
50	θ	0.4737	-0.0263	0.1151	2.1251	0.1251	0.6905
	β	0.1075	0.0075	0.0503	1.0966	0.0966	0.4272
	γ	0.1110	0.0110	0.0310	3.0438	0.0438	0.1938
	σ	0.5366	0.0366	0.1216	2.5940	0.0940	0.6200

100	θ	0.4890	-0.0110	0.0903	2.0670	0.0670	0.4628
	β	0.1035	0.0035	0.0338	1.0951	0.0951	0.3017
	γ	0.1054	0.0054	0.0193	3.0519	0.0519	0.1539
	σ	0.5185	0.0185	0.0889	2.5425	0.0425	0.4413
250	θ	0.4972	-0.0028	0.0665	2.0166	0.0166	0.2872
	β	0.1006	0.0006	0.0227	1.0665	0.0665	0.2210
	γ	0.1019	0.0019	0.0126	3.0435	0.0435	0.1115
	σ	0.5097	0.0097	0.0630	2.5241	0.0241	0.2873
500	θ	0.5017	0.0017	0.0511	2.0052	0.0052	0.1923
	β	0.1012	0.0012	0.0160	1.0511	0.0511	0.1606
	γ	0.1006	0.0006	0.0091	3.0318	0.0318	0.0825
	σ	0.5012	0.0012	0.0415	2.5051	0.0051	0.1930
1000	θ	0.5028	0.0028	0.0370	2.0010	0.0010	0.1367
	β	0.1010	0.0010	0.0105	1.0434	0.0434	0.1208
	γ	0.1002	0.0002	0.0064	3.0288	0.0288	0.0727
	σ	0.5000	0.0001	0.0289	2.5048	0.0048	0.1400

Table 1 presents the simulation outcomes corresponding to the EtGoIR distribution. It is observed that as the sample size increases, the Root Mean Square Error (RMSE) and bias associated with the parameter estimators consistently decreases. The outcome suggest that the model is consistent.

3.2 Applications

This section demonstrates the practical application of the EtGoIR distribution by utilizing it to model biomedical datasets. We compare its performance in providing a robust parametric fit to the datasets with that of the Gompertz Inverse Rayleigh (GoIR) distribution, the generalized Gompertz (GGo) distribution, the exponentiated exponential (EtEx) distribution, and the inverse Rayleigh (IR) distribution. Metrics such as the log likelihood, Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC) are employed for this comparison. To discern the most suitable model, computations of the log likelihood, AIC, and BIC values are carried out for both the proposed EtGoIR model and the alternative models used for comparison. The model exhibiting the lowest log likelihood, AIC, and BIC values is deemed the most appropriate match for the provided datasets. For this analytical endeavor, the R software is employed, facilitating the necessary calculations and comparisons.

Data set 1 has been utilized by [13] and [14]. The dataset comprises the summation of skinfold measurements from 202 athletes at the Australian Institute of Sports. It consists of the following values:
28.0, 98, 89.0, 68.9, 69.9, 109.0, 52.3, 52.8, 46.7, 82.7, 42.3, 109.1, 96.8, 98.3, 103.6, 110.2, 98.1, 57.0, 43.1, 71.1, 29.7, 96.3, 102.8, 80.3, 122.1, 71.3, 200.8, 80.6, 65.3, 78.0, 65.9, 38.9, 56.5, 104.6, 74.9, 90.4, 54.6, 131.9, 68.3, 52.0, 40.8, 34.3, 44.8, 105.7, 126.4, 83.0, 106.9, 88.2, 33.8, 47.6, 42.7, 41.5, 34.6, 30.9, 100.7, 80.3, 91.0, 156.6, 95.4, 43.5, 61.9, 35.2, 50.9, 31.8, 44.0, 56.8, 75.2, 76.2, 101.1, 47.5, 46.2, 38.2, 49.2, 49.6, 34.5, 37.5, 75.9, 87.2, 52.6, 126.4, 55.6, 73.9, 43.5, 61.8, 88.9, 31.0, 37.6, 52.8, 97.9, 111.1, 114.0, 62.9, 36.8, 56.8, 46.5, 48.3, 32.6, 31.7, 47.8, 75.1, 110.7, 70.0, 52.5, 67, 41.6, 34.8, 61.8, 31.5, 36.6, 76.0, 65.1, 74.7, 77.0, 62.6, 41.1, 58.9, 60.2, 43.0, 32.6, 48, 61.2, 171.1, 113.5, 148.9, 49.9, 59.4, 44.5, 48.1, 61.1, 31.0, 41.9, 75.6, 76.8, 99.8, 80.1, 57.9, 48.4, 41.8, 44.5, 43.8, 33.7, 30.9, 43.3, 117.8, 80.3, 156.6, 109.6, 50.0, 33.7, 54.0, 54.2, 30.3, 52.8, 49.5, 90.2, 109.5, 115.9, 98.5, 54.6, 50.9, 44.7, 41.8, 38.0, 43.2, 70.0, 97.2, 123.6, 181.7, 136.3, 42.3, 40.5, 64.9, 34.1, 55.7, 113.5, 75.7, 99.9, 91.2, 71.6, 103.6, 46.1, 51.2, 43.8, 30.5, 37.5, 96.9, 57.7, 125.9, 49.0, 143.5, 102.8, 46.3, 54.4, 58.3, 34.0, 112.5, 49.3, 67.2, 56.5, 47.6, 60.4, 34.9.

Data set 2, encompassing the remission times (in months) of a randomized collection of one hundred and twenty-eight (128) bladder cancer patients, has been utilized by [15] and [14]. The dataset comprises the following values:
0.08, 0.20, 0.40, 0.50, 0.51, 0.81, 0.90, 1.05, 1.19, 1.26, 1.35, 1.40, 1.46, 1.76, 2.02, 2.02, 2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.25, 3.31, 3.36, 3.36, 3.48, 3.52, 3.57, 3.64, 3.70, 3.82, 3.88, 4.18, 4.23, 4.26, 4.33, 4.34, 4.40, 4.50, 4.51, 4.87, 4.98, 5.06, 5.09, 5.17, 5.32, 5.32, 5.34, 5.41, 5.41, 5.49, 5.62, 5.71, 5.85, 6.25, 6.54, 6.76, 6.93, 6.94, 7.09, 7.26, 7.28, 7.32, 7.39, 7.59, 7.62, 7.63, 7.66, 7.87, 7.93, 8.26, 8.37, 8.53, 8.65, 8.66, 9.02, 9.22, 9.47, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05.

Data set 3, representing the survival times of one hundred and twenty-one (121) patients with breast cancer obtained from a large hospital during the period from 1929 to 1938, was obtained from [17]. The dataset is outlined as follows:
0.3, 0.3, 1.0, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0.

Table 2: *Summary Statistics of data*

	N	Min.	Max.	Q1	Q2	Mean	Q3	Var.	SD	Ku	Sk
Data1	202	28.00	200.80	43.85	58.60	69.02	90.35	1060.501	32.565	4.365	1.175
Data2	128	0.080	79.050	3.348	6.395	9.366	11.838	110.425	10.508	18.485	3.286
Data3	121	0.30	154.00	17.30	40.00	46.08	60.00	1259.567	35.490	3.372	1.029

Table 2 demonstrate that the three datasets exhibit a high degree of skewness.

Table 3: *The models' MLEs and performance requirements based on data set 1*

Models	$\hat{\beta}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\sigma}$	ll	AIC	BIC
EtGoIR	0.1985	369.5184	0.3036	0.1799	-953.632	1915.2650	1928.4980
GoIR	0.0031	-	0.0000	0.8601	-987.520	1981.0410	1990.9660
GGo	-0.0052	15.4031	-	0.0597	-956.086	1918.1730	1928.9200
EtEx	0.0406	8.5786	-	-	-958.006	1920.0130	1926.6300
IR	52.6054	-	-	-	-966.462	1934.9250	1938.2330

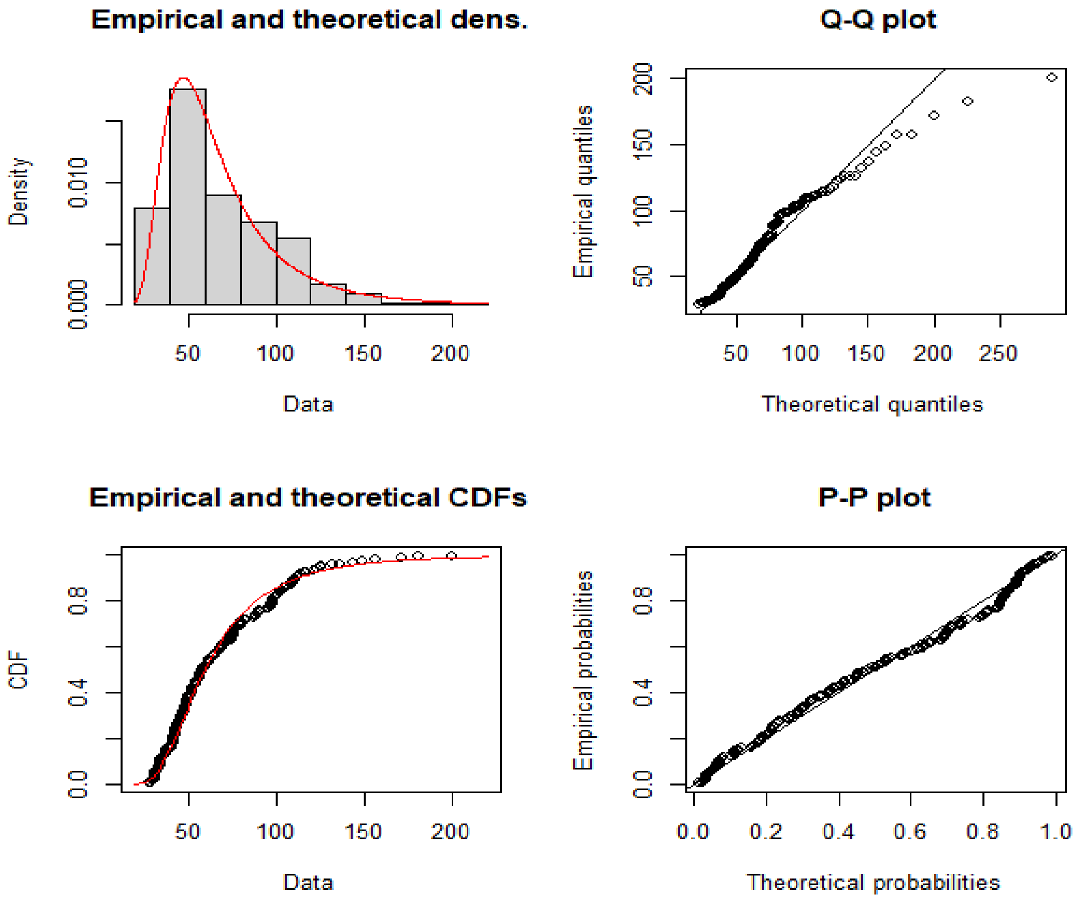

Figure 3: *Density plots for data set 1*

Table 4: *The models' MLEs and performance requirements based on data set 2*

Models	$\hat{\beta}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\sigma}$	ll	AIC	BIC
EtGoIR	0.0003	2.5796	0.0001	0.3400	-410.704	829.4088	834.1479
GoIR	0.0839	-	0.0041	0.5129	-413.575	833.1505	836.1377
GGo	-0.0224	1.5034	-	0.1678	-413.183	832.3668	835.3539
EtEx	0.1213	1.2180	-	-	-413.077	830.1552	834.8592
IR	2.2612	-	-	-	-774.341	1550.683	1553.535

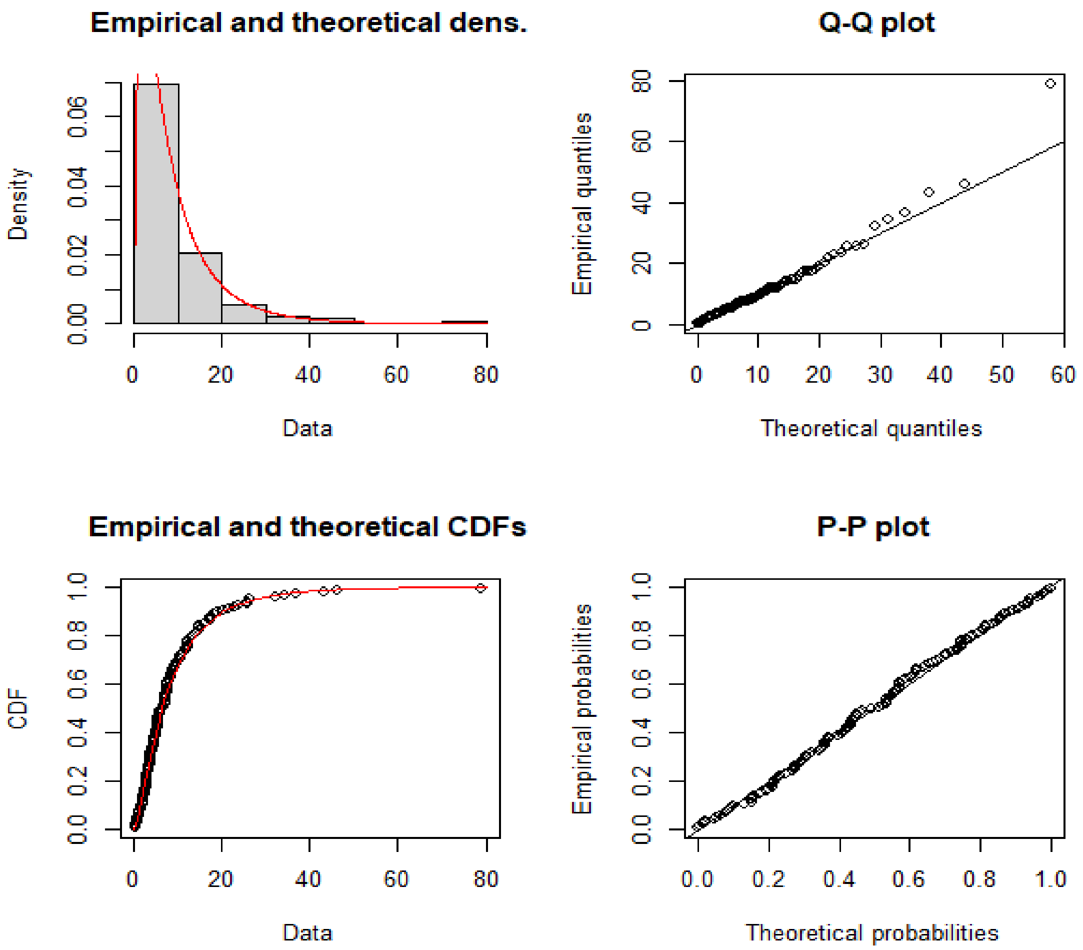

Figure 4: *Density plots for data set 2*

Table 5: *The models' MLEs and performance requirements based on data set 3.*

Models	$\hat{\beta}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\sigma}$	ll	AIC	BIC
EtGoIR	0.0000	0.5664	0.4016	0.9033	-578.7145	1165.4290	1176.6120
GoIR	0.0933	-	0.0002	0.6341	-579.9791	1165.9580	1176.7450
GGo	0.0066	1.1485	-	0.0182	-579.9435	1165.9371	1176.7274
EtEx	0.0269	1.4244	-	-	-581.7091	1167.4182	1168.2120
IR	2.2612	-	-	-	-1087.464	2176.9290	2179.7240

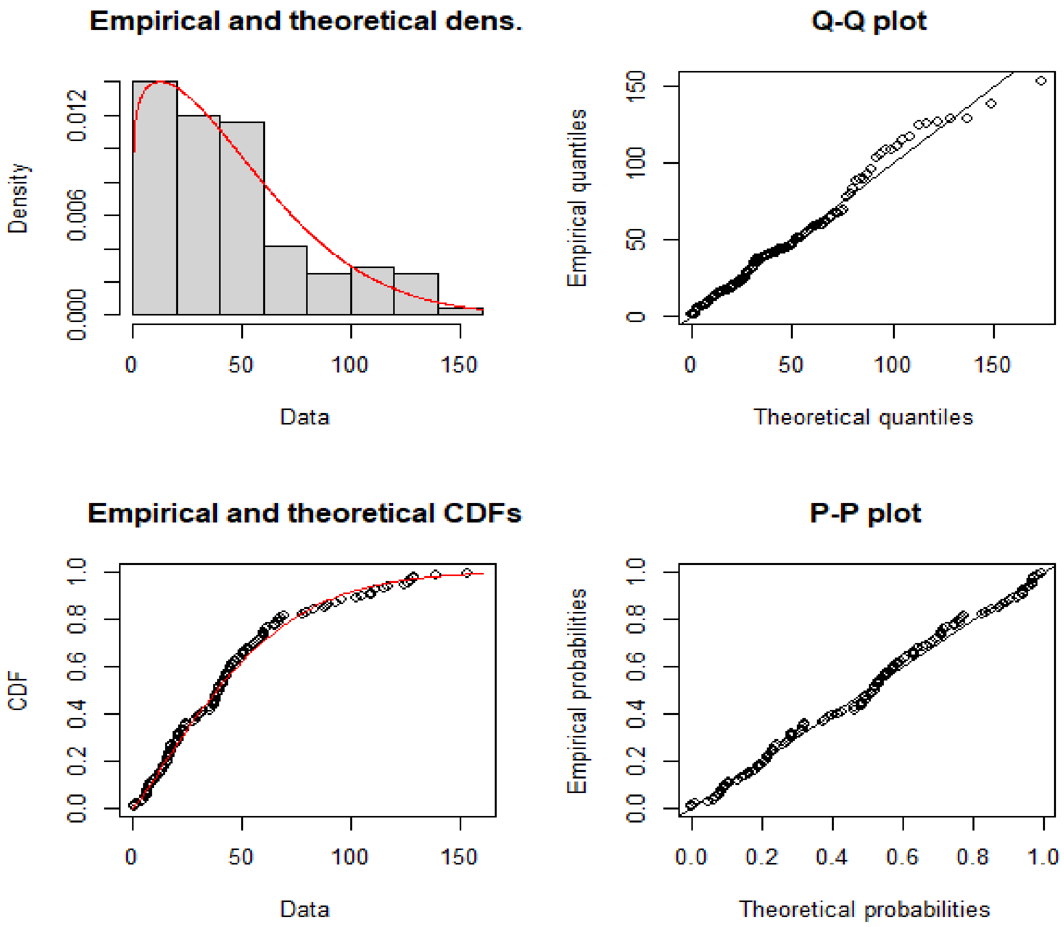

Figure 5: *Density plots for data set 3.*

Tables 3 to 5 showcase the superior ability of the proposed model to effectively fit the highly skewed datasets compared to the competing models, as indicated by the evaluation metrics employed. Figures 3 to 5 also showed that the proposed model fits the data set adequately.

IV. Discussion

This paper introduces a novel distribution termed the Exponentiated Gompertz Inverse Rayleigh (EtGoIR) distribution, extending the framework of the Gompertz Inverse Rayleigh (GoIR) distribution. The introduction of a new parameter enhances the distribution's adaptability in capturing various nuances present in biomedical datasets. The paper extensively examines the properties of the EtGoIR distribution, effectively demonstrating its practical applicability to real-life scenarios through the implementation of Maximum Likelihood Estimation (MLE). The empirical findings consistently substantiate that the proposed EtGoIR model outperforms the alternative distribution models under consideration in accurately fitting the provided datasets.

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