ANALYSIS OF MMAP/PH1,PH2/1 PREEMPTIVE PRIORITY INVENTORY RETRIAL QUEUEING SYSTEM WITH SINGLE VACATION, WORKING BREAKDOWN, REPAIR AND CLOSEDOWN

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Abstract

This paper analyzes preemptive priority inventory retrial queueing system with a single vacation, working breakdown, repair, and closedown. We assume that an arrival follows the Marked Markovian arrival process and that the server will provide them with phase-type services. The (s, S) policy to replenish the items and the replenishing duration follow an exponential distribution. In this paper, we consider two types of customers: high-priority(HP) customers and low-priority(LP) customers. Arriving HP customers should get the service if the server is idle and has a positive inventory level; otherwise, they should wait in front of the service station. Arriving LP customers get service only if there is a positive inventory level and there are no high-priority customers in the system; otherwise, go for the finite capacity size of the orbit. After the completion of service, if no one is present in the high-priority queue and orbit, the server will close down the system and then go on a single vacation. The server is idle when the vacation period ends. When the server breaks down, it only serves the present customer and operates in slow mode while it is being repaired. The number of high-priority customers in the system, the number of low-priority customers in the orbit, the inventory level, and server status may all be determined in a steady state. Numerous key performance indicators are defined, and a cost analysis is obtained. To make our mathematical concept clearer, a few numerical examples are provided.

Keywords: Queueing-inventory, (s, S) policy, Retrial, Preemptive Priority, Single Vacation, Working Breakdown, Repair, Closedown, Markovian Arrival Process, Phase-type distribution, Matrix Analytic Method.

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1. Introduction

The field of inventory retrial queueing systems has seen a rise in popularity in recent years due to developments in computer networking and communications technologies. In a queueing-inventory model, each client receives a product from the inventory upon completion of the service. Neuts [19] presented the modified Markovian point process for the first time. A number of well-known techniques fall under the large category of point processes known as MAP, including PH-renewal, Markov-modulated Poissons, and Poisson. The Markovian arrival process with several correlated and non-correlated arrival types, as well as the phase-type distribution, were both extensively clarified by Chakravarthy [8]. Neuts [20] investigated the methods used in matrix-analytic queueing theory.

Reordering products in a queueing order-demand inventory system is best done using the techniques described by Melikov and Molchanov [16]. A study by Berman et al. [6] examined a system for inventory control for a service center that uses one inventory item for each service rendered. According to their assumptions, there must always be a shortage of items for the queue to form, and demand and service rates are predictable and steady. Berman and Kim [7] developed two types of queuing-inventory models with service facilities. The first had an infinite one, while the other had a finite capacity for queuing. In their evaluation, Yadavalli et al. [25] made the assumption that reorders are readily available and that requests belong to a renewal process. The inventory system included a service station and an indefinite waiting area. Amirthakodi and Sivakumar [2] looked at retrial inventory queuing, in which there is a finite orbit size, a single server, and customer feedback. Sanjukta and Nabendu [21] looked into a carbon tax and an inventory queueing system with a partial replenishment strategy and a limited shelf life for perishable commodities.

Most of the time, it is believed that inventory and queueing models have not failed at service stations. In actuality, we regularly come across circumstances where service station malfunctions could occur. A server interruption inventory retry queueing system was covered by Krishnamoorthy et al. [11]. In their model, they took into consideration a (s, S) replenishment policy where the lead time and service time follow an exponential distribution, while the arrival follows a Poisson distribution. The retrial inventory queuing system with server failure was examined by Ushakumari [24]. When the server is processing requests or is idle, it could malfunction. If a server failure results in service disruption, users are placed into an infinitely long orbit, obliged to retry service after an arbitrary period, and so on unless the server is rendered inoperable.

The server could simply quit waiting for customers and remain unreachable in a variety of situations. It might also be completing other duties, such as maintenance or servicing more clients. Krishnamoorthy and Narayanan [12] considered a manufacturing inventory system including server vacations. They held that the manufacturing process adhered to the Markovian manufacturing method and that the service times for each customer were dispersed in a phase-type manner. The inventory queue for retrials with several vacations was analyzed by Suganya and Sivakumar [23]. They took into account a pair of servers and a limited orbit size capacity in their model. The retrial queueing system incorporating a single server, Bernoulli feedback, and vacation has been examined by Ayyappan and Gowthami [4]. They took into account both the client's arrival based on MAP and the server's service delivery based on PH distribution. Melikov et al. [17] examined the retrial queueing system that incorporates Poisson arrival, exponential service time, and delayed feedback. For their investigation, they employed both the (s, S) and (s, Q) replenishing policies.

An inventory queueing approach with MAP arrivals, PH offerings, and perishable goods was examined by Manuel et al. [15]. Additionally, they take into account their model, in which a positive customer advances one regular customer to the front of the line while a negative customer pushes one regular customer back. An inventory retrial queueing system involving two commodities was presented by Anbazhagan and Jeganathan [3]. They think of their model as having a core item and a supplement item. Jeganathan and Selvakumar [9] examined a queueing system for inventory that employed a traditional retry rate. In their work, they present an optional oscillatory client arrival procedure that is subject to Bernoulli testing and can pass through a waiting room or an infinite orbit. A two-component demand inventory retrial queueing system was examined by Abdul Reiyas and Jeganathan [1]. They took into account the (s, Q) replenishment policy while placing the order. The retrial inventory queueing model was examined by Jeganathan et al. [10] with two different kinds of clients. Mustapha and Majid [18] developed a two-phase production period production inventory model for non-immediately degrading products. For mixed demand with trade credit programs, Manisha et al. [14] have created the ideal replacement and conservation investment strategy. Ayyappan and Archana [5] discussed the non-preemptive priority queueing model with optional service and single vacation.

2. Description of the model

We take into consideration a single server with the preemptive priority inventory retrial queueing system, featuring inventory with maximum storage capacity of S units. Customers arrive via the Marked Markovian Arrival Process (MMAP) with depictions (D_0, D_1, D_2) with order size k, where the matrix $D = D_0 + D_1 + D_2$. In the system, no arrival is governed by the square matrix D_0, the arrival of high priority customers is governed by the square matrix D_1 and the arrival of low priority customers is governed by the square matrix D_2. With π representing the probability vector of D, the mean arrival rate of HP customers is $\lambda_1 = \pi D_1 e_k$ and the mean arrival rate of LP customers is $\lambda_2 = \pi D_2 e_k$. The server provides high priority(HP) and low priority(LP) services that both follow a PH-distribution with depictions (γ, P) and (ν, M) of orders l_1 and l_2, respectively. For HP and LP customers mean service rate is $\zeta_1 = [\gamma(-P)^{-1}e_{l_1}]^{-1}$ and $\zeta_2 = [\nu(-M)^{-1}e_{l_2}]^{-1}$.

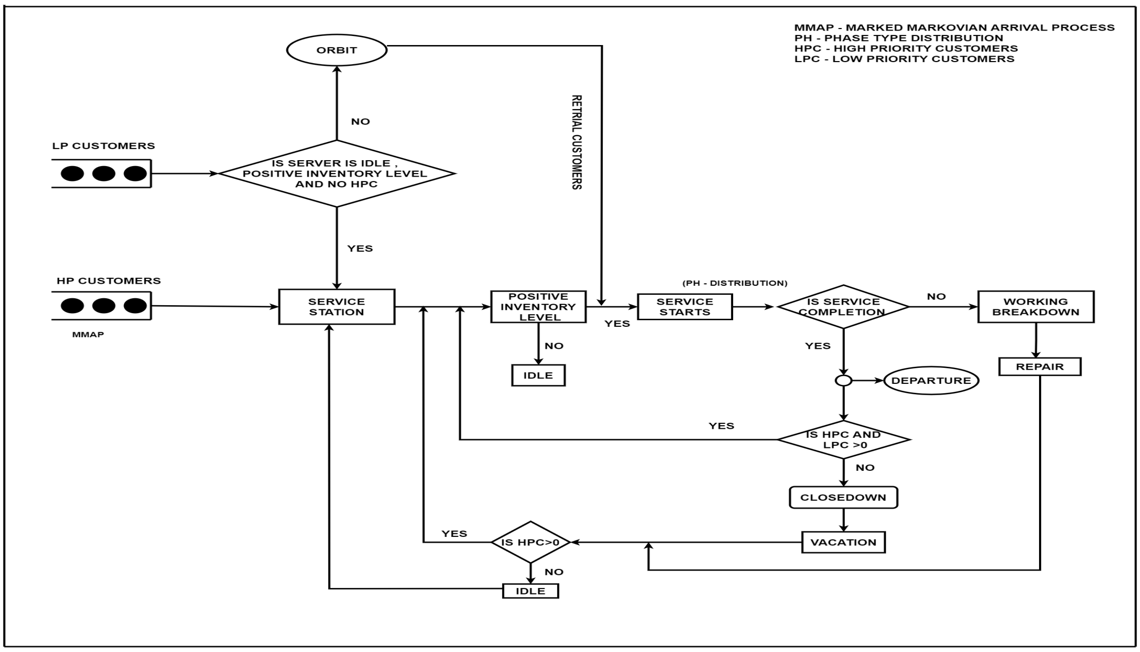

Figure 1: *A pictorial illustration of the model*

If the server breaks down while serving HP or LP customers, it will first offer a slow service mode to the impacted customers before beginning the repair procedure. The PH-distribution is followed by the slower service for HP and LP customers, together with a representation of order l_1 and l_2, respectively, represented by $(\gamma_1, \theta P)$ and $(\nu_1, \theta M)$. The breakdown time has an exponential distribution with parameter σ, and the repair process has a PH-distribution with a depiction (α, U) of order m_2. When HP customers arrive, they only interrupt their regular service if LP customer services are still in progress, and the server serves HP clients. In the event that there are no pending requests in the HP queue, the server will serve LP customers. Arriving HP customers should get the service if the server is idle and has a positive inventory level; otherwise, they should wait in front of the service station. Arriving LP customers get service only if there is a positive inventory level and there are no high-priority customers in the system; otherwise, go for the finite capacity size of the orbit, say N. After the completion of service, if no one is present in the high-priority queue and orbit, the server will close down the system and then go on a single vacation. After the completion of the vacation period, the server is idle. The closedown times follow an exponential distribution with parameter δ. The Vacation times follow the PH-distribution with depiction (β, W) of order m_1. The LP customers retrying for their service after the fixed times, the constant retrial rate follow an exponential distribution with parameter χ.

The average rate of repair and vacation is given by η and ψ respectively.

3. The Quasi Birth and Death Process for the Matrix Generations

We are going to discuss this part, which comprises the notation that forms the basis of the Quasi Birth and Death (QBD) process in our model.

- $\otimes$ - Any two different order matrices can be multiplied to create a Kronecker product, and this can be founded on the research of Steeb and Hardy [22].
- $\oplus$ - The Kronecker sum is the sum of any two of the different orders of matrices.
- I_k - The identity matrix has k dimensions.
- e - The column vector's appropriate dimension for each of its elements is 1.
- e_k - For every k elements in a column vector, the value is 1.
- $e_k(L)$ - The column vector with dimension L, where the k^{th} element is 1 and remaining elements are 0.
- $e_k'(L)$ - The transpose of $e_k(L)$.
- The arrival rate of HP and LP customers is represented by λ_i and described as $\lambda_i = \pi D_i e_k$, where i=1,2 respectively.
- The service rate for HP customers is represented by ζ_1 and described as $\zeta_1 = [\gamma(-P)^{-1}e_{l_1}]^{-1}$.
- The service rate for LP customers is represented by ζ_2 and described as $\zeta_2 = [\nu(-M)^{-1}e_{l_2}]^{-1}$.
- The vacation rate of the server is represented by ψ and described as $\psi = [\beta(-W)^{-1}e_{m_1}]^{-1}$.
- The server's rate of repair is represented by η and described as $\eta = [\alpha(-U)^{-1}e_{m_2}]^{-1}$.
- The number of HP customers in the system at time t can be represented by $N_1(t)$.
- The number of LP customers in the orbit at time t can be represented by $N_2(t)$.
- Let $V(t)$ be the state of the server at time t.

$$V(t) = \begin{cases} 0, & \text{the vacation state of the server,} \\ 1, & \text{the idle state of the server,} \\ 2, & \text{the server is offering service for HP customers,} \\ 3, & \text{the server is offering service for LP customers,} \\ 4, & \text{the server is offering slow service for HP customers,} \\ 5, & \text{the server is offering slow service for LP customers,} \\ 6, & \text{the server is under repair,} \\ 7, & \text{the server is under closedown process.} \end{cases}$$

- Let $I(t)$ be the level of inventory items at time t.
- $J_1(t)$ denotes the phases of the vacation process.
- $J_2(t)$ denotes the phases of the repair process.
- $S(t)$ denotes the phases of the service process.
- $M(t)$ denotes the phases of the arrival process.

Let $\{N_1(t), N_2(t), V(t), I(t), J_1(t), J_2(t), S(t), M(t) : t \geq 0\}$ indicate the Continuous Time Markov Chain (CTMC) with state-level independent QBD processes. The state space is as follows:

$$\Phi = l(0) \cup_{u_1=1}^{\infty} l(u_1),$$

where

$$
\begin{aligned}
l(0) =&\{(0,u_2,0,j,a_1,c) : 0 \le u_2 \le N, 0 \le j \le S, 1 \le a_1 \le m_1, 1 \le c \le k\}\\
&\cup \{(0,u_2,1,j,c) : 0 \le u_2 \le N, 0 \le j \le S, 1 \le c \le k\}\\
&\cup \{(0,u_2,3,j,d_2,c) : 0 \le u_2 \le N, 1 \le j \le S, 1 \le d_2 \le l_2, 1 \le c \le k\}\\
&\cup \{(0,u_2,5,j,d_2,c) : 0 \le u_2 \le N, 1 \le j \le S, 1 \le d_2 \le l_2, 1 \le c \le k\}\\
&\cup \{(0,u_2,6,j,a_2,c) : 0 \le u_2 \le N, 0 \le j \le S, 1 \le a_2 \le m_2, 1 \le c \le k\}\\
&\cup \{(0,u_2,7,j,c) : 0 \le u_2 \le N, 0 \le j \le S, 1 \le c \le k\},
\end{aligned}
$$

for $u_1 \ge 1$,

$$
\begin{aligned}
l(u_1) =&\{(u_1,u_2,0,j,a_1,c) : 0 \le u_2 \le N, 0 \le j \le S, 1 \le a_1 \le m_1, 1 \le c \le k\}\\
&\cup \{(u_1,u_2,1,0,c) : 0 \le u_2 \le N, 1 \le c \le k\}\\
&\cup \{(u_1,u_2,2,j,d_1,c) : 0 \le u_2 \le N, 0 \le j \le S, 1 \le d_1 \le l_1, 1 \le c \le k\}\\
&\cup \{(u_1,u_2,4,j,d_1,c) : 0 \le u_2 \le N, 0 \le j \le S, 1 \le d_1 \le l_1, 1 \le c \le k\}\\
&\cup \{(u_1,u_2,5,j,d_2,c) : 0 \le u_2 \le N, 0 \le j \le S, 1 \le d_2 \le l_2, 1 \le c \le k\}\\
&\cup \{(u_1,u_2,6,j,a_2,c) : 0 \le u_2 \le N, 0 \le j \le S, 1 \le a_2 \le m_2, 1 \le c \le k\}\\
&\cup \{(u_1,u_2,7,j,c) : 0 \le u_2 \le N, 0 \le j \le S, 1 \le c \le k\}.
\end{aligned}
$$

The QBD procedure generates an infinitesimal matrix, as provided by

$$
Q = \begin{bmatrix}
B_{00} & B_{01} & 0 & 0 & 0 & 0 & \dots\\
B_{10} & A_1 & A_0 & 0 & 0 & 0 & \dots\\
0 & A_2 & A_1 & A_0 & 0 & 0 & \dots\\
0 & 0 & A_2 & A_1 & A_0 & 0 & \dots\\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots
\end{bmatrix}.
$$

The entries in Q's block matrices are specified as follows:

$$
B_{00} = \begin{bmatrix}
{B_{00}}^{11} & {B_{00}}^{12} & 0 & 0 & 0 & 0\\
0 & {B_{00}}^{22} & {B_{00}}^{23} & 0 & 0 & 0\\
0 & {B_{00}}^{32} & {B_{00}}^{33} & {B_{00}}^{34} & 0 & {B_{00}}^{36}\\
0 & 0 & 0 & {B_{00}}^{44} & {B_{00}}^{45} & 0\\
0 & {B_{00}}^{52} & 0 & 0 & {B_{00}}^{55} & 0\\
{B_{00}}^{61} & 0 & 0 & 0 & 0 & {B_{00}}^{66}
\end{bmatrix},
$$

where

$$
{B_{00}}^{11} = \begin{bmatrix}
C_{001} & C_{002} & 0 & \dots & 0 & 0\\
0 & C_{001} & C_{002} & \dots & 0 & 0\\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots\\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\
0 & 0 & 0 & \dots & C_{001} & C_{002}\\
0 & 0 & 0 & \dots & 0 & C_{001}+C_{002}
\end{bmatrix},
$$

$$
C_{001} = \begin{bmatrix}
J_1 & 0 & \dots & 0 & 0 & \dots & J_3\\
0 & J_1 & \dots & 0 & 0 & \dots & J_3\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \dots & J_1 & 0 & \dots & J_3\\
0 & 0 & \dots & 0 & J_2 & \dots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & \dots & 0 & 0 & \dots & J_2
\end{bmatrix}, \quad C_{002} = I_{S+1} \otimes I_{m_1} \otimes D_2,
$$

$$B_{00}{}^{12} = I_{N+1} \otimes I_{S+1} \otimes W^0 \otimes I_k,$$

where $J_1 = W \oplus D_0 - \tau I_{m_1 k}, \quad J_2 = W \oplus D_0, \quad J_3 = \tau I_{m_1 k},$

$$B_{00}{}^{22} = \begin{bmatrix} C_{003} & C_{004} & 0 & \dots & 0 & 0 \\ 0 & C_{005} & C_{004} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & C_{005} & C_{004} \\ 0 & 0 & 0 & \dots & 0 & C_{006} \end{bmatrix},$$

$$C_{003} = \begin{bmatrix} J_4 & 0 & \dots & 0 & 0 & \dots & J_6 \\ 0 & J_4 & \dots & 0 & 0 & \dots & J_6 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & J_4 & 0 & \dots & J_6 \\ 0 & 0 & \dots & 0 & J_5 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & J_5 \end{bmatrix},$$

$$C_{005} = \begin{bmatrix} J_4 & 0 & \dots & 0 & 0 & \dots & J_6 \\ 0 & J_7 & \dots & 0 & 0 & \dots & J_6 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & J_7 & 0 & \dots & J_6 \\ 0 & 0 & \dots & 0 & J_8 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & J_8 \end{bmatrix},$$

$$C_{006} = \begin{bmatrix} J_9 & 0 & \dots & 0 & 0 & \dots & J_6 \\ 0 & J_{10} & \dots & 0 & 0 & \dots & J_6 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & J_{10} & 0 & \dots & J_6 \\ 0 & 0 & \dots & 0 & J_{11} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & J_{11} \end{bmatrix}$$

$$C_{004} = \begin{bmatrix} e_1'(S+1) \otimes D_2 \\ 0 \end{bmatrix},$$

where $J_4 = D_0 - \tau I_k, \quad J_5 = D_0, \quad J_6 = \tau I_k, \quad J_7 = D_0 - (\chi + \tau) I_k, \quad J_8 = D_0 - \chi I_k,$

$J_9 = (D_0 + D_2) - \tau I_k, \quad J_{10} = (D_0 + D_2) - (\chi + \tau) I_k, \quad J_{11} = (D_0 + D_2) - \chi I_k,$

$$B_{00}{}^{23} = \begin{bmatrix} C_{007} & 0 & \dots & 0 & 0 & 0 \\ C_{008} & C_{007} & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & C_{008} & C_{007} & 0 \\ 0 & 0 & \dots & 0 & C_{008} & 0 \end{bmatrix},$$

$$C_{007} = \begin{bmatrix} 0 \\ I_S \otimes \nu \otimes D_2 \end{bmatrix}, \quad C_{008} = \begin{bmatrix} 0 \\ I_S \otimes \nu \otimes \chi I_m \end{bmatrix},$$

$$B_{00}{}^{33} = \begin{bmatrix} C_{009} & C_{0010} & 0 & \dots & 0 & 0 \\ 0 & C_{009} & C_{0010} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & C_{009} & C_{0010} \\ 0 & 0 & 0 & \dots & 0 & C_{009} + C_{0010} \end{bmatrix},$$

$$C_{009} = \begin{bmatrix} J_{12} & 0 & \dots & 0 & 0 & \dots & J_{14} \\ 0 & J_{12} & \dots & 0 & 0 & \dots & J_{14} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & J_{12} & 0 & \dots & J_{14} \\ 0 & 0 & \dots & 0 & J_{13} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & J_{13} \end{bmatrix}, C_{0010} = I_S \otimes I_{l_2} \otimes D_2,$$

where $J_{12} = M \oplus D_0 - (\sigma + \tau) I_{l_2 k}$, $J_{13} = M \oplus D_0 - \sigma I_{l_2 k}$, $J_{14} = \tau I_{l_2 k}$,

$$B_{00}{}^{34} = I_{N+1} \otimes I_S \otimes e_{l_2} \otimes \nu_1 \sigma I_k, \quad B_{00}{}^{32} = \begin{bmatrix} 0 & 0 \\ 0 & I_N \otimes C_{0011} \end{bmatrix},$$

$$B_{00}{}^{36} = \begin{bmatrix} e_1(N+1) \otimes C_{0011} & 0 \end{bmatrix}, \quad C_{0011} = \begin{bmatrix} I_S \otimes M^0 \otimes I_k & 0 \end{bmatrix},$$

$$B_{00}{}^{44} = \begin{bmatrix} C_{0012} & C_{0010} & 0 & \dots & 0 & 0 \\ 0 & C_{0012} & C_{0010} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & C_{0012} & C_{0010} \\ 0 & 0 & 0 & \dots & 0 & C_{0012} + C_{0010} \end{bmatrix},$$

$$C_{0012} = \begin{bmatrix} J_{15} & 0 & \dots & 0 & 0 & \dots & J_{14} \\ 0 & J_{15} & \dots & 0 & 0 & \dots & J_{14} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & J_{15} & 0 & \dots & J_{14} \\ 0 & 0 & \dots & 0 & J_{16} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & J_{16} \end{bmatrix},$$

where $J_{15} = \theta M \oplus D_0 - \tau I_{l_2 k}$, $J_{16} = \theta M \oplus D_0$,

$$B_{00}{}^{45} = I_{N+1} \otimes C_{0013}, \quad C_{0013} = \begin{bmatrix} I_S \otimes \theta M^0 \alpha \otimes I_k & 0 \end{bmatrix},$$

$$B_{00}{}^{55} = \begin{bmatrix} C_{0014} & C_{0015} & 0 & \dots & 0 & 0 \\ 0 & C_{0014} & C_{0015} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & C_{0014} & C_{0015} \\ 0 & 0 & 0 & \dots & 0 & C_{0014} + C_{0015} \end{bmatrix},$$

$$B_{00}{}^{52} = I_{N+1} \otimes I_{S+1} \otimes U^0 \otimes I_k,$$

$$C_{0014} = \begin{bmatrix} J_{17} & 0 & \dots & 0 & 0 & \dots & J_{19} \\ 0 & J_{17} & \dots & 0 & 0 & \dots & J_{19} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & J_{17} & 0 & \dots & J_{19} \\ 0 & 0 & \dots & 0 & J_{18} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & J_{18} \end{bmatrix}, \quad C_{0015} = I_{S+1} \otimes I_{m_2} \otimes D_2,$$

where $J_{17} = U \oplus D_0 - \tau I_{m_2 k}$, $J_{18} = U \oplus D_0$, $J_{19} = \tau I_{m_2 k}$,

$$B_{00}{}^{66} = \begin{bmatrix} C_{0016} & C_{0017} & 0 & \dots & 0 & 0 \\ 0 & C_{0016} & C_{0017} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & C_{0016} & C_{0017} \\ 0 & 0 & 0 & \dots & 0 & C_{0016} + C_{0017} \end{bmatrix},$$

$$B_{00}{}^{61} = I_{N+1} \otimes I_{S+1} \otimes \beta \otimes \delta I_k,$$

$$C_{0016} = \begin{bmatrix} J_{20} & 0 & \dots & 0 & 0 & \dots & J_6 \\ 0 & J_{20} & \dots & 0 & 0 & \dots & J_6 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & J_{20} & 0 & \dots & J_6 \\ 0 & 0 & \dots & 0 & J_{21} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & J_{21} \end{bmatrix}, \quad C_{0017} = I_{S+1} \otimes D_2,$$

where $J_{20} = D_0 - (\delta + \tau) I_k$, $J_{21} = D_0 - \delta I_k$,

$$B_{01} = \begin{bmatrix} B_{01}{}^{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{01}{}^{22} & B_{01}{}^{23} & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{01}{}^{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{01}{}^{45} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{01}{}^{56} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B_{01}{}^{67} \end{bmatrix},$$

where

$$B_{01}{}^{11} = I_{N+1} \otimes I_{S+1} \otimes I_{m_1} \otimes D_1, \quad B_{01}{}^{22} = I_{N+1} \otimes e_1(S+1) \otimes D_1,$$

$$B_{01}{}^{23} = I_{N+1} \otimes C_{011}, \quad C_{011} = \begin{bmatrix} 0 \\ I_S \otimes \gamma \otimes D_1 \end{bmatrix},$$

$$B_{01}{}^{33} = I_{N+1} \otimes I_S \otimes e_{l_2} \otimes \gamma \otimes D_1, \quad B_{01}{}^{45} = I_{N+1} \otimes I_S \otimes I_{l_2} \otimes D_1,$$

$$B_{01}{}^{56} = I_{N+1} \otimes I_{S+1} \otimes I_{m_2} \otimes D_1, \quad B_{01}{}^{67} = I_{N+1} \otimes I_{S+1} \otimes D_1,$$

$$B_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{10}{}^{32} & 0 & 0 & 0 & B_{10}{}^{36} \\ 0 & 0 & 0 & 0 & B_{10}{}^{45} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$B_{10}{}^{32} = \begin{bmatrix} 0 & 0 \\ 0 & I_N \otimes C_{101} \end{bmatrix}, B_{10}{}^{36} = \begin{bmatrix} C_{101} & 0 \\ 0 & 0 \end{bmatrix}, \quad C_{101} = \begin{bmatrix} I_S \otimes P^0 \otimes I_k & 0 \end{bmatrix},$$

$$B_{10}{}^{45} = I_{N+1} \otimes C_{102}, \quad C_{102} = \begin{bmatrix} I_S \otimes \theta P^0 \alpha \otimes I_k & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} A_1{}^{11} & A_1{}^{12} & A_1{}^{13} & 0 & 0 & 0 & 0 \\ 0 & A_1{}^{22} & A_1{}^{23} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_1{}^{33} & A_1{}^{34} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_1{}^{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_1{}^{55} & A_1{}^{56} & 0 \\ 0 & A_1{}^{62} & A_1{}^{63} & 0 & 0 & A_1{}^{66} & 0 \\ A_1{}^{71} & 0 & 0 & 0 & 0 & 0 & A_1{}^{77} \end{bmatrix},$$

where

$$A_1{}^{11} = B_{00}{}^{11}, \quad A_1{}^{12} = I_{N+1} \otimes e_1(S+1) \otimes W^0 \otimes I_k,$$

$$A_1{}^{13} = I_{N+1} \otimes C_{111}, \quad C_{111} = \begin{bmatrix} 0 \\ I_S \otimes W^0 \gamma \otimes I_k \end{bmatrix},$$

$$A_1{}^{22} = \begin{bmatrix} D_0 - \tau I_k & D_2 & 0 & \dots & 0 & 0 \\ 0 & D_0 - \tau I_k & D_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & D_0 - \tau I_k & D_2 \\ 0 & 0 & 0 & \dots & 0 & (D_0 + D_2) - \tau I_k \end{bmatrix},$$

$$A_1{}^{23} = I_{N+1} \otimes e'_S(S) \otimes \gamma \otimes \tau I_k, \; A_1{}^{34} = I_{N+1} \otimes I_S \otimes e_{l_1} \otimes \gamma_1 \otimes \sigma I_k,$$

$$A_1{}^{33} = \begin{bmatrix} C_{112} & C_{113} & 0 & \dots & 0 & 0 \\ 0 & C_{112} & C_{113} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & C_{112} & C_{113} \\ 0 & 0 & 0 & \dots & 0 & C_{112} + C_{113} \end{bmatrix}, \quad C_{113} = I_S \otimes I_{l_1} \otimes D_2,$$

$$C_{112} = \begin{bmatrix} J_{22} & 0 & \dots & 0 & 0 & \dots & J_{24} \\ 0 & J_{22} & \dots & 0 & 0 & \dots & J_{24} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & J_{22} & 0 & \dots & J_{24} \\ 0 & 0 & \dots & 0 & J_{23} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & J_{23} \end{bmatrix},$$

where $J_{22} = P \oplus D_0 - (\sigma + \tau) I_{l_1 k}, \quad J_{23} = P \oplus D_0 - \sigma I_{l_1 k}, \quad J_{24} = \tau I_{l_1 k},$

$$A_1{}^{44} = \begin{bmatrix} C_{114} & C_{113} & 0 & \dots & 0 & 0 \\ 0 & C_{114} & C_{113} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & C_{114} & C_{113} \\ 0 & 0 & 0 & \dots & 0 & C_{114} + C_{113} \end{bmatrix},$$

$$C_{114} = \begin{bmatrix} J_{25} & 0 & \dots & 0 & 0 & \dots & J_{24} \\ 0 & J_{25} & \dots & 0 & 0 & \dots & J_{24} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & J_{25} & 0 & \dots & J_{24} \\ 0 & 0 & \dots & 0 & J_{26} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & J_{26} \end{bmatrix},$$

where $J_{25} = \theta P \oplus D_0 - \tau I_{l_1 k}$, $J_{26} = \theta P \oplus D_0$,

$A_1{}^{55} = B_{00}{}^{44}$, $A_1{}^{56} = B_{00}{}^{45}$, $A_1{}^{66} = B_{00}{}^{55}$,

$A_1{}^{62} = I_{N+1} \otimes e_1(S+1) \otimes U^0 \otimes I_k$, $A_1{}^{63} = I_{N+1} \otimes C_{115}$,

$$C_{115} = \begin{bmatrix} 0 \\ I_S \otimes U^0 \gamma \otimes I_k \end{bmatrix},$$

$A_1{}^{77} = B_{00}{}^{66}$, $A_1{}^{71} = B_{00}{}^{61}$,

$$A_0 = \begin{bmatrix} A_0{}^{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_0{}^{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_0{}^{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_0{}^{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_0{}^{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_0{}^{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_0{}^{77} \end{bmatrix},$$

where $A_0{}^{11} = B_{01}{}^{11}$, $A_0{}^{22} = I_{N+1} \otimes D_1$, $A_0{}^{33} = I_{N+1} \otimes I_S \otimes I_{l_1} \otimes D_1$,

$A_0{}^{44} = A_0{}^{33}$, $A_0{}^{55} = B_{01}{}^{45}$, $A_0{}^{66} = B_{01}{}^{56}$, $A_0{}^{77} = B_{01}{}^{67}$,

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_2{}^{32} & A_2{}^{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_2{}^{46} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $A_2{}^{32} = I_{N+1} \otimes e_1(S) \otimes P^0 \otimes I_k$, $A_2{}^{33} = I_{N+1} \otimes C_{211}$,

$$C_{211} = \begin{bmatrix} 0 & 0 \\ I_{S-1} \otimes P^0 \gamma \otimes I_k & 0 \end{bmatrix}, \quad A_2{}^{46} = B_{10}{}^{45}.$$

4. Stationary Analysis

We analyze our model in a few consistent system configurations.

4.1. Criteria for stability

Let us define the matrix A as follows: $A = A_0 + A_1 + A_2$, signifying that it is an irreducible infinitesimal generator matrix with dimensions of $((N+1)(S+1)m_1k + (N+1)k + 2(N+1)Sl_1k + (N+1)Sl_2k + (N+1)(S+1)m_2k + (N+1)(S+1)k)$.

The vector $\varkappa$ represents the stationary probability vector of A that achieving the criteria $\varkappa A = 0$ and $\varkappa e = 1$. The vector $\varkappa$ is divided by $\varkappa = (\varkappa_0, \varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4, \varkappa_5, \varkappa_6) = (\varkappa_{000}, \varkappa_{001}, \ldots, \varkappa_{00S}, \varkappa_{010}, \varkappa_{011}, \ldots, \varkappa_{01S}, \ldots, \varkappa_{0N0}, \varkappa_{0N1}, \ldots, \varkappa_{0NS}, \varkappa_{100}, \varkappa_{110}, \ldots, \varkappa_{0N0}, \varkappa_{201}, \ldots, \varkappa_{20S}, \varkappa_{211}, \ldots, \varkappa_{21S}, \ldots, \varkappa_{2N1}, \ldots, \varkappa_{2NS}, \varkappa_{301}, \ldots, \varkappa_{30S}, \varkappa_{311}, \ldots, \varkappa_{31S}, \ldots, \varkappa_{3N1}, \ldots, \varkappa_{3NS}, \varkappa_{401}, \ldots, \varkappa_{40S}, \varkappa_{411}, \ldots, \varkappa_{41S}, \ldots, \varkappa_{4N1}, \ldots, \varkappa_{4NS}, \varkappa_{500}, \varkappa_{501}, \ldots, \varkappa_{50S}, \varkappa_{510}, \varkappa_{511}, \ldots, \varkappa_{51S}, \ldots, \varkappa_{5N0}, \varkappa_{5N1}, \ldots, \varkappa_{5NS}, \varkappa_{600}, \varkappa_{601}, \ldots, \varkappa_{60S}, \varkappa_{610}, \varkappa_{611}, \ldots, \varkappa_{61S}, \ldots, \varkappa_{6N0}, \varkappa_{6N1}, \ldots, \varkappa_{6NS})$, where $\varkappa_0$ has a dimension of $(N+1)(S+1)m_1k$, $\varkappa_1$ has a dimension of $(N+1)k$, $\varkappa_2$ has a dimension of $(N+1)Sl_1k$, $\varkappa_3$ has a dimension of $(N+1)Sl_1k$, $\varkappa_4$ has a dimension of $(N+1)Sl_2k$, $\varkappa_5$ has a dimension of $(N+1)(S+1)m_2k$ and $\varkappa_6$ has a dimension of $(N+1)(S+1)k$. When examining the Markov process within the framework of QBD, our model's stability should satisfy the essential and sufficient requirements of $\varkappa A_0 e < \varkappa A_2 e$. Upon performing certain algebraic reductions, the stability condition $\varkappa A_0 e < \varkappa A_2 e$ is determined to be

$$\begin{aligned}
&\sum_{u_2=0}^{N}\sum_{j=0}^{S} \varkappa_{0u_2j}(e_{m_1} \otimes D_1 e_k) + \sum_{u_2=0}^{N} \varkappa_{1u_20}(D_1 e_k) + \sum_{u_2=0}^{N}\sum_{j=1}^{S} \varkappa_{2u_2j}(e_{l_1} \otimes D_1 e_k) \\
&+ \sum_{u_2=0}^{N}\sum_{j=1}^{S} \varkappa_{3u_2j}(e_{l_1} \otimes D_1 e_k) + \sum_{u_2=0}^{N}\sum_{j=1}^{S} \varkappa_{4u_2j}(e_{l_2} \otimes D_1 e_k) + \sum_{u_2=0}^{N}\sum_{j=0}^{S} \varkappa_{5u_2j}(e_{m_2} \otimes D_1 e_k) \\
&+ \sum_{u_2=0}^{N}\sum_{j=0}^{S} \varkappa_{6u_2j}(D_1 e_k) \; < \; \sum_{u_2=0}^{N}\sum_{j=1}^{N} \varkappa_{2u_2j}(P^0 \otimes e_k) + \sum_{u_2=0}^{N}\sum_{j=1}^{N} \varkappa_{3u_2j}(\theta P^0 \otimes e_k).
\end{aligned}$$

4.2. Analysis of Stationary Probability Vector

Let ϕ represent the stationary probability vector for Q, and this is divided as $\phi = (\phi_0, \phi_1, \phi_2, \ldots)$. Mention that ϕ_0 has a dimension of $(N+1)(S+1)m_1k + 2(N+1)(S+1)k + 2(N+1)Sl_2k + (N+1)(S+1)m_2k$ and $\phi_1, \phi_2, \ldots$ have a dimension of $(N+1)(S+1)m_1k + (N+1)k + 2(N+1)Sl_1k + (N+1)Sl_2k + (N+1)(S+1)m_2k + (N+1)(S+1)k$ and the vector ϕ satisfies $\phi Q = 0$ and $\phi e = 1$.

Additionally , after the stability requirement of the model is met, the stationary probability vector ϕ can be obtained by applying the following equation:

$$\phi_{u_1} = \phi_1 R^{u_1-1}, \; u_1 \geq 1.$$

The matrix quadratic equation $R^2A_2 + RA_1 + A_0 = 0$ is satisfied by the minimal non-negative solution R based on Neuts [20]. The matrix quadratic equation yields the rate matrix. The order of the rate matrix R is given by $((N+1)(S+1)m_1k + (N+1)k + 2(N+1)Sl_1k + (N+1)Sl_2k + (N+1)(S+1)m_2k + (N+1)(S+1)k)$ and it fulfills the condition $RA_2e = A_0e$.

By solving the following equations, the sub vectors ϕ_0 and ϕ_1 can be determined.

$$\phi_0 B_{00} + \phi_1 B_{10} = 0,$$

$$\phi_0 B_{01} + \phi_1 (A_1 + RA_2) = 0,$$

Subject to the normalizing condition

$$\phi_0 e_0 + \phi_1 (I - R)^{-1} e_1 = 1,$$

where $e_0 = e_{(N+1)(S+1)m_1k+2(N+1)(S+1)k+2(N+1)Sl_2k+(N+1)(S+1)m_2k}$ and
$e_1 = e_{(N+1)(S+1)m_1k+(N+1)k+2(N+1)Sl_1k+(N+1)Sl_2k+(N+1)(S+1)m_2k+(N+1)(S+1)k}$.

According to Latouche and Ramaswami [13], by utilizing important stages in the logarithmic reduction process, the R matrix can be produced analytically .

5. Measures of System Performance

- Average size of the HP customers in the system

$$E_{sys} = \sum_{u_1=1}^{\infty} u_1 \phi_{u_1} e.$$

- Average size of the LP customers in the orbit

$$\begin{aligned}
E_{orb} = &\sum_{u_1=0}^{\infty} \sum_{u_2=1}^{N} \sum_{j=0}^{S} \sum_{a_1=1}^{m_1} \sum_{c=1}^{k} u_2 \phi_{u_1 u_2 0 j a_1 c} + \sum_{u_2=1}^{N} \sum_{j=0}^{S} \sum_{c=1}^{k} u_2 \phi_{0 u_2 1 j c} \\
&+ \sum_{u_1=1}^{\infty} \sum_{u_2=1}^{N} \sum_{c=1}^{k} u_2 \phi_{u_1 u_2 1 0 c} + \sum_{u_1=1}^{\infty} \sum_{u_2=1}^{N} \sum_{j=1}^{S} \sum_{d_1=1}^{l_1} \sum_{c=1}^{k} u_2 \phi_{u_1 u_2 2 j d_1 c} \\
&+ \sum_{u_2=1}^{N} \sum_{j=1}^{S} \sum_{d_2=1}^{l_2} \sum_{c=1}^{k} u_2 \phi_{0 u_2 3 j d_2 c} + \sum_{u_1=1}^{\infty} \sum_{u_2=1}^{N} \sum_{j=1}^{S} \sum_{d_1=1}^{l_1} \sum_{c=1}^{k} u_2 \phi_{u_1 u_2 4 j d_1 c} \\
&+ \sum_{u_1=0}^{\infty} \sum_{u_2=1}^{N} \sum_{j=1}^{S} \sum_{d_2=1}^{l_2} \sum_{c=1}^{k} u_2 \phi_{u_1 u_2 5 j d_2 c} + \sum_{u_1=0}^{\infty} \sum_{u_2=1}^{N} \sum_{j=0}^{S} \sum_{a_2=1}^{m_2} \sum_{c=1}^{k} u_2 \phi_{u_1 u_2 6 j a_2 c} \\
&+ \sum_{u_1=0}^{\infty} \sum_{u_2=1}^{N} \sum_{j=0}^{S} \sum_{c=1}^{k} u_2 \phi_{u_1 u_2 7 j c}.
\end{aligned}$$

- Expected size of the inventory items

$$\begin{aligned}
E_{inv} = &\sum_{u_1=0}^{\infty} \sum_{u_2=0}^{N} \sum_{j=1}^{S} \sum_{a_1=1}^{m_1} \sum_{c=1}^{k} j \phi_{u_1 u_2 0 j a_1 c} + \sum_{u_2=0}^{N} \sum_{j=1}^{S} \sum_{c=1}^{k} j \phi_{0 u_2 1 j c} \\
&+ \sum_{u_1=1}^{\infty} \sum_{u_2=0}^{N} \sum_{j=1}^{S} \sum_{d_1=1}^{l_1} \sum_{c=1}^{k} j \phi_{u_1 u_2 2 j d_1 c} + \sum_{u_2=0}^{N} \sum_{j=1}^{S} \sum_{d_2=1}^{l_2} \sum_{c=1}^{k} j \phi_{0 u_2 3 j d_2 c} \\
&+ \sum_{u_1=1}^{\infty} \sum_{u_2=0}^{N} \sum_{j=1}^{S} \sum_{d_1=1}^{l_1} \sum_{c=1}^{k} j \phi_{u_1 u_2 4 j d_1 c} + \sum_{u_1=0}^{\infty} \sum_{u_2=0}^{N} \sum_{j=1}^{S} \sum_{d_2=1}^{l_2} \sum_{c=1}^{k} j \phi_{u_1 u_2 5 j d_2 c} \\
&+ \sum_{u_1=0}^{\infty} \sum_{u_2=0}^{N} \sum_{j=1}^{S} \sum_{a_2=1}^{m_2} \sum_{c=1}^{k} j \phi_{u_1 u_2 6 j a_2 c} + \sum_{u_1=0}^{\infty} \sum_{u_2=0}^{N} \sum_{j=1}^{S} \sum_{c=1}^{k} j \phi_{u_1 u_2 7 j c}.
\end{aligned}$$

- Expected reorder rate

$$\begin{aligned}
E_R = &\sum_{u_2=0}^{N} \sum_{d_2=1}^{l_2} \sum_{c=1}^{k} \phi_{0 u_2 3(s+1) d_2 c} (M^0 \otimes I_k) e + \sum_{u_1=0}^{\infty} \sum_{u_2=0}^{N} \sum_{d_2=1}^{l_2} \phi_{u_1 u_2 5(s+1) d_2 c} (\theta M^0 \alpha \otimes I_k) e \\
&+ \sum_{u_2=0}^{N} \sum_{d_1=1}^{l_1} \sum_{c=1}^{k} \phi_{1 u_2 2(s+1) d_1 c} (P^0 \otimes I_k) e + \sum_{u_1=2}^{\infty} \sum_{u_2=0}^{N} \sum_{d_1=1}^{l_1} \sum_{c=1}^{k} \phi_{u_1 u_2 2(s+1) d_1 c} (P^0 \gamma \otimes I_k) e \\
&+ \sum_{u_1=1}^{\infty} \sum_{u_2=0}^{N} \sum_{d_1=1}^{l_1} \sum_{c=1}^{k} \phi_{u_1 u_2 2(s+1) d_1 c} (\theta P^0 \alpha \otimes I_k) e.
\end{aligned}$$

- Probability for the vacation state of the server

$$P_{vac} = \sum_{u_1=0}^{\infty} \sum_{u_2=0}^{N} \sum_{j=0}^{S} \sum_{a_1=1}^{m_1} \sum_{c=1}^{k} \phi_{u_1 u_2 0 j a_1 c}.$$

- Probability for the idle state of the server

$$P_{idle} = \sum_{u_2=0}^{N} \sum_{j=0}^{S} \sum_{c=1}^{k} \phi_{0 u_2 1 j c} + \sum_{u_1=1}^{\infty} \sum_{u_2=0}^{N} \sum_{c=1}^{k} \phi_{u_1 u_2 1 0 c}.$$

- The probability that HP customers receive normal mode service from the server

$$P_{HNB} = \sum_{u_1=1}^{\infty} \sum_{u_2=0}^{N} \sum_{j=1}^{S} \sum_{d_1=1}^{l_1} \sum_{c=1}^{k} \phi_{u_1 u_2 2 j d_1 c}.$$

- The probability that LP customers receive normal mode service from the server

$$P_{LNB} = \sum_{u_2=0}^{N} \sum_{j=1}^{S} \sum_{d_2=1}^{l_2} \sum_{c=1}^{k} \phi_{0 u_2 3 j d_2 c}.$$

- The probability that HP customers receive slow mode service from the server

$$P_{HSB} = \sum_{u_1=1}^{\infty} \sum_{u_2=0}^{N} \sum_{j=1}^{S} \sum_{d_1=1}^{l_1} \sum_{c=1}^{k} \phi_{u_1 u_2 4 j d_1 c}.$$

- The probability that LP customers receive slow mode service from the server

$$P_{LSB} = \sum_{u_1=0}^{\infty} \sum_{u_2=0}^{N} \sum_{j=1}^{S} \sum_{d_2=1}^{l_2} \sum_{c=1}^{k} \phi_{u_1 u_2 5 j d_2 c}.$$

- Probability of the server is in repair process

$$P_{rep} = \sum_{u_1=0}^{\infty} \sum_{u_2=0}^{N} \sum_{j=0}^{S} \sum_{a_2=1}^{m_2} \sum_{c=1}^{k} \phi_{u_1 u_2 6 j a_2 c}.$$

- Probability of the server is in closedown process

$$P_{cd} = \sum_{u_1=0}^{\infty} \sum_{u_2=0}^{N} \sum_{j=0}^{S} \sum_{c=1}^{k} \phi_{u_1 u_2 7 j c}.$$

- The rate of effective retrials

$$\mathfrak{R} = \chi \sum_{u_2=1}^{N} \sum_{j=1}^{S} \sum_{c=1}^{k} \phi_{0 u_2 1 j c}$$

6. Analysis of Cost function

We have assumed that every cost factor (per unit of time) correlates to a distinct system measure while developing the expense function for our model.

- E_I - The cost of inventory for retaining each unit of goods.
- E_{H_1} - Keeping a HP customer 's cost in the system for each unit of time.
- E_{H_2} - Keeping a LP customer 's cost in the system for each unit of time.
- E_S - Initial costs for each order.

$$TC(s, S) = E_I E_{inv} + E_{H_1} E_{sys} + E_{H_2} E_{orb} + E_S E_R$$

7. Numerical Results

Using both numerical and graphical illustrations, we will be studying the behavior of the models in the section that follows. The next three are various MAP representations with the same mean value of 1 across all arrival processes. Chakravarthy [8] used these three arrival value sets as input data in their literature.

- **A-ER(Arrival in Erlang):**

$$D_0 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix} \quad D_1 = \begin{bmatrix} 0 & 0 \\ 1.2 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 0 \\ 0.8 & 0 \end{bmatrix}.$$

- **A-EX(Arrival in Exponential):**

$$D_0 = \begin{bmatrix} -1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.6 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.4 \end{bmatrix}.$$

- **A-HE(Arrival in Hyper-exponential):**

$$D_0 = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1.026 & 0.114 \\ 0.1026 & 0.0114 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.684 & 0.076 \\ 0.0684 & 0.0076 \end{bmatrix}.$$

The service, vacation, and repair processes each have three distinct phase-type distributions that we should take into consideration. We will use the notations X-ER, X-EX and X-HE respectively for Erlang, exponential and hyper-exponential cases dealing with X-type distribution where $X = S, V, R$ depending on whether the services, vacations or repairs are under consideration.

- **X-ER(Erlang):**

$$\gamma = \nu = \beta = \alpha = (1, 0), \qquad P = M = W = U = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}.$$

- **X-EX(Exponential):**

$$\gamma = \nu = \beta = \alpha = [1], \qquad P = M = W = U = [1].$$

- **X-HE(Hyper-exponential):**

$$\gamma = \nu = \beta = \alpha = (0.8, 0.2), \qquad P = M = W = U == \begin{bmatrix} -2.8 & 0 \\ 0 & -0.28 \end{bmatrix}.$$

7.1. Illustrative Example 1

We explored the effects of repair rate (η) versus the average size of HP customers in the system(E_{sys}). In order to attain system stability, we fix $\lambda = 1$, $\zeta_1 = 10$, $\zeta_2 = 8$, $\psi = 8$, $\sigma = 1$, $\tau = 5$, $\chi = 4$, $\delta = 4$, $\theta = 0.6$, $s = 3$, $S = 6$, $N = 5$.

- We combine the arrival and service time categories in Tables 1 through 3 to investigate the repair rate versus the average size of HP customers in the system.
- When the repair rate (η) rises, the corresponding the average size of HP customers in the system(E_{sys}) reduces.
- When comparing arrival times to all other arrivals, the E_{sys} drops quickly for hyper-exponential arrivals, and slowly for Erlang arrivals. Similarly, for service durations, the E_{sys} decreases more slowly in Erlang services than it does with hyper-exponential services.

7.2. Illustrative Example 2

We explored the effects of HP service rate (ζ_1) versus the Total Cost(TC) of the system. In order to attain system stability, we fix $\lambda = 1$, $\zeta_2 = 8$, $\psi = 8$, $\eta = 6$, $\sigma = 1$, $\tau = 5$, $\chi = 4$, $\delta = 4$, $\theta = 0.6$, $s = 3$, $S = 6$, $N = 5$, $E_I = 50$, $E_{H_1} = 200$, $E_{H_2} = 180$, $E_R = 220$.

- We combine the arrival and service time categories in Tables 4 through 6 to investigate the HP service rate versus the total cost of the system
- When the HP service rate (ζ_1) rises, the corresponding the total cost of the system(TC) reduces.
- When comparing arrival times to all other arrivals, the TC drops quickly for hyper-exponential arrivals, and slowly for Erlang arrivals. Similarly, for service durations, the TC decreases more slowly in Erlang services than it does with hyper-exponential services.

7.3. Illustrative Example 3

We explored the effects of retrial rate (χ) versus the average size of LP customers in the orbit(E_{orb}). In order to attain system stability, we fix $\lambda = 1$, $\zeta_1 = 10$, $\zeta_2 = 8$, $\psi = 8$, $\sigma = 1$, $\tau = 5$, $\eta = 6$, $\delta = 4$,

$\theta = 0.6,\ s = 3,\ S = 6,\ N = 5.$

- We combine the arrival and service time categories to investigate the rate of retrial versus the average size of LP customers in the orbit, using Figures 2 through 4.
- When the retrial rate (χ) rises, the corresponding average size of LP customers in the orbit(E_{orb}) reduces.
- When comparing arrival times to all other arrivals, the E_{orbit} drops quickly for hyper-exponential arrivals, and slowly for Erlang arrivals.

Table 1: *Repair rate(η) vs E_{sys} - X-ER*

	X-ER		
η	A-ER	A-EX	A-HE
6.0	0.10813513	0.12190924	0.14286803
6.5	0.10786026	0.12152199	0.14226519
7.0	0.10763579	0.12120252	0.14176864
7.5	0.10744949	0.12093492	0.14135332
8.0	0.10729269	0.12070781	0.14100134
8.5	0.10715912	0.12051288	0.14069960
9.0	0.10704411	0.12034388	0.14043834
9.5	0.10694416	0.12019608	0.14021010
10.0	0.10685657	0.12006582	0.14000914
10.5	0.10677924	0.11995019	0.13983096

Table 2: *Repair rate(η) vs E_{sys} - X-EX*

	X-EX		
η	A-ER	A-EX	A-HE
6.0	0.10901178	0.12413720	0.14698476
6.5	0.10867336	0.12369032	0.14629182
7.0	0.10839815	0.12332339	0.14572339
7.5	0.10817062	0.12301736	0.14524978
8.0	0.10797985	0.12275870	0.14484984
8.5	0.10781790	0.12253750	0.14450816
9.0	0.10767894	0.12234641	0.14421327
9.5	0.10755857	0.12217984	0.14395643
10.0	0.10745340	0.12203347	0.14373094
10.5	0.10736082	0.12190393	0.14353155

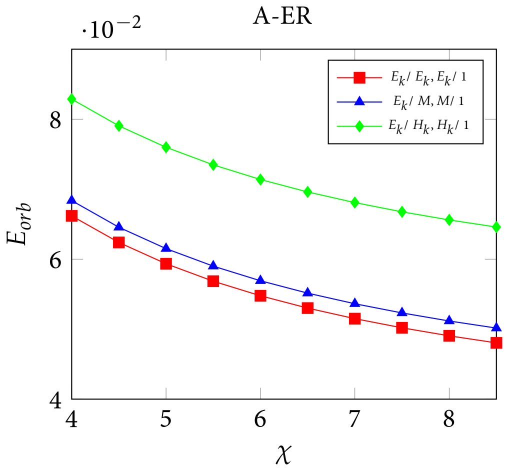

Figure 2: *Retrial rate(χ) vs E_{orb} - A-ER*

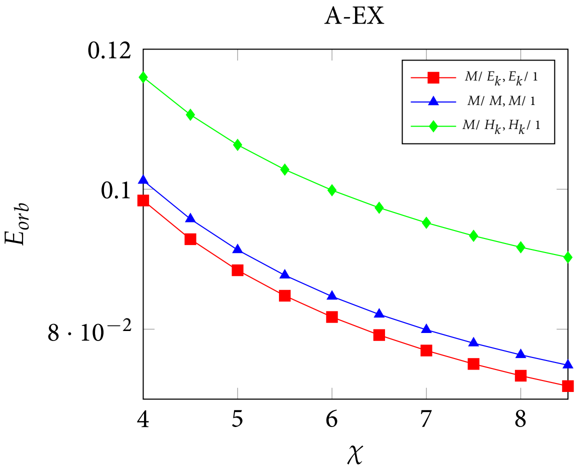

Figure 3: *Retrial rate(χ) vs E_{orb} - A-EX*

Table 3: *Repair rate(η) vs E_{sys} - X-HE*

	X-HE		
η	A-ER	A-EX	A-HE
6.0	0.11918175	0.13742625	0.16861900
6.5	0.11850292	0.13665603	0.16748394
7.0	0.11795708	0.13603251	0.16656203
7.5	0.11751074	0.13551942	0.16580136
8.0	0.11714044	0.13509118	0.16516511
8.5	0.11682931	0.13472934	0.16462655
9.0	0.11656496	0.13442027	0.16416588
9.5	0.11633814	0.13415374	0.16376814
10.0	0.11614181	0.13392192	0.16342189
10.5	0.11597051	0.13371874	0.16311819

Table 4: *HP service rate(ζ_1) vs TC - X-ER*

	X-ER		
ζ_1	A-ER	A-EX	A-HE
10	287.97326332	298.59933362	316.43100759
11	287.74939522	298.08848910	315.15693318
12	287.58577777	297.70317492	314.18547489
13	287.46248681	297.40469160	313.42429235
14	287.36716153	297.16827941	312.81429994
15	287.29182156	296.97748060	312.31615614
16	287.23113722	296.82099764	311.90277542
17	287.18144377	296.69085871	311.55497162
18	287.14015651	296.58130309	311.25881669
19	287.10541130	296.48808035	311.00398146

Table 5: *HP service rate(ζ_1) vs TC - X-EX*

	X-EX		
ζ_1	A-ER	A-EX	A-HE
10	288.98571417	299.98654358	318.45145147
11	288.70159130	299.40909492	317.07970441
12	288.49311376	298.97229601	316.03161949
13	288.33556222	298.63313169	315.20927312
14	288.21349742	298.36397016	314.54968265
15	288.11689338	298.14638167	314.01074509
16	288.03902055	297.96767769	313.56338397
17	287.97523158	297.81888191	313.18695269
18	287.92223650	297.69349308	312.86643407
19	287.87765496	297.58670481	312.59067408

Table 6: *HP service rate(ζ_1) vs TC - X-HE*

	X-HE		
ζ_1	A-ER	A-EX	A-HE
10	295.87738733	307.40357235	328.41423547
11	295.31869946	306.58861243	326.77655347
12	294.89650447	305.96029062	325.50031090
13	294.56917993	305.46410267	324.48262461
14	294.30990735	305.06435501	323.65537379
15	294.10075941	304.73681683	322.97187661
16	293.92937439	304.46453218	322.39919670
17	293.78699896	304.23532420	321.91350697
18	293.66729077	304.04024987	321.49719185
19	293.56556118	303.87261147	321.13697796

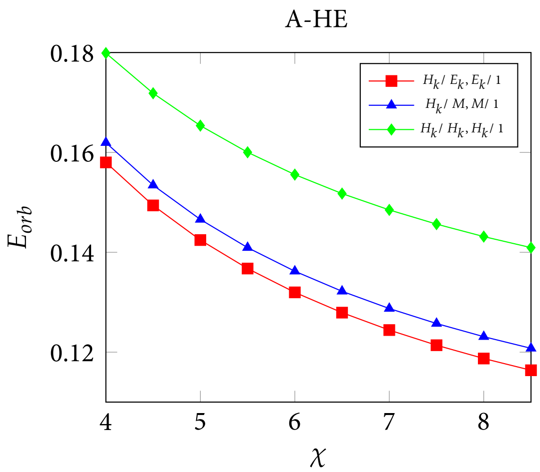

Figure 4: *Retrial rate(χ) vs E_{orb} - A-HE*

8. Conclusion

The present study investigated a retrial inventory queuing system that incorporates MMAP arrivals for HP and LP customers, services, vacations, and repairs, all of which follow phase-type distribution, (s, S) replenishment inventory policy, working breakdown, and closedown. We examined the system's stability criteria as well as the invariant probability vector. We analyzed the active period and also offered cost evaluations and system performance measures. Employing numeric values of arrivals and services in this model, we computed the average size of HP customers in the system for different values of repair rate and the total cost of the system for different values of service rate. The two-dimensional plots show the average size of LP customers in the orbit for different values of retrial rate. The average size of HP customers in the system for various values of vacation and service rates is depicted in the three-dimensional graphs. Every table and graph shows the stability of the system. We also expand our research to include multi-servers with two commodity inventory queueing systems.

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