COSINE MARSHAL-OLKIN-G FAMILY OF DISTRIBUTION: PROPERTIES AND APPLICATIONS

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Abstract

Trigonometric distributions have recently been emphasized due to it applicability and relevance for modeling different phenomena. This article contributes to the existing literature on trigonometric family by introducing and investigating new trigonometric family of distribution which is developed by compounding the cosine family of distribution with Marshall-olkin family of distribution to form a new Cosine Marshall-Olkin family of distribution (CMO). Graphical, numerical and analytical approach was explored to study the properties and applicability of the new CMO family of distribution. Special representations and important reliability properties and other statistical properties were defined. Simulation study was conducted in order to have an insight on the estimates of the three parameters model using maximum products of spacing (MPS). Emphases on the greater flexibility of the new CMO family of distribution beyond the cosine-G family and other top models of the Cosine related family was made through Weibull distribution. The results revealed the superiority of the Cosine Marshall-Olkin Weibull model (CMO-W) over others via two data sets.

Keywords: Cosine-G family, Marshall-Olkin-G family, Maximum Products of Spacing, Hazard function, Survival function.

I. Introduction

Recently, many authors have introduced various approaches to develop flexible continuous distributions from classical continuous distributions. The statisticians' attentions have been drawn to various applications of these continuous distributions in environment, physics, medicine, biology, finance, insurance, engineering and economy to mention few. The classical distributions are induced by adding parameter(s) to enhance the asymmetry, kurtosis, tails properties, central and dispersion parameters. This idea is considered as generalization of the classical distributions. These generalized distributions belong to particular families defined by transformation of the baseline cumulative distribution function (cdf). The values of the newly introduced parameter(s) can enhance the statistical capacities of the baseline distribution. for instance, families such as Weibull-G [1], Exp-G [2], Topp-Leone generated (TL-G) [3] Type I Half Logistic-G [4], new power

TL-G [5], Type II half Logistic-G [6], truncated inverted Kumaraswamy-G [7], a new alpha power transformed-G [8], a new extended alpha power transformed-G [9], type II power TL-G [10], Odd Beta prime-G [11].

A recent approach involves defining families of distributions by using the trigonometric transformation, be it parametric or not. Kumar et al. [12] and Souza [13] launched this trigonometric family exploring the use of the sine function, resulting to the sine-G family. The [14] and [15] extended the exponential and weibull distribution through sine-G family. The non trigonometric compounding families of distributions seen in the literature include but not limited to [16], [17], it extension is found in [18], [19]. The trigonometric compounded families include [20], [21], [22], [23], [24], [25] [26], [27], [28], [29], [30], [31], [32], [33].

The Marshall-olkin-G family of distribution was proposed by [34] and it was used to extended flexibility of Exponential and weibull distribution The Cosine –G family of distribution was proposed by [35]. Now, this article intends to compound the two families to form a new family of distribution called Cosine Marshall-olkin-G family of distribution.

The motivations behind CMO-G family are to develop models with improved shapes for the pdf and hazard function, improve symmetrical and asymmetrical distributions, construct heavy-tailed distributions, improve the flexibility of the baseline model through skewness, kurtosis, mean and variance, provide better fits than other Cosine family of distribution with the same baseline distribution and possibly with the same number of parameters and more complexity.

II. Methods

2.1 The Marshal-Olkin-G Family of Distribution

Definition 1: Suppose $X \sim MO\left(x;\theta,\xi\right)$ with corresponding cdf and pdf given by:

$$H_{MO}\left(x;\theta,\xi\right)=\frac{G\left(x;\xi\right)}{\theta+\left(1-\theta\right)G\left(x;\xi\right)} \tag{1}$$

and

$$h_{MO}\left(x;\theta,\xi\right)=\frac{\theta g\left(x;\xi\right)}{\left(\theta+\left(1-\theta\right)G\left(x;\xi\right)\right)^{2}} -\infty<x<\infty \text{ where } \theta>0 \text{, and it is a shape parameter} \tag{2}$$

2.2 The Cosine-G Family of Probability Distribution

Definition 2: Suppose $X \sim COS\left(x;\Psi\right)$ with corresponding cdf and pdf given by:

$$F\left(x;\Psi\right)=1-\cos\left[\frac{\pi}{2}H\left(x\right)\right] \tag{3}$$

and

$$f\left(x;\Psi\right)=\frac{\pi}{2}h\left(x\right)sin\left[\frac{\pi}{2}H\left(x\right)\right] \tag{4}$$

2.3 The proposed Cosine Marshal Olkin-G family of distribution

Definition 3: Suppose $X \sim CMO(x;\theta,\xi)$ with cdf expressed below, where $\theta > 0$ and θ is a shape parameter and ξ is a baseline vector parameter is defined as the Cosine Marshal-Olkin-G Family

$$F_{CMO}(x;\theta) = 1 - \cos\left[\frac{\pi}{2}\left(\frac{G(x)}{\theta + (1-\theta)G(x)}\right)\right] \tag{5}$$

It is important to note that for any baseline distribution, signified as $G(x)$, *CMO* cdf satisfy the following;

a. $g(x) = \frac{dG(x)}{dx}$

b. $\int_0^\infty g(x)dx = 1$

c. The survival function $1 - G(x)$

Definition 4: Suppose $X \sim CMO(x;\theta,\xi)$ with pdf expressed below, where $\theta > 0$ and θ is a shape parameter and ξ is a baseline vector parameter is defined as the Cosine Marshal-Olkin-G Family

$$f_{CMO}(x;\theta) = \frac{\pi}{2}\frac{\theta g(x)}{\left(\theta + (1-\theta)G(x)\right)^2} sin\left[\frac{\pi}{2}\left(\frac{G(x)}{\theta + (1-\theta)G(x)}\right)\right] \tag{6}$$

2.4 Special Representation

The pdf of the proposed Cosine Marshall-olkinG family can be expanded using the tailor series and binomial expansion; thus

$$f_{CMO}(x;\theta) = \frac{\pi}{2}\frac{\theta g(x)}{\left(\theta + (1-\theta)G(x)\right)^2} sin\left[\frac{\pi}{2}\left(\frac{G(x)}{\theta + (1-\theta)G(x)}\right)\right]$$

$$sin\left[\frac{\pi}{2}\left(\frac{G(x)}{\theta + (1-\theta)G(x)}\right)\right] = \sum_{i=0}^{\infty}\frac{(-1)^i}{(2i+1)!}\frac{\pi^{2i+1}}{2^{2i+1}}G(x)^{2i}\left(\theta + (1-\theta)G(x)\right)^{-2i}$$

Consider $\left(\theta + (1-\theta)G(x)\right)^{-2i}$ and $\left(\theta + (1-\theta)G(x)\right)^{-2}$

$$\left(\theta + (1-\theta)G(x)\right)^{-2(i+1)} = \sum_{j=0}^{\infty}\theta^j(-1)^j\begin{pmatrix}2(i+1)-1+j\\ j\end{pmatrix}(1-\theta)^j G(x)^{2ij}$$

$$f_{CMO}(x;\theta) = \sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}}{(2i+1)!}\frac{\pi^{2i+2}}{2^{2i+2}}\begin{pmatrix}2(i+1)-1+j\\ j\end{pmatrix}\theta^{j+1}(1-\theta)^j g(x)G(x)^{2ij}$$

Hence the expansion of the pdf is expressed as

$$f_{CMO}\left(x;\theta\right)=\sum_{i,j=0}^{\infty}\Psi_{i,j}\,g\left(x\right)G\left(x\right)^{2ij} \tag{7}$$

Where

$$\Psi_{i,j}=\frac{\left(-1\right)^{i+j}}{\left(2i+1\right)!}\frac{\pi^{2i+2}}{2^{2i+2}}\binom{2\left(i+1\right)-1+j}{j}\theta^{j+1}\left(1-\theta\right)^{j}$$

The cdf can also be expanded as follows:

$$F_{CMO}\left(x;\theta\right)=1-cos\left[\frac{\pi}{2}\left(\frac{G\left(x\right)}{\theta+\left(1-\theta\right)G\left(x\right)}\right)\right]=1-\sum_{k=0}^{\infty}\frac{\left(-1\right)^{k}}{\left(2k\right)!}\frac{\pi^{2k}}{2^{2k}}\left(\frac{G\left(x\right)}{\theta+\left(1-\theta\right)G\left(x\right)}\right)^{2k}$$

$$=1-\sum_{k=0}^{\infty}\frac{\left(-1\right)^{i}}{\left(2k\right)!}\frac{\pi^{2k}}{2^{2k}}G\left(x\right)^{2k}\left(\theta+\left(1-\theta\right)G\left(x\right)\right)^{-2k}$$

$$\left(\theta+\left(1-\theta\right)G\left(x\right)\right)^{-2k}=\sum_{l=0}^{\infty}\theta^{l}\left(-1\right)^{l}\binom{2\left(k+1\right)-1+l}{l}\left(1-\theta\right)^{l}G\left(x\right)^{l}$$

$$F_{CMO}\left(x;\theta\right)=1-\sum_{k,l=0}^{\infty}\frac{\left(-1\right)^{k+l}}{\left(2k\right)!}\frac{\pi^{2k}}{2^{2k}}\binom{2\left(k+1\right)-1+l}{l}\theta^{l}\left(1-\theta\right)^{l}G\left(x\right)^{2k+l}$$

Therefore,

$$F_{CMO}\left(x;\theta,\xi\right)=1-\sum_{k,l=0}^{\infty}\Phi_{k,l}G\left(x\right)^{2k+l} \tag{8}$$

Where

$$\Phi_{k,l}=\frac{\left(-1\right)^{k+l}}{\left(2k\right)!}\frac{\pi^{2k}}{2^{2k}}\binom{2\left(k+1\right)-1+l}{l}\theta^{l}\left(1-\theta\right)^{l}$$

Definition 5: Suppose $X\sim CMO\left(x;\theta,\xi\right)$ with cdf and pdf well defined, where $\theta>0$ and is a shape parameter and ξ is a baseline vector parameter. Then the survival function of X, signified by $SF_{CMO}\left(x;\theta,\xi\right)=1-F_{CMO}\left(x;\theta\right)$, the survival function for the *CMO* family of distribution, can be represented by $SF_{CMO}\left(x;\theta,\xi\right)=\cos\left[\frac{\pi}{2}\left(\frac{G\left(x\right)}{\theta+\left(1-\theta\right)G\left(x\right)}\right)\right]$

Definition 6: Suppose $X\sim CMO\left(x;\theta,\xi\right)$ with cdf and pdf well defined, where $\theta>0$ and θ is a shape parameter and ξ is a baseline vector parameter. Then the hazard rate function of X, signified by $HRF_{CMO}\left(x;\theta,\xi\right)=f_{CMO}\left(x;\theta,\xi\right)/SF_{CMO}\left(x;\theta\right)$, the hazard rate function for the *CMO* family of distribution, can be represented by

$$HRF_{CMO}\left(x;\theta,\xi\right)=\frac{\pi}{2}\frac{\theta g\left(x\right)}{\left(\theta+\left(1-\theta\right)G\left(x\right)\right)^{2}}\tan\left[\frac{\pi}{2}\left(\frac{G\left(x\right)}{\theta+\left(1-\theta\right)G\left(x\right)}\right)\right] \tag{10}$$

Definition 7: Suppose $X \sim CMO(x;\theta,\xi)$ with cdf and pdf well defined, where $\theta > 0$ and θ is a shape parameter and ξ is a baseline vector parameter. Then the Qunatile function of X, signified by
$QF_{CMO}(x;\theta,\xi) = F_{CMO}^{-1}(x;\theta,\xi)$, the Qunatile function for the *CMO* family of distribution can be obtained as follows:

$$\mathrm{u} = 1-\cos\left[\frac{\pi}{2}\left(\frac{G(x)}{\theta+(1-\theta)G(x)}\right)\right],\ u \in (0,1),$$

$$\Phi(\mathrm{u}) = \mathrm{G}^{-1}\left\{\frac{\theta\theta\cos^{-1}(1-u)}{\frac{\pi}{2}-(1-\theta)\theta\cos^{-1}(1-u)}\right\} \tag{11}$$

Definition 8: Suppose $X \sim CMO(x;\theta,\xi)$ with cdf and pdf well defined, where $\theta > 0$ and θ is a shape parameter and ξ is a baseline vector parameter. Then the r^{th} Moments of X can be obtained as follow

$$\mu_r^{'} = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

$$\mu_r^{'} = \sum_{i,j=0}^{\infty} \Psi_{\mathrm{i,j}} \int_0^{\infty} x^r g(x) G(x)^{2ij}\,dx$$

$$\mu_r^{'} = \sum_{i,j=0}^{\infty} \Psi_{\mathrm{i,j}}\,\Phi \tag{12}$$

where

$$\Phi = \int_0^{\infty} x^r g(x) G(x)^{2ij}\,dx$$

Definition 9: Suppose $X \sim CMO(x;\theta,\xi)$ with cdf and pdf well defined, where $\theta > 0$ and θ is a shape parameter and ξ is a baseline vector parameter. The r^{th} Moment generating function of X is obtained through

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

Thus, the moment generating function of the Cosine Marshall-olkin-G family of distribution is given by:

$$M_x(t) = \sum_{i,j=0}^{\infty} \Psi_{\mathrm{i,j}} \int_0^{\infty} e^{tx}\, g(x) G(x)^{2ij}\,dx$$

$$M_x(t) = \sum_{i,j=0}^{\infty} \Psi_{\mathrm{i,j}}\,\Upsilon \tag{13}$$

where

$$\Upsilon = \int_0^{\infty} e^{tx}\, g(x) G(x)^{2ij}\,dx$$

Definition 10: Suppose $X \sim CMO(x;\theta,\xi)$ with cdf and pdf well defined, where $\theta > 0$ and θ is a shape parameter and ξ is a baseline vector parameter. The entropy is obtained as given below

$$I_\theta(x) = \frac{1}{1-\theta}\log\int_0^{\infty} f(x)^\theta\,dx$$

$$f(x)^\theta = \left(\sum_{i,j=0}^{\infty} \Psi_{i,j}\, g(x) G(x)^{2ij}\right)^\theta$$

$$f(x)^{\theta} = \left(\sum_{i,j=0}^{\infty}\Psi_{i,j}\right)^{\theta}\left(g(x)G(x)^{2ij}\right)^{\theta}$$

Let $\omega = g(x)G(x)^{2ij}$

Therefore,

$$f(x)^{\theta} = \left(\sum_{i,j=0}^{\infty}\Psi_{i,j}\right)^{\theta}\omega^{\theta} \tag{14}$$

$$I_{\theta}(x) = \frac{1}{1-\theta}\left[\left(\sum_{i,j=0}^{\infty}\Psi_{i,j}\right)^{\theta}\int_{0}^{\infty}\omega^{\theta}\,dx\right]$$

2.5 Cosine Marshall-olkinWeibull Distribution

Supposed the baseline distribution is Weibull distribution with cdf and pdf given by:

$$G(x) = 1 - e^{-\left(\frac{x}{\lambda}\right)^{\alpha}} \tag{15}$$

$$g(x) = \frac{\alpha}{\lambda}\left(\frac{x}{\lambda}\right)^{\alpha-1} e^{-\left(\frac{x}{\lambda}\right)^{\alpha}} \tag{16}$$

Where α is a shape parameter and λ is a scale parameter, then the cumulative distribution, probability distribution, hazard and survival function of the Cosine Marshall-olkinWeibull (CMO-W) distribution is given as: $F_{CMO}(x;\theta) = 1 - \cos\left[\frac{\pi}{2}\left(\frac{1-e^{-\left(\frac{x}{\lambda}\right)^{\alpha}}}{\theta + (1-\theta)\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)}\right)\right]$ (17)

and the associated pdf is given as:

$$f_{CMO}(x;\theta) = \frac{\frac{\pi}{2}\frac{\theta\alpha}{\lambda}\left(\frac{x}{\lambda}\right)^{\alpha} e^{-\left(\frac{x}{\lambda}\right)^{\alpha}}}{\left[\theta + (1-\theta)\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)\right]^{2}}\sin\left\{\frac{\pi}{2}\left[\frac{1-e^{-\left(\frac{x}{\lambda}\right)^{\alpha}}}{\left(\theta + (1-\theta)\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)\right)}\right]\right\}$$

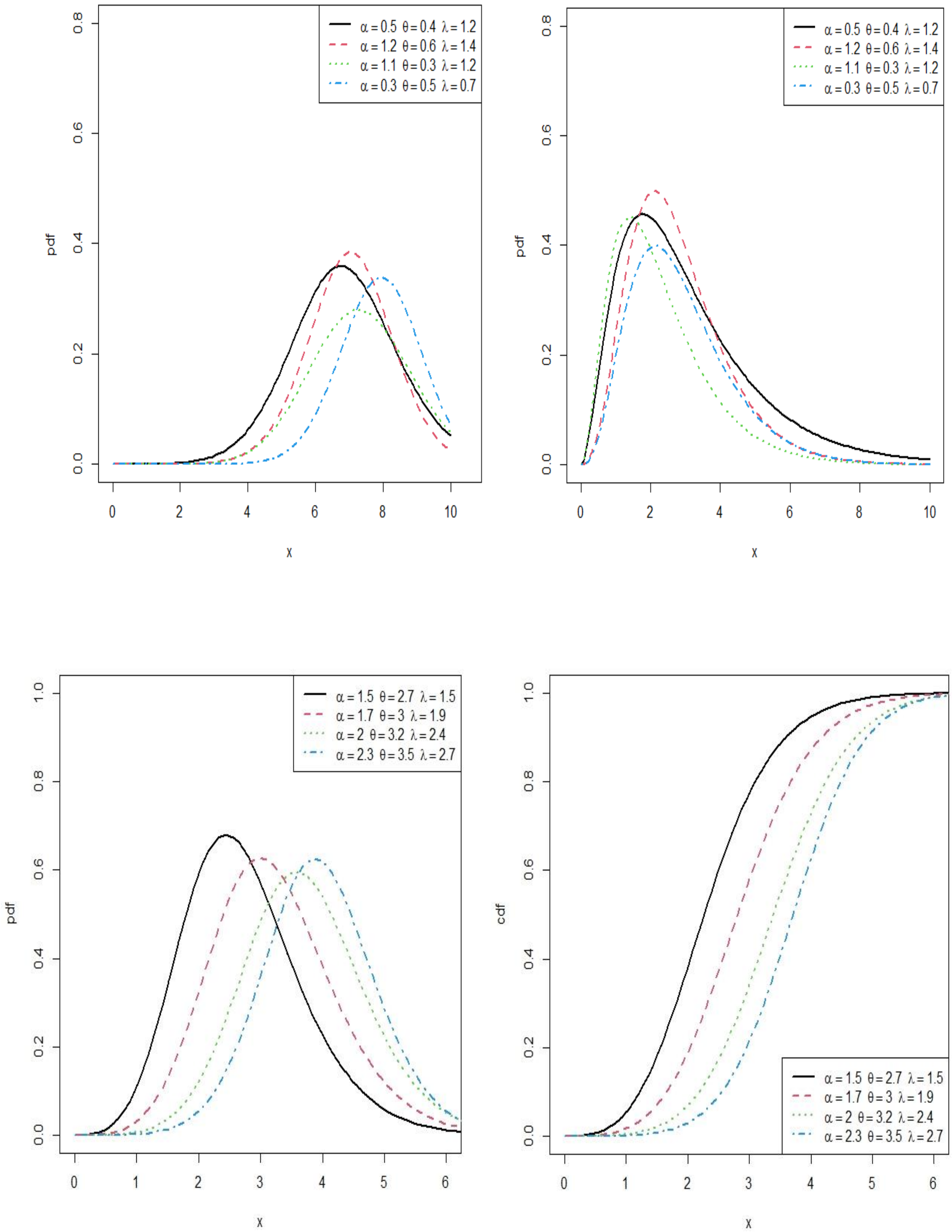

Figure 1: *Plots of pdf and cdf of CMO-W distribution*

The figure 1 above reveals left skewness, right skewness and approximately symmetric pdf shapes. The cdf shape converges to one, validating the CMO-W distribution.

The Hazard and reliability function of the CMO-W distribution is obtained as:

$$HRF_{CMO}\left(x;\theta,\xi\right)=\frac{\pi\theta\left(\frac{\alpha}{\lambda}\left(\frac{x}{\lambda}\right)^{\alpha-1}e^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)}{2\left(\theta+\left(1-\theta\right)\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)\right)^{2}}\tan\left[\frac{\pi}{2}\left(\frac{1-e^{-\left(\frac{x}{\lambda}\right)^{\alpha}}}{\theta+\left(1-\theta\right)\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)}\right)\right] \tag{19}$$

and

$$SF_{CMO}\left(x;\theta,\xi\right)=\cos\left[\frac{\pi}{2}\left(\frac{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)}{\theta+\left(1-\theta\right)\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\right)}\right)\right] \tag{20}$$

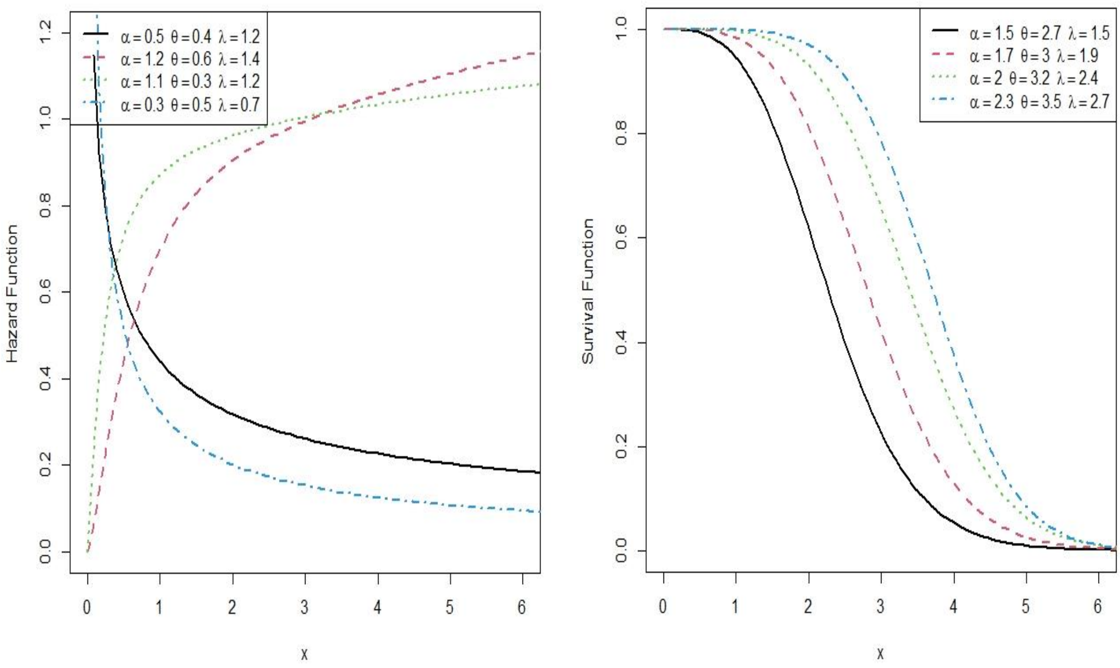

Figure 2: *Plots of hazard and reliability function of CMO-W distribution*

The figure 2 above reveals the shapes of the hazard and reliability function the hazard shapes obviously shows increasing and decreasing failure rate, and the reliability shapes shows a drop from one to zero with varying values of parameters

III. Results

3.1 Simulation study

In this section, we provide, we provides the simulation of parameters of the CMO-W distribution using Maximum products of spacing estimation method. Random numbers were systematically generated from fixed values of the parameters $\theta = 0.5, \lambda = 2, \alpha = 1$, $\theta = 0.7, \lambda = 2.2, \alpha = 1$ and $\theta = 0.6, \lambda = 2.3, \alpha = 1$ and $\theta = 0.8, \lambda = 2.1, \alpha = 1$ based on 10,000 replications. The sample sizes (n) considered are 20, 50, 100, 250, 500 and1000. The result is displayed in Table 1 and Table 2

Table 1: The MPSs parameter estimates (Est. value), Biases and RMSEs of various parameters values

n	Parameters	Est. value	Bias	RMSE	Est. value	Bias	MSE
20	θ	0.6195	0.1195	0.4675	0.8346	0.1346	0.5855
	λ	2.1325	0.1325	0.8886	2.3199	0.1199	0.9108
	α	0.9536	-0.0464	0.2357	0.9563	-0.0437	0.2327
50	θ	0.5984	0.0984	0.3570	0.8354	0.1354	0.5037
	λ	2.0474	0.0474	0.7208	2.2352	0.0352	0.7881
	α	0.9581	-0.0419	0.1652	0.9608	-0.0392	0.1722
100	θ	0.5719	0.0719	0.2906	0.7981	0.0981	0.3977
	λ	2.0375	0.0375	0.6079	2.2232	0.0232	0.6503
	α	0.9703	-0.0297	0.1247	0.9712	-0.0288	0.1281
250	θ	0.5572	0.0572	0.2226	0.7788	0.0788	0.3004
	λ	2.0014	0.0014	0.4941	2.2114	0.0114	0.5370
	α	0.9786	-0.0214	0.0970	0.9787	-0.0213	0.1010
500	θ	0.5318	0.0318	0.1674	0.7592	0.0592	0.2422
	λ	2.0092	0.0092	0.3890	2.2032	0.0032	0.4317
	α	0.9873	-0.0127	0.0745	0.9832	-0.0168	0.0801
1000	θ	0.5197	0.0197	0.1254	0.7326	0.0326	0.1810
	λ	2.0041	0.0041	0.2914	2.2011	0.0010	0.3362
	α	0.9918	-0.0082	0.0547	0.9904	-0.0096	0.0608

Table 2: The MPSs parameter estimates (Est. value), Biases and RMSEs of various parameters values

n	Parameters	Est. value	Bias	RMSE	Est. value	Bias	RMSE
20	θ	0.7259	0.1259	0.5302	0.9459	0.1459	0.6535
	λ	2.4416	0.1416	0.9796	2.2125	0.1125	0.8606
	α	0.9567	-0.0433	0.2366	0.9607	-0.0393	0.2402
50	θ	0.7174	0.1174	0.4218	0.9525	0.1525	0.5519
	λ	2.3376	0.0376	0.8233	2.1128	0.0128	0.7333
	α	0.9593	-0.0407	0.1690	0.9588	-0.0412	0.1711
100	θ	0.6863	0.0863	0.3426	0.9240	0.0981	0.1240
	λ	2.3231	0.0231	0.6725	2.1063	0.0232	0.0063
	α	0.9703	-0.0297	0.1253	0.9698	-0.0288	-0.0302
250	θ	0.0722	0.0722	0.0722	0.9008	0.1008	0.3486
	λ	-0.0181	-0.0181	-0.0181	2.0640	-0.0360	- 0.4893
	α	-0.0226	-0.0226	-0.0226	0.9752	0.0248	0.1006
500	θ	0.6444	0.0444	0.2047	0.8628	0.0628	0.2716
	λ	2.2928	-0.0072	0.4473	2.0824	-0.0176	0.4125
	α	0.9851	-0.0149	0.0772	0.9840	-0.0160	0.0808
1000	θ	0.6300	0.0300	0.1558	0.8433	0.0433	0.2073
	λ	2.2845	-0.0155	0.3418	2.0783	-0.0217	0.3206
	α	0.9897	-0.0103	0.0585	0.9886	-0.0114	0.0624

3.2 Applications

Application of the CMO-W distribution to two real life data sets are provided and revealing it applicability in practice along with comparison with its comparators. The proposed Cosine Marshall-olkin-Weibull distribution (CMO-W) is compared with four other Cosine extended Weibull distributions, namely: Cosine Topp–Leone Weibull (CTL-W) distribution [36], Extended Cosine Weibull (ECS-W) distribution [37], New Alpha Power Cosine-Weibull (NACos-W) distribution [38] and Cosine Weibull (C-W) distribution [39].

The information criteria explored to investigate the goodness-of- fit of the distribution appropriate for the data are Akaike's Information Criterion (AIC), Consistent Akaike's Information Criterion (CAIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC). The computation can be seen as follows

$$\text{AIC} = -2\ell + 2p,$$

$$\text{CAIC} = -2\ell + \frac{2np}{n-p-1},$$

$$BIC = -2\ell + p\log(n),$$

$$\text{HQIC} = -2\ell + 2p\log(\log(n)),$$

where ℓ is the maximized log likelihood of the parameter vector $\Omega = (\theta, \lambda, \alpha)$, p is the number of estimated parameters and n is the number of observations. The best fitted model is selected based on minimum value obtained through the information criteria measures.

Dataset 1:
"The data set shown below represents the civil engineering data with 85 hailing times, previously used by Kotz and Dorp (2004):"
4.79, 4.75, 5.40, 4.70, 6.50, 5.30, 6.00, 5.90, 4.80, 6.70, 6.00, 4.95, 7.90, 5.40, 3.50, 4.54, 6.90, 5.80, 5.40, 5.70, 8.00, 5.40, 5.60, 7.50, 7.00, 4.60, 3.20, 3.90, 5.90, 3.40, 5.20, 5.90, 4.40, 5.20, 7.40, 5.70, 6.00, 3.60, 6.20, 5.70, 5.80, 5.90, 6.00, 5.15, 6.00, 4.82, 5.90, 6.00, 7.30, 7.10, 4.73, 5.90, 3.60, 6.30, 7.00, 5.10, 6.00, 6.60, 4.40, 6.80, 5.60, 5.90, 5.90, 8.60, 6.00, 5.80, 5.40, 6.50, 4.80, 6.40, 4.15, 4.90, 6.50, 8.20, 7.00, 8.50, 5.90, 4.40, 5.80, 4.30, 5.10, 5.90, 4.70, 3.50, 6.80.

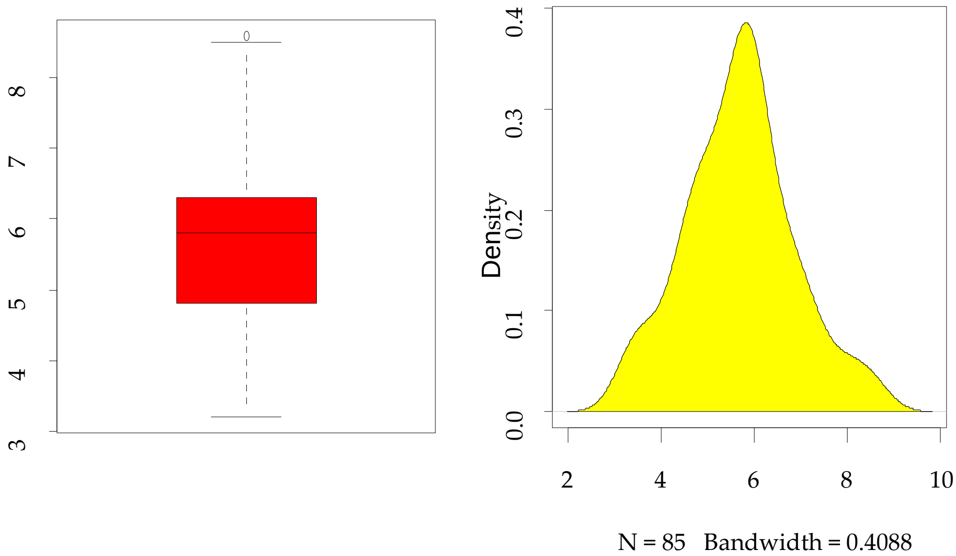

Figure 3: The boxplot and kernel density of the data set 1

Dataset 2:
"The data set shown below represents the strength of carbon fibers tested under tension at gauge lengths of 10mm, previously used Bi and Gui (2017):"
1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

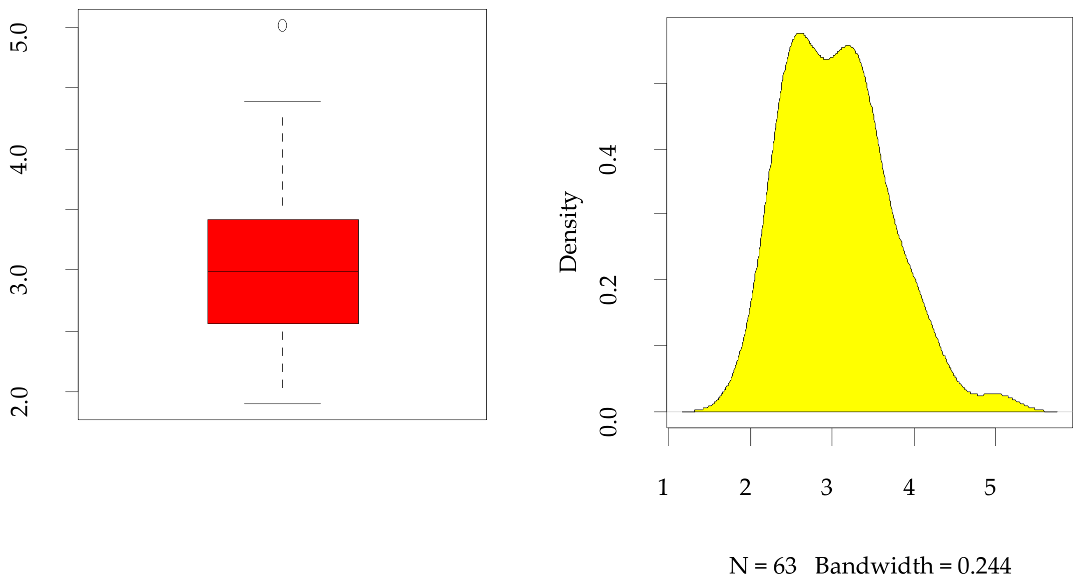

Figure 4: The boxplot and kernel density of the data set 2

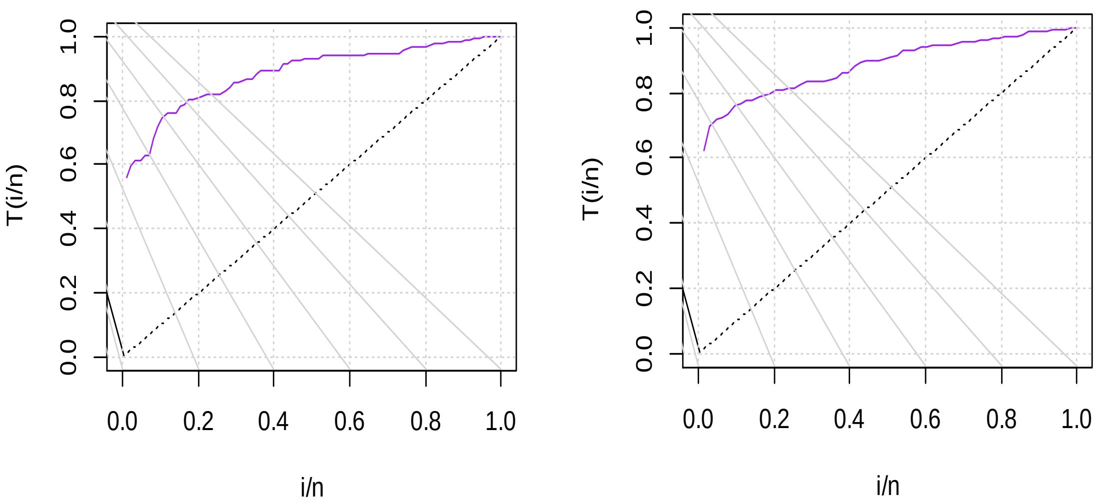

Figure 5: *The TTT plot of data set 1 and 2*

Table 3: MPSs, Log-likelihoods and Goodness of Fits Statistics for the Data Set 1

Distributions	λ	α	θ	β	LL	AIC	CAIC	BIC	HQIC
CMO-W	3.9229	2.2488	2.8840		-110.2788	226.5576	226.8539	233.8856	229.5051
CTL-W	0.0026	1.3745		3.2049	-132.9372	271.8744	272.1707	279.2024	274.8219
ECS-W	3.8988	0.0382	0.7231		-258.3757	522.7514	523.0477	530.0794	525.6989
NACos-W		4.8815	3.1622	0.0026	-193.4512	392.9024	393.1987	400.2304	395.8499
C-W		2.8953	0.0094	0.1471	-138.7963	283.5926	283.8889	290.9206	286.5401

Table 4: MPSs, Log-likelihoods and Goodness of Fits Statistics for the Data Set 2

Distributions	λ	α	θ	β	LL	AIC	CAIC	BIC	HQIC
MO-W	4.4779	6.1830	0.0415		-82.1587	170.3174	170.7242	176.7468	172.8461
CTL-W	0.3398	12.5851		1.4887	-86.6096	179.2192	179.6260	185.6486	181.7479
ECS-W	0.0027	0.9870	4.4644		-84.5562	175.1124	175.5192	181.5418	177.6411
NACos-W		8.1270	0.0128	2.6986	-85.2837	176.5674	176.9742	182.9968	179.0961
C-W		0.5119	6.9721	0.0020	-86.0652	178.1304	178.5372	184.5598	180.6591

IV. Discussion

We introduce a novel Cosine Marshall-Olkin family of distribution and its properties, therein, we extended the Weibull distribution to form a new sub-model known as Cosine Marshall-Olkin Weibull distribution. We conducted a comprehensive study of the new Cosine Marshall-Olkin Weibull distribution properties. Furthermore, we investigate the consistency and efficiency of the estimates obtained from the parameters of the novel distribution. We employ the maximum products of spacing estimation technique, which enabled us to access the values of the parameters effectively. To demonstrate the applicability of the proposed distribution, we provide insights on its performance using two real-life datasets. The analysis reveals that the new model outperforms other trigonometric family of distribution with the same baseline.

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