RELIABILITY ESTIMATION OF STRESS-STRENGTH MODEL USING FUZZY DISTORTION FUNCTION UNDER UNCERTAINTY IN ENVIRONMENTAL FACTORS

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Abstract

In the reliability estimation of stress-strength models, external factors such as temperature, humidity, etc. may influence the distribution of stress and strength random variables. In traditional reliability analysis, these external factors are accounted for by introducing a real-valued distortion function, which replaces the original distribution with a distorted one. However, it's important to note that the effect of these external factors is not always adequately represented by a single real-valued function. To address this issue, we propose the use of fuzzy numbers within the distortion function. In this paper, we introduce the concept of a "fuzzy distortion function" to incorporate the uncertainty stemming from external factors when estimating the reliability of stress-strength relationships. We present a methodology for estimating fuzzy reliability by employing this fuzzy distortion function. Through an illustrative example, we demonstrate how this approach to estimating fuzzy reliability offers a wider range of possibilities for system reliability and provides more comprehensive insights into the system's behaviour. Throughout our exploration, we have delved into the diverse properties inherent in fuzzy distortion functions. These properties highlight the versatility and adaptability of such functions in capturing uncertainty within data sets. Moreover, we have scrutinized several methods for constructing fuzzy distortion functions from pre-existing ones. By examining these methods, we gain valuable insights into how fuzzy distortion functions can be tailored to specific contexts and applications, thereby enhancing the accuracy and robustness of reliability analysis in complex systems. Additionally, in the conventional stress-strength model, reliability is determined without considering the uncertainty in the parameters of the distribution function. The drawback of existing methods in the literature is that they do not consider the uncertainty or fuzziness in the parameters of the distribution. Therefore, we estimate the system reliability in the presence of fuzzy parameters in the distribution function of corresponding random variables. The method we discuss in this paper provides a reliability estimate of the given system under realistic situations. A sensitivity analysis study is carried out to examine the behaviour of mean square errors (MSE) of estimated system reliability under various scenarios. It is observed that MSE can be significantly reduced by a suitable choice of parameters in the membership function of fuzzy parameters.

Keywords: Reliability, Fuzzy reliability, Distortion function, Fuzzy triangular number

1. Introduction

An et al. [1] used the universal generating function (UGF) approach to develop a discrete SSI model. And they handled strength and stress as discrete random variables. Considering a unilateral dependence of strength on stress found in some real-world circumstances. Huang et al. [2] provide a discrete SSI model with SDS based on a UGF approach. This model treats a

structure's SDS as a discrete random variable that, depending on the amplitude of the applied stress, has a different conditional probability mass function (pmf). In his study of reliability of stress-strength for a general coherent system, Eryilmaz [3] offered the exact formula as well as approximations and limitations. The estimation process for exponential stress-strength distributions was further illustrated by the author. In order to estimate the reliability of stress-strength model with multicomponent, Rao et al. [4] showed that the BLUE method of estimation shows the least MSE when compared to exact MLE, Method of Moments and TMMLE.
Kizilaslan [5] considered the system of multi-component with k statistically identical and independently distributed components of strength and each component subjected to a shared random stress. In order to determine the system's reliability in both known and unknown instances for the common second parameter λ, he used both classical and Bayesian methodologies. The system of multicomponent with k uniformly distributed and statistically independent strength components was explored by Kizilaslam [6]. Each element in the system was exposed to a common random stress. When the common scale parameter λ is known in some situations but not in others, the reliability of the system can be further assessed using frequentist and Bayesian approaches.
Dey et al. [7] compared the reliability of Bayes estimators and MLEs with respect to the mean squared errors and the average biases in their study of reliability of the stress-strength model of multicomponent for 238 two parameter Kumaraswamy distribution, then the distribution of strength and stress is the same. When both the stress and strength variations follow the same population, Rao et al. [8] investigated the reliability of stress-strength model with multicomponent for exponentiated Weibull distribution. Additionally, the predicted asymptotic confidence interval for the reliability of several components under stress. For the investigation of structural dependability using Copulas, Zhang et al. [9] presented a stress-strength time-varying correlation interference model.

By assuming that both the strength and stress variables follow a Chen distribution with a common shape parameter that may or may not be known, Tanmay et al. [10] obtained point and interval estimates of reliability of the multi-component stress-strength model of a s-out-of-j system using both Bayesian and classical approaches. The multi-component stress-strength reliability was evaluated by Amal et al. [11] based on the recorded data. When the stress and strength variables follow separate Weibull distributions with distinct scale parameters, the system's dependability is established. When samples are taken from distributions of stress and strength, and measurements are made in terms of upper record values, the reliability in MSS is evaluated using the maximum likelihood and Bayesian techniques of estimation. Amer et al. [12] assessed the reliability when the variables strength and stress are independent and follow the exponentiated Pareto distribution. The simple random sampling (SRS), median ranked set sampling (MRSS) and ranked set sampling (RSS) methods are used to calculate the maximum likelihood estimators in R. In four separate circumstances, the dependability estimate based on MRSS is taken into consideration. When the strength and stress variables are modeled by two separate but not identically distributed random variables from the generalized inverted exponential distributions, Amal et al. [13] assessed reliability of the stress-strength model. When evaluating the stress strength reliability estimator, MRSS is primarily used as opposed to RSS and SRS. A fresh addition to stress-strength models was made by Saber et al. [14]. The extended exponential distribution is used to apply the new model. The asymptotic distribution, the Bayesian estimation and the maximum likelihood estimator are derived. Shubham et al. [15] investigated both conventional and Bayesian techniques of reliability estimation of stress-strength model with multi-component and arrived at a maximum likelihood estimate of dependability. Additionally, the confidence intervals for asymptotic, boot-p, and boot-t data were built. Zhang et al. [16] investigated how well the multi-component stress-strength model, which includes one stress and two associated strength components from a parallel system, could predict dependability.

The rest of the paper is organized as follows. Section 2 gives basic definitions connected with fuzzy numbers and distortion functions. Section 3 introduces the concept of fuzzy distortion function. Section 3.1 describes with the estimation of the fuzzy reliability using fuzzy distortion function and a numerical illustration of this method is presented. Section 3.2 deals with some interesting properties of distortion function. Section 3.3 deals with some basic methods for the construction of distortion function. Section 4 deals with the estimation of reliability of the stress strength model using weighted distributions. In Section 5, we present a sensitivity analysis study to check the behaviour of MSE of reliability estimate.Section 6 concludes with presenting the findings of this study.

2. Some basic definitions

In this section, we now provide some key definitions that are necessary to comprehend the findings in the sections that follow.

Definition 1. (see [17]) A fuzzy set D defined on a set S is a mapping from S to the unit interval $[0,1]$, denoted by

$$D = \{(x, \mu_D(x))\,; \quad x \in S\}$$

or

$$D = \{x, \mu_D(x)\}\,,$$

where $\mu_D(x)$ is the membership function of the set D.

Definition 2. (see [17]) A fuzzy set D defined on the real line $\Re$ is convex if and only if $\forall\, x_1 \in S, \forall\, x_2 \in S$ and $\forall\, \lambda \in [0,1]$, there holds

$$\mu_D\left(\lambda x_1 + (1-\lambda)x_2\right) \geq \min\left(\mu_D\left(x_1\right), \mu_D\left(x_2\right)\right),$$

or equivalently, a fuzzy set is said to be convex if all of its cut sets are convex. If the $\geq$ sign is replaced by $>$ sign, then we say that the fuzzy set is strictly convex.

Definition 3. (see [17]) For any $\alpha \in [0,1]$, an α -cut set of D, denoted by D_α, is a classic set defined by

$$D_\alpha = \{x \in S, \mu_D(x) \geq \alpha\}\,.$$

Obviously, $D_{\alpha_1} \subseteq D_{\alpha_2}$ if $\alpha_1 \geq \alpha_2$.

Definition 4. (see [17]) If the membership function of fuzzy number D is determined by

$$\mu_D(x) = \begin{cases} 0 & ; x \leq a_1, \\ (x-a_1)\,/\,(a_2-a_1) & ; a_1 \leq x \leq a_2, \\ (a_3-x)\,/\,(a_3-a_2) & ; a_2 \leq x \leq a_3, \\ 0 & ; x \geq a_3. \end{cases} \quad x, a_1, a_2, a_3, \in \Re,$$

then D is referred to as a triangular fuzzy number, denoted $D = (a_1, a_2, a_3)$.
Suppose $D = (a_1, a_2, a_3)$. Then

$$D_a = \left[a_1 + \alpha\left(a_2 - a_1\right), a_3 - \alpha\left(a_3 - a_2\right)\right].$$

Definition 5. (see [18]) A function $\nu(u)$ is called a distortion function if the following conditions hold:
(i) $\nu(u)$ is a non-decreasing function on the interval [0, 1],
(ii) $\nu(0) = 0$ and $\nu(1) = 1$,
(iii) except a finite number of points, $\varphi(u) = \frac{d}{du}\nu(u)$ exists on the interval [0, 1].

Definition 6. (see [19]) Let V denote the strength random variable of the system and W denote the stress random variable. If V and W are independent with respective distribution functions G and F, then the traditional stress-strength reliability can be estimated as

$$R = P\{V > W\} = \iint_{v>w} dF(w)dG(v). \tag{1}$$

3. The concept of fuzzy distortion function

In traditional stress-strength model, we are not incorporating the uncertainty in the external factor. So it is necessary to incorporate the uncertainty in the external factor. Consider a traditional stress-strength model, where system reliability is estimated by using a stress-strength relation. The system's stress is represented by the random variable W, which has the cumulative distribution function F. With a cumulative distribution function of G, the random variable V represents the system's strength. Then, the reliability P of the stress-strength model of the system is given by $P = P(W < V)$. The drawback of this model is that the reliability estimate is unrealistic, since the normal working condition of the system is not considered. In other words, the uncertainty in the external factors is not considered. In general, reliability of the system is affected by environmental factors. To make it more clear, let us consider an example: think about creating a bridge in a city. Let W denote the weight stress of the bridge with a distribution of F and V represent the leg strength of a bridge with a distribution of G. With time, environmental elements including vibration, humidity, and high temperatures are exposed to the random strength and random stress. Now it is to be observed that the effect of external factor modelled by distortion function $\nu(.)$ need not be a simple real valued function. In other words, the distortion force can not be considered as a constant force acting on the system at any given time. Hence, one may not obtain the realistic results. Hence, it is necessary to incorporate this vagueness in the external factors. It is not possible to represent the uncertainty in the external factor by a single real valued distortion function. Therefore, we let the distortion function can take fuzzy value also. In that concern we introduce the concept of fuzzy distortion function, which is more practicable.
We are defining the fuzzy distortion function as follows,

Fuzzy distortion function
A function ν from $[0, 1]$ to set of fuzzy numbers in $[0, 1]$, is called fuzzy distortion function if the following conditions hold:
1. For fixed $\alpha \in [0, 1]$, the functions $A_\alpha(u)$ and $B_\alpha(u)$ are non decreasing where A_α and B_α are end points of the α-cut $\nu_\alpha(u) = [A_\alpha(u), B_\alpha(u)]$,
2. $\nu(0)$ is a fuzzy number with 0 having only membership value 1,
3. $\nu(1)$ is a fuzzy number with 1 having only membership value 1,
4. For fixed $\alpha \in [0, 1]$, the functions $A_\alpha(u)$ and $B_\alpha(u)$ are differentiable except a finite number of points on the interval $[0, 1]$.

Then we can have some interesting result for these fuzzy distortion functions.
Theorem The end points of $\alpha-$cuts of fuzzy distortion functions are real distortion function. That is, for fixed $\alpha \in [0, 1]$, the above functions $A_\alpha(u)$ and $B_\alpha(u)$ are real distortion functions
Proof: For Fixed $\alpha \in [0, 1]$, the function $A_\alpha(u)$ is non decreasing. Since $\nu(0)$ is a fuzzy number with 0 having only membership value 1, it shows that $A_\alpha(0) = 0$. Similarly $\nu(1)$ is a fuzzy number with 1 having only membership value, shows that $A_\alpha(1) = 1$. Finally from the definition of fuzzy distortion function, it is clear that $A_\alpha(u)$ is differentiable With the exception of a few points on the range $[0, 1]$. Hence $A_\alpha(u)$ satisfy all the conditions for distortion function.
Similarly one can prove that $B_\alpha(u)$ is a distortion function. Since $\nu(0)$ is a fuzzy number with 0 having only membership value 1, it shows that $B_\alpha(0) = 0$. Similarly $\nu(1)$ is a fuzzy number with 1 having only membership value, shows that $B_\alpha(1) = 1$. Finally from the definition of fuzzy distortion function, it is clear that $B_\alpha(u)$ is differentiable with the exception of a few points on the range $[0, 1]$. Hence $B_\alpha(u)$ satisfy all the conditions for distortion function.

3.1. Estimation of fuzzy reliability using fuzzy distortion function

In traditional stress-strength model, we are not incorporating the uncertainty in the external factor. So it is necessary to incorporate the uncertainty in the external factor. It is not possible to represent the uncertainty in the external factor by a single real valued distortion function. In that

concern we are defining the fuzzy distortion function as follows, which is more reliable.
Let the random variables V and W stand in for the system's strength with a cumulative distribution function of G and stress with a cumulative distribution function of F, respectively. Let $\nu(.)$ be a fuzzy distortion function. Then fuzzy reliability of the system can be estimated as follows.
For fixed $\alpha \in [0,1]$, we get two real valued function $A_\alpha(u)$ and $B_\alpha(u)$. Since both $A_\alpha(u)$ and $B_\alpha(u)$ are real distortion function.
Consider the function $A_\alpha(u)$, then we can estimate the system reliability using the distortion function $A_\alpha(u)$. Let $R_{\alpha,a}$ denote the reliability estimated using the function $A_\alpha(u)$ and is given by

$$R_{\alpha,a} = \int_0^1 A_\alpha\left(F\left(G^{-1}(u)\right)\right) dA_\alpha(u). \tag{2}$$

Let $R_{\alpha,b}$ denote the reliability estimated using the function $B_\alpha(u)$ and is given by

$$R_{\alpha,b} = \int_0^1 B_\alpha\left(F\left(G^{-1}(u)\right)\right) dB_\alpha(u). \tag{3}$$

We can estimate the $\alpha-$ cuts of fuzzy reliability of the system as follows

$$R_\alpha = [R_{\alpha,a}, R_{\alpha,b}]. \tag{4}$$

Similarly for each $\alpha \in [0,1]$, we can estimate the $\alpha-$cut of R.

3.1.1 Illustration

Let the strength of a bridge leg be represented by the random variable V with distribution G. The bridge's tension and weight are represented by the random variable W with distribution F. As time goes on, the random strength and stress are subjected to temperature-related external conditions. Suppose u^3,u^2 and V are the distortion function corresponding to the varying temperature. Let $\nu(u) = [u^3, u^2, u]$ be the distortion function. Then $A_\alpha(u)$ and $B_\alpha(u)$ can be estimated as follows

$$A_\alpha = u^3 + \alpha\left[u^2 - u^3\right] \tag{5}$$

and

$$B_\alpha = u - \alpha\left[u - u^2\right]. \tag{6}$$

The system's stress is represented by the random variable W, which has the cumulative distribution function $F(x) = 1 - e^{-\lambda_1 x}$. With a cumulative distribution function of $G(y) = 1 - e^{-\lambda_2 y}$, the random variable V represents the system's strength.

Then $A_{\alpha,a}$ and $B_{\alpha,a}$ can be estimated as follows

$$A_{\alpha,a} = \int_0^1 \left((1-\alpha)\left(1-(1-u)^{\lambda_1/\lambda_2}\right)^3 + \alpha\left(1-(1-u)^{\lambda_1/\lambda_2}\right)^2\right)\left(3(1-\alpha)u^2 + 2\alpha u\right) du \tag{7}$$

and

$$B_{\alpha,b} = \int_0^1 \left((1-\alpha)\left[1-(1-u)^{\lambda_1/\lambda_2}\right] + \alpha\left[1-(1-u)^{\lambda_1/\lambda_2}\right]^2\right)\left((1-\alpha) + 2\alpha u\right) du. \tag{8}$$

For each α, we can estimate the α-cut of R as

$$R_\alpha = [A_{\alpha,a}, B_{\alpha,b}]. \tag{9}$$

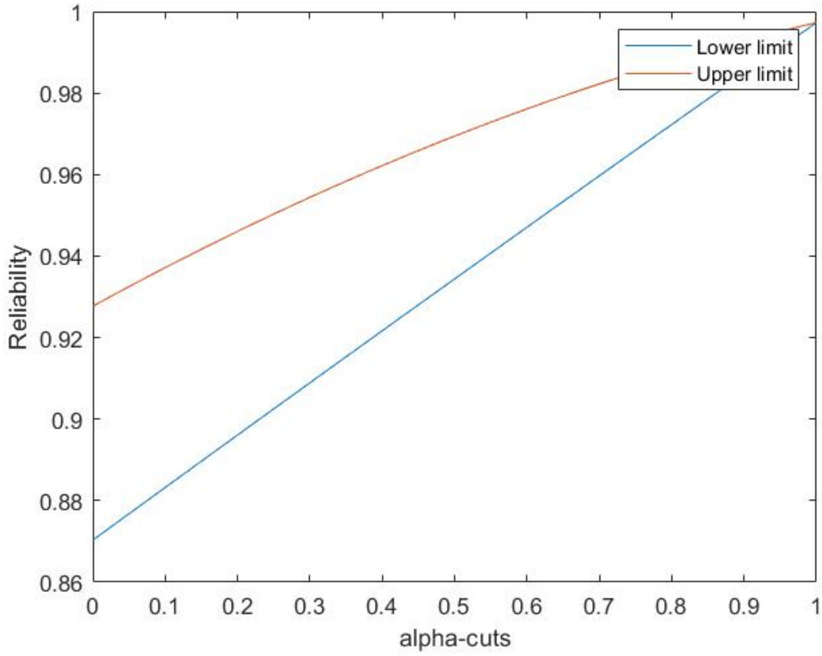


Figure 1: *Fuzzy system reliability of for various values of α- cuts when $\lambda_1 = 0.33724$ and $\lambda_2 = 0.02628$.*

3.2. Characterization and some properties of fuzzy distortion function

Result 1: The average of two distortion functions is again a distortion function.
Proof : Let $\nu_1(t)$ and $\nu_2(t)$ be two distortion function.
Define $f(t) = \frac{\nu_1(t)+\nu_2(t)}{2}$. Since both $\nu_1(t)$ and $\nu_2(t)$ are non-decreasing functions. Clearly $f(t)$ is a non-decreasing function.
Since $\nu_1(0) = 0$ and $\nu_2(0) = 0$, so we have $f(0) = 0$. Similarly $\nu_1(1) = 1$ and $\nu_2(1) = 1$, shows that $f(1) = 1$.
Since both $\nu_1(t)$ and $\nu_2(t)$ are differentiable with the exception of a few points, it holds for $f(t)$ also.

Result 2: The average of finite number of distortion functions is again a distortion function.
Proof: Let $\nu_1(t), \nu_2(t), ..., \nu_n(t)$ be n distortion functions.
Define $f(t) = \frac{\nu_1(t)+\nu_2(t)+...+\nu_n(t)}{n}$. We have $\nu_1(t), \nu_2(t), ..., \nu_n(t)$ are non-decreasing functions. Then $f(t)$ is also a non-decreasing function.
We have $\nu_1(0) = 0, \nu_2(0) = 0, ..., \nu_n(0) = 0$, so we have $f(0) = 0$. Similarly $\nu_1(1) = 1, \nu_2(1) = 1, ..., \nu_n(1) = 1$, imply $f(1) = 1$.
Since all the functions $\nu_1(t), \nu_2(t), ..., \nu_n(t)$ are differentiable with the exception of a limited number of points, then the function $f(t)$ is also differentiable with the exception of a few points.

Result 3: The product of two distortion functions is again a distortion function.
Proof : Let $\nu_1(t)$ and $\nu_2(t)$ be two distortion function.
Define $f(t) = \nu_1(t).\nu_2(t)$. Since both $\nu_1(t)$ and $\nu_2(t)$ are non-decreasing functions. Clearly $f(t)$ is a non-decreasing function.
Since $\nu_1(0) = 0$ and $\nu_2(0) = 0$, so we have $f(0) = 0$. Similarly $\nu_1(1) = 1$ and $\nu_2(1) = 1$, imply $f(1) = 1$.
Since both $\nu_1(t)$ and $\nu_2(t)$ are differentiable with the exception of a few points, it holds for $f(t)$ also.

Result 4: The product of finite number distortion functions is again a distortion function.
Proof: Let $\nu_1(t), \nu_2(t), ..., \nu_n(t)$ be n distortion functions.
Define $f(t) = \nu_1(t).\nu_2(t)...\nu_n(t)$. We have $\nu_1(t), \nu_2(t), ..., \nu_n(t)$ are non-decreasing functions. Then $f(t)$ is also a non-decreasing function.
We have $\nu_1(0) = 0, \nu_2(0) = 0, ..., \nu_n(0) = 0$, so we have $f(0) = 0$. Similarly $\nu_1(1) = 1, \nu_2(1) = 1, ..., \nu_n(1) = 1$, imply $f(1) = 1$.
Since all the functions $\nu_1(t), \nu_2(t), ..., \nu_n(t)$ are differentiable with the exception of a limited number of points, hence it holds for $f(t)$ also.

3.3. Methods for construction of fuzzy distortion function

Result 1 Let $\nu_1(t)$, $\nu_2(t)$ and $\nu_3(t)$ are three distortion function with $\nu_1(t) \leq \nu_2(t) \leq \nu_3(t)$. Then $\nu(t)$ defined by $\nu(t) = [\nu_1(t), \nu_2(t), \nu_3(t)]$ is a fuzzy distortion function.
Proof: Define $\nu(t) = [\nu_1(t), \nu_2(t), \nu_3(t)]$. Then

$$A_\alpha(t) = \nu_1(t) + \alpha(\nu_2(t) - \nu_1(t)) \tag{10}$$

and

$$B_\alpha(t) = \nu_3(t) - \alpha(\nu_3(t) - \nu_2(t)). \tag{11}$$

First we have to prove that $A_\alpha(t)$ and $B_\alpha(t)$ are non-decreasing. For that we made a rearrange as $A_\alpha(t) = (1-\alpha)\nu_1(t) + \alpha\nu_2(t)$ and $B_\alpha(t) = (1-\alpha)\nu_2(t) + \alpha\nu_3(t)$.
Since $\nu_1(t)$, $\nu_2(t)$ and $\nu_3(t)$ are non-decreasing functions and $1-\alpha > 0$. Both $A_\alpha(t)$ and $B_\alpha(t)$ are non-decreasing.
We have $\nu_1(0) = 0, \nu_2(0) = 0$ and $\nu_3(0) = 0$.
Then

$$A_\alpha(0) = (1-\alpha)\nu_1(0) + \alpha(\nu_2(0) = (1-\alpha)0 + \alpha 0 = 0, \tag{12}$$

and

$$B_\alpha(0) = (1-\alpha)\nu_2(0) + \alpha(\nu_3(0) = (1-\alpha)0 + \alpha 0 = 0. \tag{13}$$

Similarly $\nu_1(1) = 1, \nu_2(1) = 1$ and $\nu_3(1) = 1$.
Then

$$A_\alpha(1) = (1-\alpha)\nu_1(1) + \alpha(\nu_2(1) = (1-\alpha)1 + \alpha 1 = 1 \tag{14}$$

and

$$B_\alpha(1) = (1-\alpha)\nu_2(1) + \alpha(\nu_3(1) = (1-\alpha)1 + \alpha 1 = 1. \tag{15}$$

Since the three functions $\nu_1(t), \nu_2(t)$ and $\nu_3(t)$ are differentiable with the exception of a limited number of points, $A_\alpha(t)$ and $B_\alpha(t)$ are also differentiable with with the exception of a few points. Hence the function $\nu(t)$ satisfy all the conditions for fuzzy distortion function.

Result 2 Let $\nu_1(t)$, $\nu_2(t)$, $\nu_3(t)$ and $\nu_4(t)$ are four distortion function with $\nu_1(t) \leq \nu_2(t) \leq \nu_3(t) \leq \nu_4(t)$. Then $\nu(t)$ defined by $\nu(t) = [\nu_1(t), \nu_2(t), \nu_3(t), \nu_4(t)]$ is a fuzzy distortion function

Proof: Define $\nu(t) = [\nu_1(t), \nu_2(t), \nu_3(t), \nu_4(t)]$. Then

$$A_\alpha(t) = \nu_1(t) + \alpha(\nu_2(t) - \nu_1(t)) \tag{16}$$

and

$$B_\alpha(t) = \nu_4(t) - \alpha(\nu_4(t) - \nu_3(t)). \tag{17}$$

First we will prove that $A_\alpha(t)$ and $B_\alpha(t)$ are non-decreasing. For that we made a rearrange as $A_\alpha(t) = (1-\alpha)\nu_1(t) + \alpha(\nu_2(t)$ and $B_\alpha(t) = (1-\alpha)\nu_4(t) + \alpha\nu_3(t)$.
Since $\nu_1(t)$, $\nu_2(t)$, $\nu_3(t)$ and $\nu_4(t)$ are non-decreasing functions and $1-\alpha > 0$. Both $A_\alpha(t)$ and

$B_\alpha(t)$ are non-decreasing.
We have $\nu_1(0) = 0, \nu_2(0) = 0, \nu_3(0) = 0$ and $\nu_4(0) = 0$,
Then

$$A_\alpha(0) = (1-\alpha)\nu_1(0) + \alpha(\nu_2(0) = (1-\alpha)0 + \alpha 0 = 0, \tag{18}$$

and

$$B_\alpha(0) = (1-\alpha)\nu_4(0) + \alpha\nu_3(0) = (1-\alpha)0 + \alpha 0 = 0. \tag{19}$$

Similarly $\nu_1(1) = 1, \nu_2(1) = 1$, $\nu_3(1) = 1$ and $\nu_4(1) = 1$.
Then

$$A_\alpha(1) = (1-\alpha)\nu_1(1) + \alpha(\nu_2(1) = (1-\alpha)1 + \alpha 1 = 1 \tag{20}$$

and

$$B_\alpha(1) = (1-\alpha)\nu_4(1) + \alpha\nu_3(1) = (1-\alpha)1 + \alpha 1 = 1. \tag{21}$$

Since the four functions $\nu_1(t), \nu_2(t)$, $\nu_3(t)$ and $\nu_4(t)$ are differentiable except a finite number of points, $A_\alpha(t)$ and $B_\alpha(t)$ are also differentiable with the exception of a few points.
It is proved that the function $\nu(t)$ satisfy all the conditions for fuzzy distortion function.

4. Estimation of stress-strength reliability using the weighted probability density function

Consider a traditional stress-strength model, where system reliability is estimated by using a stress-strength relation. Let the random variable Y represent the strength of the system with cumulative distribution function G and the random variable X represent the stress of the system with cumulative distribution function F. Then, the stress-strength reliability P of the system is given by $P = P(X < Y)$. The drawback of this model is that the reliability estimate is unrealistic, since the normal working condition of the system is not considered. In other words, the fuzziness of parameters of the distribution is not taken into account, whereas, on the other hand, state of the system is highly dependent on the state of the parameters in the lifetime distribution. This is due to the fact that there exists an uncertainty in the parameters of distribution function. Hence, we incorporate this uncertainty factor by suitably modifying the density function and obtain the weighted probability density function and which can be used to get reliability estimate of the given system. Hence, we proceed as follows:

Let the random variable X have the probability density function $f(x, \theta)$. For the function $H(\theta)$, $\theta \in \Theta$, where Θ is the domain of definition of the parameter θ, the weighted probability density function of X is defined by

$$f(x) = \int_\Theta H^*(\theta) f(x, \theta) d\theta, \tag{22}$$

where $H^*(\theta)$ is called the pseudo-membership function and is defined by

$$H^*(\theta) = \frac{H(\theta)}{\int_\Theta H(\theta) d\theta}. \tag{23}$$

The definition and construction of above membership function is explained in detail by authors in [20].
Now let X be the stress random variable having the probability density function $f(x, \theta_1)$, $\theta_1 \in \Theta_1$, where Θ_1 is the domain of definition of the parameter θ_1. Then the weighted probability density function (*wpdf*) of X (see, [20] for further information) is given by

$$f^*(x) = \int_{\theta_1 \in \Theta_1} H^*(\theta_1) \cdot f(x, \theta_1) d\theta_1 \tag{24}$$

Let Y be the strength random variable having the probability density function $g(x, \theta_2), \theta_2 \in \Theta_2$, where Θ_2 is the domain of definition of the parameter θ_2. Then the weighted probability density function of Y, similar to the definition of *wpdf* of X, is given by

$$g^*(x) = \int_{\theta_2 \in \Theta_2} H^*(\theta_2) \cdot g(x, \theta_2) d\theta_2 \tag{25}$$

Let X and Y have exponential distribution with parameters θ_1 and θ_2 respectively. Then the stress random variable X has probability density function $f(x) = \theta_1 e^{-\theta_1 x}$, $\theta_1 > 0$ and the strength random variable Y has probability density function $g(y) = \theta_2 e^{-\theta_2 y}$, $\theta_2 > 0$.
Let $H_1(\theta_1)$ and $H_2(\theta_2)$ represent the membership function for the parameters θ_1 and θ_2 respectively. Then we may write (See, [21])

$$H_1(\theta_1) = \begin{cases} \frac{1+\cos\left[a\pi\left(\theta_1 - \frac{1}{b}\right)\right]}{2} & \text{if } \frac{1}{b} - \frac{1}{a} \le \theta_1 \le \frac{1}{b} + \frac{1}{a} \\ 0, & \text{otherwise} \end{cases} \tag{26}$$

$$H_2(\theta_2) = \begin{cases} \frac{1+\cos\left[c\beta\left(\theta_2 - \frac{1}{d}\right)\right]}{2} & \text{if } \frac{1}{d} - \frac{1}{c} \le \theta_2 \le \frac{1}{d} + \frac{1}{c} \\ 0, & \text{otherwise} \end{cases} \tag{27}$$

It is easy to check from the definition of pseudo-membership function given in equation (23) that

$$H_1^*(\theta_1) = \frac{a}{2} \begin{cases} \frac{1+\cos\left[a\beta\left(\theta_1 - \frac{1}{b}\right)\right]}{2} & \text{if } \frac{1}{b} - \frac{1}{a} \le \theta_1 \le \frac{1}{b} + \frac{1}{a} \\ 0, & \text{otherwise} \end{cases} \tag{28}$$

$$H_2^*(\theta_2) = \frac{c}{2} \begin{cases} \frac{1+\cos\left[c\beta\left(\theta_2 - \frac{1}{d}\right)\right]}{2} & \text{if } \frac{1}{d} - \frac{1}{c} \le \theta_2 \le \frac{1}{d} + \frac{1}{c} \\ 0, & \text{otherwise} \end{cases} \tag{29}$$

Then reliability of the system under stress-strength model is given by

$$\begin{aligned} R^* &= P(X^* < Y^*) \\ &= \iint_{x<y} f^*(x) g^*(y) dx dy \\ &= \int_0^\infty \int_0^y f^*(x) g^*(y) dx dy \\ &= \int_0^\infty \int_0^y \left[\int_{\theta_1 \in \Theta_1} H_1^*(\theta_1) f(x, \theta_1) d\theta_1 \int_{\theta_2 \in \Theta_2} H_2^*(\theta_2) g(y, \theta_2) d\theta_2 \; dx dy \right. \\ &= \int_{\theta_1 \in \Theta_1} \int_{\theta_2 \in \Theta_2} H_1^*(\theta_1) H_2^*(\theta_2) \left[\int_0^\infty \int_0^y f(x, \theta_1) g(y, \theta_2) dx dy \right] d\theta_2 d\theta_1 \\ &= \int_{\theta_1 \in \Theta_1} \int_{\theta_2 \in \Theta_2} \frac{ac}{4} \left[1 + \cos\left(a\pi \left(\theta_1 - \frac{1}{b} \right) \right) \right] \left[1 + \cos\left(c\pi \left(\theta_2 - \frac{1}{d} \right) \right) \right] \frac{\theta_1}{\theta_1 + \theta_2} d\theta_2 d\theta_1, \end{aligned} \tag{30}$$

where the parameters θ_1 and θ_2 are non-negative. Since the closed form expression for indefinite integrals do not exit in (30), we resort to evaluate the integrals using numerical integration. The integrals are computed using two-dimensional quadrature method with the help of the operator quad2d in MATLAB for numerical integration. The following section gives the details of numerical results.

4.1. Numerical Results

In this section, we present some numerical results to illustrate the reliability of the system via equation (30) for various choices of the parameters, namely, a, b, c, and d in equation (28) and (29), which are the part of the integral in (30). The Table 1 illustrates the results obtained in Section 4.

Table 1 Reliability estimation using the weighted probability density function

a	b	c	d	Reliability
9.5	9.3	450.0	444.0	0.9758
9.6	9.0	452.0	451.0	0.9774
12.0	10.0	456.0	450.0	0.9759
11.0	9.0	456.0	454.0	0.9785
10.0	8.0	457.0	454.0	0.9810
11.0	7.0	456.0	455.0	0.9840
19.0	9.0	456.0	455.0	0.9816
10.0	8.0	456.0	454.0	0.9810

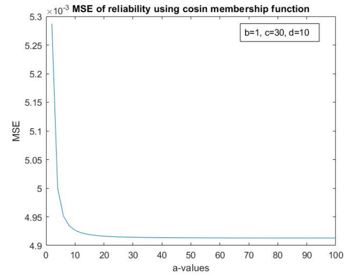
It is observed from the computational experience that for the exponential lifetime distribution of the system, the fuzziness of the parameters modelled using membership function has significant advantage in describing the vagueness in the system parameters. Note that the above results are obtained from reliability equation given in (30), where the closed form expression of reliability does not exist, and a numerical integration is carried out. Hence, the choice of membership function works well under the situation that the exact expression for reliability is not possible to compute. This is evident from the examples computed in the above Table 1, where the minimum reliability obtained is about 97%. The reliability of the system without using the weighted probability density function with $\theta_1 = 0.00227$ and $\theta_2 = 0.00447$ is estimated as 0.3368. But from Table 1, it is observed that when we use weighted probability density function with parameters $a = 9.5$, $b = 9.3$, $c = 450.0$ and $d = 444.0$, the reliability of the system is estimated as 0.9758. It is observed that, when we use the weighted probability density function, the reliability of the system is increased by 60%.

5. Sensitivity analysis

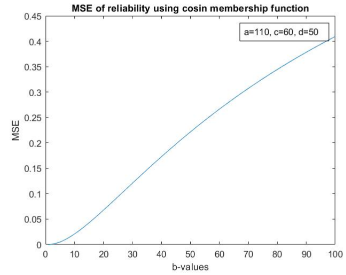
In previous sections, several reliability estimates developed by incorporating the fuzziness in the data involve number of parameters which come from either membership functions or from distortion functions applied for estimating the stress-strength reliability of the system under consideration. Therefore it is necessary to study the worthiness of these reliabilities estimated in terms of their mean square errors (MSE). Hence this section discuss MSE of the reliability estimate developed in Section 4.

We study via numerical computation, how MSE is sensitive to changes in the parameters a, b, c and d of the membership function of parameters of distribution functions. For illustration, we let $\theta_1 = 0.2128$ and $\theta_2 = 0.0045$. First, we vary one parameter keeping the remaining parameters fixed. There will be four different cases. Figure 2 shows the variation of MSE when one of the parameters is varied while keeping other parameters fixed. In Figure 2, for the fixed combination $b = 1$, $c = 30$, $d = 10$ (see, Graph (i)), and varying the parameter a, it can be observed that there is a sudden decrease in MSE. Similar observation is true for set of $a = 10$, $b = 9$, $c = 450$ and for varying d (see, graph (iv)). Note that from Graph (ii) and (iii) we see that the MSE is showing increasing trend.

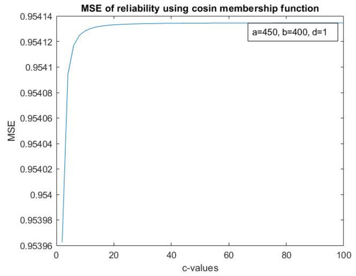
Next, study the changes in MSE by varying two parameters while keeping any two of the four remaining parameters held fixed. Figure 3 illustrate one such case for the fixed combination of $b = 10$, $c = 60$ while varying the remaining parameters. It is observed that there is a gradual decrease in MSE.



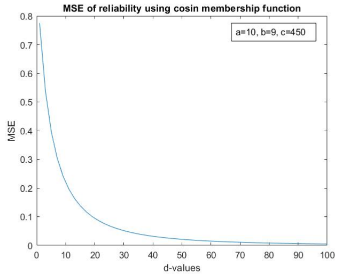
Graph(i)



Graph(ii)



Graph(iii)



Graph(iv)

Figure 2: MSE of reliability estimator under membership function, when one parameter is varied and remaining parameters are held fixed.

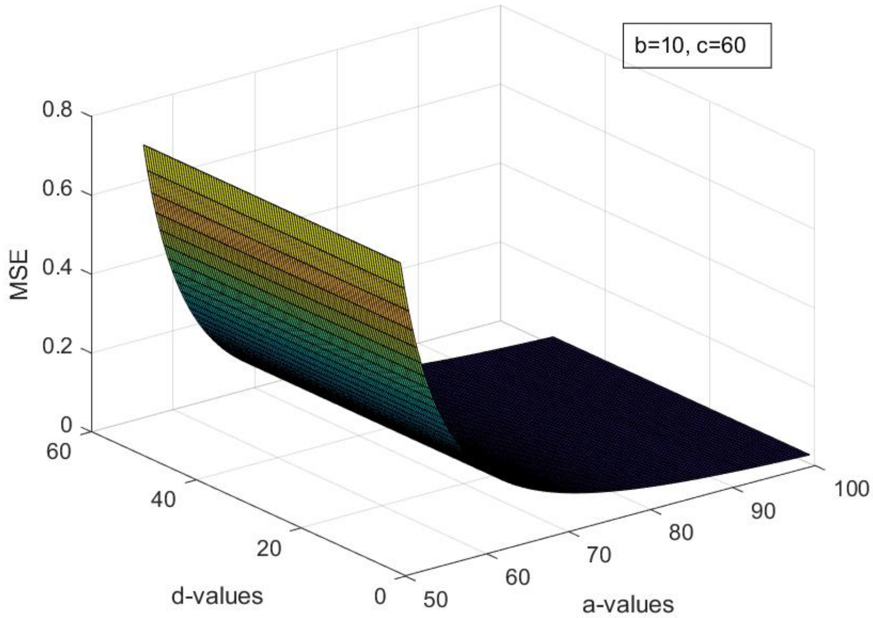


Figure 3 MSE of reliability using membership function when b and c are fixed.

Finally, from this sensitivity analysis study, it is noted that the accuracy of reliability estimate depends upon the choice of the parameters in the membership function, which can again depends upon availability of the type of data. Further, observe that the reliability estimates obtained are based upon numerical integration, since the closed form of expression does not exist. However, we strongly believe that the choice of membership in modelling vagueness play very important role in obtaining fuzzy reliability estimate.

6. Conclusions

In the conventional stress-strength model, reliability is determined without considering the variability in environmental factors. In our research, we introduced the notion of a fuzzy distortion function to account for the uncertainty in these environmental factors. Our approach involved assessing the reliability of the stress-strength model by employing the fuzzy distortion function. We illustrated this method with an example and also examined various characteristics of the distortion function. Additionally, we derived techniques for constructing a fuzzy distortion function based on existing actual distortion functions. The drawbacks of existing methods in the literature are that it does not consider the uncertainty or fuzziness in data and nature to estimate the system reliability under realistic situations. But in this work system reliability is estimated using the weighted probability density function by incorporating the fuzziness in data, which is practical and realistic. Finally, a sensitivity analysis study of MSE of reliability estimate obtained using cosine membership function is presented. It is observed that MSE as a function of parameters of membership function of a fuzzy parameter, can be minimized significantly by careful choice of parameters of membership functions.

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