RT&A, No 3 (79)
Volume 19, September 2024

ANALYSIS OF TWO NON-IDENTICAL UNIT SYSTEM HAVING SAFE AND UNSAFE FAILURES WITH REBOOTING AND PARAMETRIC ESTIMATION IN CLASSICAL AND BAYESIAN PARADIGMS

Poonam Sharma and Pawan Kumar

•

Department of Statistics, University of Jammu, J&K
sharmapoonam1038@gmail.com, pkk_skumar@yahoo.co.in

Abstract

The present paper aims at the study of a two non-identical system model having safe and unsafe failures and rebooting. The focus centers on the analysis w.r.t important reliability measures and estimation of parameters in Classical and Bayesian paradigms. At first one of the units is operational whereas other one is confined to standby mode. Any unit may suffer safe or unsafe failure. A safe failure is immediately taken up for remedial action by a repairman available with the system all the time, while the case of unsafe failure cannot be dealt directly but first rebooting is performed to convert the unsafe failure to safe failure mode so as to start repair normally. A switching device is used to make the repaired and standby units operational. The lifetime of both the units and switching device are taken to be exponentially distributed random variables whereas the distribution of repair times are assumed to be general. Regenerative point technique is employed to derive assosciated measures of effectiveness. To make the study more elaborative and visually attractive, some of the derived characteristics have been studied graphically too. A simulation study has also been undertaken to exhibit the behaviour of obtained characteristics in Classical and Bayesian setup. Valuable inferences about MLE and Bayes estimates have been drawn from the tables and graphs for varying values of failure and repair parameters.

Keywords: Reliability, Availability, Mean Time to System Failure, Regenerative Point Technique, Rebooting, Coverage Proabability, Bayesian Estimation, Maximum Likelihood Estimation.

1. Introduction

Reliability is a fundamental concept that underpins the dependability and consistency of systems, processes, products, or services. It is the assurance that something will perform its purposeful function or deliver expected outcomes consistently and without failure over a specified period or under specific conditions. In a world where technological advancements and complex interdependencies are ever-increasing, reliability has become a critical factor in determining the success, safety, and satisfaction of individuals, businesses, and societies at large. We observe that machine failure, which results in significant losses, frequently follows unit failure. The incorporation of standby units is one strategy for enhancing reliability. Also, there are cases where the root cause of a unit failure is not immediately identified, leading to inadequate coverage that must be fixed by rebooting. Depending on the complexity, the length of the reboot time varies from system to system. Recent times have seen extensive and rigorous research on reliability, availability, standby systems, inadequate coverage, reboot, etc. Sharma & Kumar[1] examined the concept of two similar units with one switching device and imperfect coverage. In case of unsafe failure, repair cannot begin immediately but first rebooting is done which transforms the unsafe failure to safe failure and then repair is carried out. Trivedi[2] gave the concept of reboot in his work "Probability & Statistics With Reliability, Queuing and Computer Science Applications" Gupta et al.[3] carried a study about two dissimilar unit parallel system accompanied by correlated lifetimes. The system stops functioning when both of the units fail. Wang & Chen [4] provided a comparative analysis by computing the availability of three systems with General Repair times and Reboot delay. Pham [5] performed analysis of reliability of a system with high voltage having dependent failures and insufficient coverage. A high voltage (HV) system that consists of a power supply and two transmitters is considered. Also a model of the HV system and a detailed development of the reliability function are presented. Ke et al. [6] examined a resolvable system with insufficient coverage and reboot. As a unit fails, it can be immediately detected, and replaced with a coverage probability c. Kumar P & Jain M [7] proposed the machine having multi-components with service interruption, imperfect coverage, and reboot .Kadyan & Malik [8] performed a stochastic study on non-identical units with cold standby units operating at the same time. The idea of Classical and Bayesian estimation in a two non-identical unit parallel system is given by Saxena et.al[9], where the Bayesian estimates are calculated by taking different priors. Also, a comparative study is done to determine the performance of Maximum likelihood estimation and Bayesian estimation methods. Kishan & Jain [10] put forth the idea of study of system model both in classical and Bayesian perspectives and some important measures of reliability characteristics of a two nonidentical unit standby system model with repair, inspection and post repair are obtained using regenerative point technique.

Keeping above ideas in mind, this paper deals with the performance measures and estimation of parameters of a two non-identical units system with switching device and rebooting having safe and unsafe failures. Switch is used to turn on the unit from standby to operational mode and initially is assumed to be in good condition. Unsafe failures occur when the cause of any of the breakdowns is unknown and can be resolved by rebooting. Reboot delay times and failure times for both units and switch are assumed to be exponentially distributed, whereas the repair time distributions are taken to be general in nature. Other measures, such as mean time to system failure, reliability, availability, and expected number of repairs, have been calculated using the regenerating point technique. Furthermore, a simulation study is carried out to examine the given system model in both the Classical and Bayesian setups. Finally, numerous noteworthy conclusions are drawn from the tables and graphs.

2. System Description and Assumptions

• The system is composed of two non-identical units, A and B, coupled by a switch, S.

• Initially, one of the units is functioning, while the other remains in standby mode. A switch assists to turn on the repaired and standby components. During the early stage, switch is supposed to be in operable condition.

• There may be both safe and unsafe failures among the units but only a regular switch failure. If any of the unit fails safely, it can be identified with coverage probability c, and repaired instantly if the repairman is present.

• In the event of an unsafe failure, repair can't begin instantly; instead, a reboot is first performed to convert the unsafe failure to a safe failure, followed by a usual normal repair. Reboot delay periods are taken as exponentially distributed random variables with varying parameters.

• The system has a dedicated repair facility and is constantly accessible to repair and reboot failed items. Switch repair has priority over failed items in the system.

• The failure times of the units and switch follow an exponential distribution, whereas the repair time distributions are general.

• A repaired item functions as new.

3. Notations and Symbols

α_1: Failure rate of Unit A
α_2: Failure rate of Unit B
α_3: Failure rate of Switch
c: Coverage probability
γ_1: Rebooting delay rate for unsafe failure of Unit A
γ_2: Rebooting delay rate for unsafe failure of Unit B.
$F_1(.)$: Repair rate of unit A
$F_2(.)$: Repair rate of Unit B
$F_3(.)$: Repair rate of Switch

3.1. SYMBOLS FOR THE STATES OF THE SYSTEM

A_0/B_0 :Units in operative mode
A_r/B_r :Units under repair
A_s/B_s :Units in standby mode
S_g/S_r :Switch under good/repair condition
A_{wr}/B_{wr} :Units waiting for repair
A_{usf}/B_{usf}:Units having unsafe failure

Using the symbols provided above, the achievable states of the system are:

$S_0 = [A_0, B_s, S_g]$ $S_1 = [A_r, B_0, S_g]$ $S_2 = [A_{usf}, B_g, S_g]$
$S_3 = [A_0, B_s, S_r]$ $S_4 = [A_{usf}, B_g, S_{wr}]$ $S_5 = [A_{wr}, B_0, S_r]$
$S_6 = [A_{wr}, B_{usf}, S_{wr}]$ $S_7 = [A_{wr}, B_{wr}, S_r]$ $S_8 = [A_{wr}, B_{usf}, S_g]$
$S_9 = [A_r, B_{wr}, S_g]$ $S_{10} = [A_0, B_r, S_g]$ $S_{11} = [A_{usf}, B_{wr}, S_g]$
$S_{12} = [A_g, B_{wr}, S_r]$ $S_{13} = [A_{wr}, B_g, S_r]$

The transition diagram of the model is shown in Figure 1.

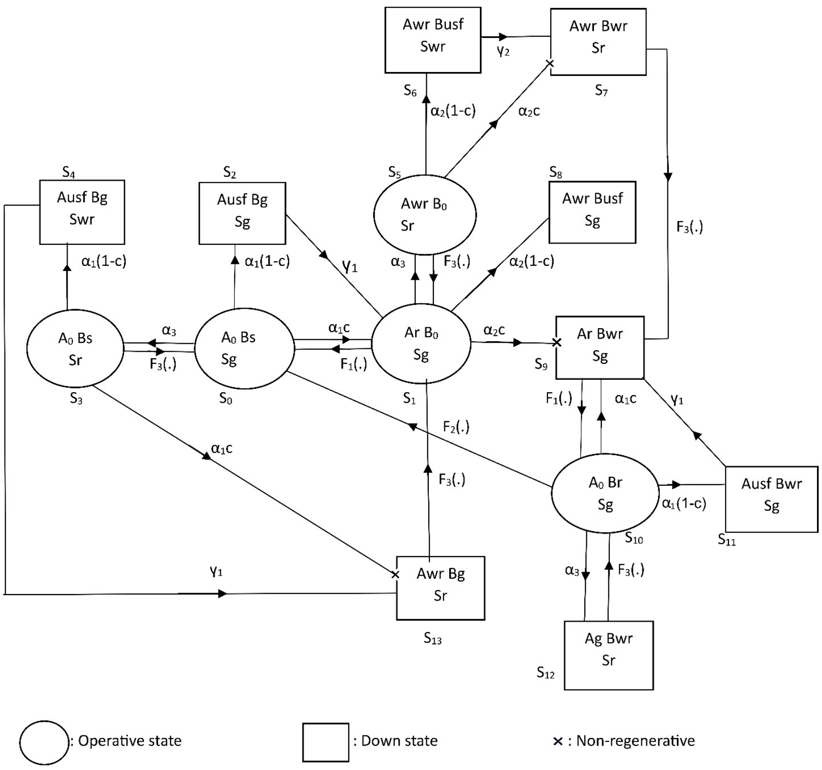

Figure 1: *Transition Diagram*

4. Transition Probabilities and Sojourn Times

The long-run or of the steady state probabilities are obtained as under,

$p_{ij} = \lim_{t\to\infty} Q_{ij}(t) = \int q_{ij}(t)dt \qquad p_{ij}^{(k)} = \lim_{t\to\infty} Q_{ij}^{(k)}(t) \qquad \text{and} \qquad p_{ij}^{(k,l)} = \lim_{t\to\infty} Q_{ij}^{k,l}(t).$

In particular we have

$$p_{01}(t) = \int \alpha_1 c e^{-\alpha_1 c t} e^{-\alpha_1(1-c)t} e^{-\alpha_3 t} dt = \frac{\alpha_1 c}{(\alpha_1+\alpha_3)}$$

Similarly,

$p_{02} = \frac{\alpha_1(1-c)}{\alpha_1+\alpha_3}$

$p_{03} = \frac{\alpha_3}{\alpha_1+\alpha_3}$

$p_{15} = \frac{\alpha_3}{\alpha_2+\alpha_3}[1-\widetilde{F_1}(\alpha_2+\alpha_3)]$

$p_{18} = \frac{\alpha_2(1-c)}{\alpha_2+\alpha_3}[1-\widetilde{F_1}(\alpha_2+\alpha_3)]$

$p_{10} = \widetilde{F_1}(\alpha_2+\alpha_3)$

$p_{1,10}^{(9)} = \frac{\alpha_2 c}{\alpha_2+\alpha_3}[1-\widetilde{F_1}(\alpha_2+\alpha_3)]$

$p_{30} = \widetilde{F_3}(\alpha_1)$

$p_{34} = (1-c)(1-\widetilde{F_3}(\alpha_1))$

$p_{31}^{(13)} = c(1-\widetilde{F_3}(\alpha_1))$

$p_{51} = \widetilde{F_3}(\alpha_2)$

$p_{56} = (1-c)[1-\widetilde{F_3}(\alpha_2)]$

$p_{59}^{(7)} = c[1-\widetilde{F_3}(\alpha_2)]$

$p_{10,11} = \frac{\alpha_1(1-c)}{\alpha_1+\alpha_3}[1-\widetilde{F_2}(\alpha_1+\alpha_2)]$

$p_{10,9} = \frac{\alpha_1 c}{\alpha_1+\alpha_3}[1-\widetilde{F_2}(\alpha_1+\alpha_3)]$

$p_{10,12} = \frac{\alpha_3}{\alpha_1+\alpha_3}[1-\widetilde{F_2}(\alpha_1+\alpha_3)]$

Thus, the following relationships can be established

$p_{01} + p_{02} + p_{03} = 1$

$p_{30} + p_{34} + p_{31}^{(13)} = 1$

$p_{15} + p_{18} + p_{10} + p_{1,10}^{(9)} = 1$

$p_{51} + p_{56} + p_{59}^{(7)} = 1$

$p_{10,0} + p_{10,9} + p_{10,11} + p_{10,12} = 1$

$p_{67} = p_{79} = p_{89} = p_{13,1} = p_{4,13} = p_{11,9} = p_{21} = p_{12,10} = p_{9,10} = 1$

4.1. Mean Sojourn times

In reliability, Mean Sojourn time ψ_i, is the expected length of time a system spends in a certain state before moving to another. There is never any transition from S_i to any other state, as long as the system is in state S_i. We utilize this knowledge to determine ψ_i for state S_i. Given T_i as the sojourn time in state S_i, the mean sojourn time ψ_i is as follows.

$$\psi_i = E[T_i] = \int P(T_i > t)dt$$

Hence, using the above formula following values for mean sojourn time are obtained:

$\psi_0 = \frac{1}{(\alpha_1+\alpha_3)}$

$\psi_1 = \frac{1}{(\alpha_2+\alpha_3)}[1-\widetilde{F_1}(\alpha_2+\alpha_3)]$

$\psi_3 = \frac{1}{\alpha_1}[1-\widetilde{F_3}(\alpha_1)]$

$\psi_5 = \frac{1}{\alpha_2}[1-\widetilde{F_3}(\alpha_1)]$

$\psi_2 = \psi_4 = \psi_{11} = \frac{1}{\gamma_1}$

$\psi_6 = \psi_8 = \frac{1}{\gamma_2}$

$\psi_9 = \int \bar{F_1}(t)dt$

$\psi_{10} = \frac{1}{(\alpha_1+\alpha_3)}[1-\widetilde{F_2}(\alpha_1+\alpha_3)]$

$\psi_7 = \psi_{12} = \psi_{13} = \int \bar{F_3}(t)dt$

5. Analysis of Reliability and MTSF

Let random variable T_i represents the life time of system when it initiate from state $S_i \in E_i$, the system's reliability is determined by:

$R_i(t) = P[T_i > t]$

To calculate $R_i(t)$, we treat failed states as absorbing states.
The recursive relations between $R_i(t)$ can be established using probabilistic arguments by referring to the state transition diagram. Using the Laplace transform and determining the set of equations for $R_0^*(s)$, we get

$$R_0^*(s) = \frac{N_1(s)}{D_1(s)} \tag{1}$$

[1] where,

$$N_1(s) = Z_0^* + Z_3^* q_{03}^* + Z_5^* q_{01}^* q_{15}^* - Z_0^* q_{15}^* q_{51}^* - Z_3^* q_{03}^* q_{15}^* q_{51}^*$$

$$D_1(s) = 1 - q_{01}^* q_{10}^* - q_{15}^* q_{51}^* - q_{03}^* q_{30}^* + q_{03}^* q_{15}^* q_{30}^* q_{51}^*$$

We obtain the system's reliability by taking the inverse Laplace transform of (1).
To obtain MTSF, we use the given formula

$$E(T_0) = \int R_0(t)dt = \lim_{s\to 0} R_0^*(s) = \frac{N_1(0)}{D_1(0)} \tag{2}$$

where,

$$N_1(0) = \psi_0 + \psi_1 p_{01} + \psi_5 p_{01} p_{15} - \psi_0 p_{15} p_{51} - \psi_3 p_{03} p_{15} p_{51}$$

and

$$D_1(0) = 1 - p_{01}p_{10} - p_{15}p_{51} - p_{03}p_{30} + p_{03}p_{15}p_{30}p_{51}$$

Since we've $q_{ij}^*(0) = p_{ij}$ and $\lim_{s\to 0} Z_i^*(s) = \int Z_i(t)dt = \psi_i$

6. Availability Analysis

The probability that a system is able to perform its intended task at time 't' if it initiates from $S_i \in E_i$ is known as Availability. Point wise availability refers to a system's availability at a specified time. It is a measure of system performance that reflects whether a system is potentially operational and able to provide the expected service at a given time. Using stochastic reasoning, recurrence relations between different point-wise availabilities are established. Using the Laplace transformations and solving the equations for $A_0^*(s)$, we obtain

$$A_0^*(s) = \frac{N_2(s)}{D_2(s)}$$

where,

$$N_2(s) = Z_0^* q_{15}^* q_{51}^* b_1 + Z_3^*(q_{02}^* - q_{03}^* b_1) - (Z_0^* + Z_1^* Y_1 + Z_5^* Y_3)(q_{10,9}^* q_{9,10}^*) + Z_{10}^* q_{1,10}^{(9)*} Y_1 - (Z_0^* + Z_1^* Y_1) \\ (q_{10,12}^* q_{12,10}^*) - Z_3^* q_{03}^* + Z_5^* Y_1 + (Z_1^* + Z_5^* q_{15}^*) q_{03}^* q_{31}^{(13)*} q_{11,9}^* q_{10,11}^* q_{9,10}^* + Z_{10}^* q_{9,10}^* b_3 (Z_1^* + Z_5^* q_{15}^*) \\ (Z_0^* + Z_1^* Y_2) q_{11,9}^* q_{10,11}^* q_{9,10}^* + Z_{10}^* q_{18}^* q_{89}^* q_{9,10}^* Y_1 + q_{03}^* q_{34}^* q_{131}^* q_{4,13}^* b_2 + q_{15}^* q_{9,10}^* Z_{10}^* b_4 + (Z_1^* + Z_3^*) \\ q_{03}^* + Z_5^* q_{01}^* q_{15}^*$$

Here,

[1]Limits of integeration whenever they are 0 to ∞ are not mentioned.

$Y_1 = q_{01}^* + q_{02}^* q_{21}^* + q_{03}^* q_{31}^{(13)*} + q_{34}^* q_{13,1}^* q_{4,13}^*)$

$Y_2 = q_{01}^* + q_{02}^* q_{21}^* \; q_{03}^* q_{34}^* q_{131}^* q_{4,13}^*$

$Y_3 = q_{02}^* q_{15}^* q_{21}^* + q_{03}^* q_{15}^* q_{31}^{(13)*} \; q_{03}^* q_{15}^* q_{34}^* \; q_{13,1}^* q_{4,13}^*$

$b_1 = 1 - q_{10,9}^* q_{9,10}^* - q_{10,12}^* \; q_{12,10}^* - q_{11,9}^* q_{9,10}^* q_{10,11}^*$

$b_2 = Z_1^* q_{15}^* Z_5^* + q_{15}^* (q_{59}^{(7)*} q_{9,10}^* \; Z_{10}^* + q_{56}^* \; q_{67} q_{9,10}^* Z_{10}^*)$

$b_3 = q_{18}^* q_{89}^* + q_{15}^* \; (q_{59}^{(7)*} + q_{56}^* q_{67}^*)$

$b_4 = (q_{59}^{(7)*} (q_{01}^* + q_{02}^* \; q_{21}^*) + q_{01}^* q_{56}^* q_{67}^*)$

and,

$$D_2(s) = q_{10,0}^* - q_{10}^* q_{10,0}^* + q_{01}^* q_{15}^* q_{10,0}^* - (q_{56}^* + q_{59}^{(7)*})[-q_{15}^* q_{10,0}^* (q_{01}^* + q_{02}^* + q_{03}^* (q_{31}^{(13)*} + q_{34}^*)) - q_{02}^* q_{10,0}^*] \\ -q_{03}^* q_{10,0}^* (q_{30}^* - q_{30}^* q_{13}^* q_{51}^* + q_{10}^* q_{31}^{(13)*} + q_{10}^* q_{34}^*) - q_{15}^* q_{51}^* q_{10,0}^* - q_{10,0}^* (q_{34}^* + q_{31}^{(13)*})(q_{03}^* (q_{1,10}^{(9)*} + q_{18}^*)) \quad (3)$$

The steady state availability is given as under

$$A_0 = \lim_{t \to \infty} A_0(t) = \lim_{s \to 0} \; sA_0^*(s) = \frac{N_2(0)}{D_2(0)}$$

Furthermore, its a well known fact that $q_{ij}(t)$ is the pdf of the time of transition from state S_i to S_j and $q_{ij}(t)$ is the probability of a transition from state S_i to state S_j during the interval (t, t + dt) , thus

$q_{ij}^*(s)|s = 0 = q_{ij}^*(0) = p_{ij}$

We also know that

$\lim_{s \to 0} Z_i^*(s) = \int Z_i(t) dt = \psi_i$

Therefore,

$$N_2(0) = \psi_0 p_{15} p_{51} b_1 + \psi_3 (p_{02} - p_{03} b_1) - (\psi_0 + \psi_1 Y_1 + \psi_5 Y_3)(p_{10,9} p_{9,10}) + \psi_{10} p_{1,10}^{(9)} Y_1 - (\psi_0 + \psi_1 Y_1) \\ (p_{10,12} p_{12,10}) - \psi_3 p_{03} + \psi_5 Y_1 + (\psi_1 + \psi_5 p_{15}) p_{03} p_{31}^{(13)} p_{11,9} p_{10,11} p_{9,10} + \psi_{10} p_{9,10} b_3 (\psi_1 + \psi_5 p_{15}) \\ (\psi_0 + \psi_1 Y_2) p_{11,9} p_{10,11} p_{9,10} + \psi_{10} p_{18} p_{89} p_{9,10} Y_1 + p_{03} p_{34} p_{13,1} p_{4,13} b_2 + p_{15} p_{9,10} \psi_{10} b_4 + (\psi_1 + \psi_3) \\ p_{03} + \psi_5 p_{01} p_{15}$$

Here,

$Y_1 = p_{01} + p_{02} p_{21} + p_{03} (p_{31}^{(13)} + p_{34} p_{13,1} p_{4,13})$

$Y_2 = p_{01} + p_{02} p_{21} \; p_{03} p_{34} p_{13,1} p_{4,13}$

$Y_3 = p_{02} p_{15} p_{21} + p_{03} p_{15} p_{31}^{(13)} \; p_{03} p_{15} p_{34} \; p_{13,1} p_{4,13}$

$b_1 = 1 - p_{10,9} p_{9,10} - p_{10,12} \; p_{12,10} - p_{11,9} p_{9,10} p_{10,11}$

$b_2 = \psi_1 p_{15} \psi_5 + p_{15} (p_{59}^{(7)} p_{9,10} \; \psi_{10} + p_{56} \; p_{67} p_{9,10} \psi_{10})$

$b_3 = p_{18}p_{89} + p_{15}\ (\overset{(7)}{p_{59}} + p_{56}p_{67})$

$b_4 = (\overset{(7)}{p_{59}}(p_{01} + p_{02}\ p_{21}) + p_{01}p_{56}p_{67})$

$$D_2(0) = p_{10,0} - p_{01}p_{10,0} + p_{01}p_{15}p_{10,0} - (p_{56} + \overset{(7)}{p_{59}})[-p_{15}p_{10,0}(p_{01} + p_{02} + p_{03}(\overset{(13)}{p_{31}} + p_{34})) - p_{02}p_{10,0}] - p_{03}p_{10,0}(p_{30} - p_{30}p_{13}p_{51} + p_{10}\overset{(13)}{p_{31}} + p_{10}p_{34}) - p_{15}p_{51}p_{10,0} - p_{10,0}(p_{34} + \overset{(13)}{p_{31}})(p_{03}(\overset{(9)}{p_{1,10}} + p_{18}))$$

For a given system, the steady-state probability of its long-term operation is given by

$$A_0 = \lim_{t\to\infty} A_0(t) = \lim_{s\to 0}\ sA_0^*(s)$$

$$\lim_{s\to 0}\frac{sN_2(s)}{D_2(s)} = \lim_{s\to 0}N_2(s)\lim_{s\to 0}\frac{s}{D_2(s)}$$

Since as s $\to 0$, $D_2(s)$ becomes zero. Therefore, applying L'Hospital's rule, A_0 becomes

$$A_0 = \frac{N_2(0)}{D_2'(0)} \tag{4}$$

where,

$$D_2'(0) = p_{10,0}(\psi_0 + \psi_1) - p_{10,0}p_{03}[(\psi_3)(1 - p_{15}p_{51}) + p_{30}(\psi_1 + p_{15}\psi_5) - p_{34} + p_{34}p_{15}p_{51}(\psi_4 + \psi_{13}) + p_{18}p_{30}\psi_8 + \psi_9(p_{10}(1 - p_{30}) - p_{30}p_{15}) - p_{15}p_{10,0}[(p_{51}\psi_0 - \psi_5 - \psi_6(p_{56}(1 - p_{03}p_{30}))) + \psi_9 (1 - p_{51}) + p_{02}p_{51}\psi_2] - p_{15}p_{51}[(1 - p_{03}p_{30})(\psi_{10} + \psi_{11}p_{10,11}) - \psi_4] + p_{10,12}\psi_{12} + \psi_9(p_{10,9} + p_{10,11} - p_{03}p_{30}[(1 - p_{10})(\psi_{10} + \psi_{11}p_{10,11}) + \psi_{12}p_{10,12}(1 - p_{15}p_{51}) + \psi_9(p_{10,9} + p_{10,11} - p_{10}(1 - p_{10,12}) + p_{18}p_{10,0})] - p_{10}(\psi_{10} + \psi_{11}p_{10,11}) + \psi_{12}(p_{10,12})(1 - p_{34}) + \psi_9[(1 - p_{03})(p_{10,9} + p_{10,11}) + p_{03}(1 - p_{10,12} + p_{34})] + p_{10,0}[(p_{02}\psi_2 + p_{18}(\psi_8 + \psi_9))] + p_{10,11}(\psi_{11} + \psi_9) + p_{10,12}\psi_{12} + \psi_{10} \tag{5}$$

Using $N_2(0)$ and $D_2'(0)$ in equation[4], the expression for A_0 can be determined. The system's expected uptime for (0,t] is provided by

$$\mu_{up}(t) = \int_0^t A_0(u)du$$

So that,

$$\mu_{up}^*(s) = \frac{A_0^*(s)}{s}$$

7. Busy Period Analysis

$B_i(t)$ is defined as the probability that, at time t=0, the system, which begins in the regenerative state Si$\in$ E, is undergoing repair as a result of a unit failure. To estimate these probabilities, we utilize simple probabilistic logics, on taking Laplace transformation and solving the consequent set of equations for $B_0^*(s)$, we have

$$B^*{}_0(s) = \frac{N_3(s)}{D_2(s)}$$

$$N_3(s) = q_{03}^{*}q_{34}^{*}[\psi_4 - q_{15}^{*}q_{51}^{*}b_1 - q_{10,9}^{*}q_{9,10}^{*}b_5 - q_{10,12}^{*}q_{12,10}^{*}b_6 + q_{13,1}^{*}q_{4,13}^{*}b_7 + q_{1,10}^{(9)*}(\psi_{10} + q_{10,9}^{*}\psi_9 + q_{10,11}^{*}\psi_{11} + q_{11,9}^{*}q_{10,11}^{*}\psi_9) + q_{15}^{*}q_{56}^{*}b_8 + q_{15}^{*}q_{59}^{(7)*}(\psi_9 + q_{9,10}^{*}\psi_{10} + q_{9,10}^{*}q_{10,11}^{*}\psi_{11}) + \psi_1(1 - q_{11,9}^{*}q_{9,10}^{*}q_{10,11}^{*})] + \psi_4(1 - q_{11,9}^{*}q_{9,10}^{*}q_{10,11}^{*}) + q_{1,10}^{(9)*}q_{01}^{*}b_9 + q_{02}^{*}q_{21}^{*}b_9 + q_{03}^{*}q_{31}^{(13)*}b_9 + q_{31}^{(13)*}[q_{03}^{*}q_{18}^{*}(\psi_8 b_1)] + q_{15}^{*}q_{56}^{*}[q_{03}^{*}\psi_6 b_1 + q_{03}^{*}q_{67}^{*}q_{9,10}^{*}(\psi_{10} + q_{10,11}^{*}\psi_{11})] + q_{15}^{*}q_{59}^{(7)*}(q_{03}^{*}q_{9,10}^{*}(\psi_{10} + q_{10,11}^{*}\psi_{11}) - q_{03}^{*}q_{10,12}^{*}q_{12,10}^{*}\psi_9) + q_{03}^{*}\psi_1 b_1 + q_{18}^{*}[q_{02}^{*}q_{21}^{*}(\psi_8 b_1) + q_{89}^{*}q_{9,10}^{*}(\psi_{10} + q_{10,11}^{*}\psi_{11})] + q_{01}^{*}\psi_8 b_1 + q_{15}^{*}[q_{01}^{*}q_{56}^{*}(\psi_6 b_1 + q_{67}^{*})\psi_9(1 - q_{10,12}^{*}q_{12,10}^{*}) + q_{67}^{*}q_{9,10}^{*}(\psi_{10} + q_{10,11}^{*}\psi_{11})] + q_{01}^{*}q_{59}^{(7)*}(\psi_9(1 - q_{10,12}^{*}q_{12,10}^{*}) + q_{9,10}^{*}\psi_{10}) + q_{02}^{*}q_{21}^{*}[\psi_6(q_{51}^{*} - q_{56}^{*}b_1)] + q_{59}^{(7)*}\psi_9(1 - q_{10,12}^{*}q_{12,10}^{*}) + q_{56}^{*}q_{67}^{*}b_{10} + q_{02}^{*}q_{51}^{*}(-\psi_2 b_1 + q_{03}^{*}q_{59}^{(7)*}q_{31}^{(13)*}\psi_9) + q_{9,10}^{*}b_{11} - q_{02}^{*}(q_{10,9}^{*} + q_{11,9}^{*}q_{10,11}^{*})(\psi_2 + \psi_1 q_{21}^{*}) + q_{01}^{*}\psi_1 + q_{02}^{*}(\psi_2 + q_{21}^{*}\psi_1)(1 - q_{10,12}^{*}q_{12,10}^{*})$$

here,

$b_1 = 1 - q_{10,9}^{*}q_{9,10}^{*} - q_{10,12}^{*}q_{12,10}^{*} - q_{11,9}^{*}q_{9,10}^{*}q_{10,11}^{*}$

$b_5 = \psi_4 - q_{13,1}^{*}q_{4,13}^{*}(\psi_1 + q_{18}^{*}\psi_8 + q_{15}^{*}q_{56}^{*}\psi_6)$

$b_6 = \psi_4 - q_{13,1}^{*}q_{4,13}^{*}(\psi_1 q_{18}^{*}\psi_8 + q_{18}^{*}q_{89}^{*}\psi_9 + q_{15}^{*}q_{59}^{(7)*}\psi_9 + q_{15}^{*}q_{56}^{*}q_{67}^{*}\psi_9)$

$b_7 = q_{18}^{*}(\psi_8 + q_{89}^{*}\psi_9 + q_{89}^{*}q_{9,10}^{*}\psi_{10} + q_{89}^{*}q_{9,10}^{*}q_{10,11}^{*}\psi_{11} - q_{11,9}^{*}q_{9,10}^{*}q_{10,11}^{*}\psi_8)$

$b_8 = \psi_6 + q_{67}^{*}\psi_9 + q_{67}^{*}q_{9,10}^{*}\psi_{10} - q_{10,12}^{*}q_{12,10}^{*}\psi_6 - q_{11,9}^{*}q_{9,10}^{*}q_{10,11}^{*}\psi_6 + q_{67}^{*}q_{9,10}^{*}q_{10,11}^{*}\psi_{11}$

$b_9 = \psi_{10} + \psi_9 q_{10,9}^{*} + q_{10,11}^{*}(\psi_{11} + q_{11,9}^{*}\psi_9)$

$b_{10} = \psi_9 + q_{9,10}^{*}\psi_{10} + q_{9,10}^{*}q_{10,11}^{*}\psi_{11} - q_{10,12}^{*}q_{12,10}^{*}\psi_9$

$b_{11} = q_{01}^{*}(\psi_1(q_{10,9}^{*} + q_{11,9}^{*}q_{10,11}^{*})) + q_{15}^{*}q_{59}^{(7)*}q_{10,11}^{*}\psi_{11}$

and, D_2(s) is same as given in equation [3].
The probability that the repairman will be busy in the long run is as follows:

$$B_0 = \lim_{t\to\infty} B_0(t) = \lim_{s\to 0}\ sB_0^{*}(s) = \frac{N_3(0)}{D_2^{'}(0)}$$

where,

$$N_3(0) = p_{03}p_{34}[\psi_4 - p_{15}p_{51}b_1 - p_{10,9}p_{9,10}b_5 - p_{10,12}p_{12,10}b_6 + p_{13,1}p_{4,13}b_7 + p_{1,10}^{(9)}(\psi_{10} + p_{10,9}\psi_9 + p_{10,11}\psi_{11} + p_{11,9}p_{10,11}\psi_9) + p_{15}p_{56}b_8 + p_{15}p_{59}^{(7)}(\psi_9 + p_{9,10}\psi_{10} + p_{9,10}p_{10,11}\psi_{11}) + \psi_1(1 - p_{11,9}p_{9,10}p_{10,11})] + \psi_4(1 - p_{11,9}p_{9,10}p_{10,11}) + p_{1,10}^{(9)}p_{01}b_9 + p_{02}p_{21}b_9 + p_{03}p_{31}^{(13)}b_9 + q_{31}^{(13)}[p_{03}p_{18}(\psi_8 b_1)] + p_{15}p_{56}[p_{03}\psi_6 b_1 + p_{03}p_{67}p_{9,10}(\psi_{10} + p_{10,11}\psi_{11})] + p_{15}p_{59}^{(7)}(p_{03}p_{9,10}(\psi_{10} + p_{10,11}\psi_{11}) - p_{03}p_{10,12}p_{12,10}\psi_9) + p_{03}\psi_1 b_1 + p_{18}[p_{02}p_{21}(\psi_8 b_1) + p_{89}p_{9,10}(\psi_{10} + p_{10,11}\psi_{11})] + p_{01}\psi_8 b_1 + p_{15}[p_{01}p_{56}(\psi_6 b_1 + p_{67})\psi_9(1 - p_{10,12}p_{12,10}) + p_{67}p_{9,10}(\psi_{10} + p_{10,11}\psi_{11})] + p_{01}p_{59}^{(7)}(\psi_9(1 - p_{10,12}p_{12,10}) + p_{9,10}\psi_{10}) + p_{02}p_{21}[\psi_6(p_{51} - p_{56}b_1)] + p_{59}^{(7)}\psi_9(1 - p_{10,12}p_{12,10}) + p_{56}p_{67}b_{10} + p_{02}p_{51}(-\psi_2 b_1 + p_{03}p_{59}^{(7)}p_{31}^{(13)}\psi_9) + p_{9,10}b_{11} - p_{02}(p_{10,9} + p_{11,9}p_{10,11})(\psi_2 + \psi_1 p_{21}) + p_{01}\psi_1 + p_{02}(\psi_2 + p_{21}\psi_1)(1 - p_{10,12}p_{12,10})$$

here,

$b_1 = 1 - p_{10,9}p_{9,10} - p_{10,12}p_{12,10} - p_{11,9}p_{9,10}p_{10,11}$

$b_5 = \psi_4 - p_{13,1}p_{4,13}(\psi_1 + p_{18}\psi_8 + p_{15}p_{56}\psi_6)$

$b_6 = \psi_4 - p_{13,1}p_{4,13}(\psi_1 p_{18}\psi_8 + p_{18}p_{89}\psi_9 + p_{15}p_{59}^{(7)}\psi_9 + p_{15}p_{56}p_{67}\psi_9)$

$b_7 = p_{18}(\psi_8 + p_{89}\psi_9 + p_{89}p_{9,10}\psi_{10} + p_{89}p_{9,10}p_{10,11}\psi_{11} - p_{11,9}p_{9,10}p_{10,11}\psi_8)$

$b_8 = \psi_6 + p_{67}\psi_9 + p_{67}p_{9,10}\psi_{10} - p_{10,12}p_{12,10}\psi_6 - p_{11,9}p_{9,10}p_{10,11}\psi_6 + p_{67}p_{9,10}p_{10,11}\psi_{11}$

$b_9 = \psi_{10} + \psi_9 p_{10,9} + p_{10,11}(\psi_{11} + p_{11,9}\psi_9)$

$b_{10} = \psi_9 + p_{9,10}\psi_{10} + p_{9,10}p_{10,11}\psi_{11} - p_{10,12}p_{12,10}\psi_9$

$b_{11} = p_{01}(\psi_1(p_{10,9} + p_{11,9}p_{10,11})) + p_{15}p_{59}^{(7)}p_{10,11}\psi_{11}$

and $D_2^{'}(0)$ is same as obtained in [5].
During (0,t], the repairman's expected busy time is given by

$$\mu_b(t) = \int_0^t B_0(u)du$$

So that,

$$\mu_b^*(s) = \frac{B_0^*(s)}{s}$$

8. Expected Number of Repairs

When the system begins from regenerative state S_i, $V_i(t)$ is described as the expected number of repairs over the time range (0,t] of the failed units. Furthermore, given the definition of $V_i(t)$, the recurrence relations can be framed easily and, taking their Laplace- Stieltjes transformations and solving the consequent set of equations for $\widetilde{V}_0(s)$, we get
$\widetilde{V}_0(s) = N_4(s)/D_3(s)$
where,

$$\begin{aligned} N_4(s) = {} & \widetilde{Q}_{02}\widetilde{Q}_{21}[-\widetilde{Q}_{10}b_2 - \widetilde{Q}_{1,10}^{(9)}b_2\widetilde{Q}_{18}\widetilde{Q}_{89}\widetilde{Q}_{9,10}b_2 - \widetilde{Q}_{15}\widetilde{Q}_{59}^{(7)}\widetilde{Q}_{9,10}b_2 - \widetilde{Q}_{15}\widetilde{Q}_{56}\widetilde{Q}_{67}\widetilde{Q}_{9,10}b_2] \\ & + \widetilde{Q}_{03}\widetilde{Q}_{34}[\widetilde{Q}_{13,1}\widetilde{Q}_{4,13}b_1 - \widetilde{Q}_{10}b_1 - \widetilde{Q}_{1,10}^{(9)}b_2 + \widetilde{Q}_{15}\widetilde{Q}_{51}b_1 - \widetilde{Q}_{18}\widetilde{Q}_{89}\widetilde{Q}_{9,10}b_2 - \widetilde{Q}_{15}\widetilde{Q}_{56} \\ & \widetilde{Q}_{67}\widetilde{Q}_{9,10}b_2 - \widetilde{Q}_{15}\widetilde{Q}_{59}^{(7)}\widetilde{Q}_{9,10}b_2] + \widetilde{Q}_{18}\widetilde{Q}_{89}[-\widetilde{Q}_{01}\widetilde{Q}_{9,10}b_2 - \widetilde{Q}_{03}\widetilde{Q}_{9,10}\widetilde{Q}_{31}^{(13)}b_2]\widetilde{Q}_{15} \\ & [-\widetilde{Q}_{01}\widetilde{Q}_{59}^{(7)}\widetilde{Q}_{9,10}b_2 - \widetilde{Q}_{01}\widetilde{Q}_{56}\widetilde{Q}_{67}\widetilde{Q}_{9,10}b_2] - \widetilde{Q}_{01}\widetilde{Q}_{10}b_1 \end{aligned}$$

here,

$b_1 = 1 - \widetilde{Q}_{10,9}\widetilde{Q}_{9,10} - \widetilde{Q}_{10,12}\widetilde{Q}_{12,10} - \widetilde{Q}_{11,9}\widetilde{Q}_{9,10}\widetilde{Q}_{10,11}$

$b_2 = 1 - \widetilde{Q}_{10,0} - \widetilde{Q}_{10,12}\widetilde{Q}_{12,10}$

and $D_3(s)$ is written by replacing $\overset{*}{q}_{ij}$ and $q_{ij}^{(k)*}$ by $\widetilde{Q}_{ij}$ and $\widetilde{Q}_{ij}^{(k)}$ respectively in equation[3].
The expected number of repairs per unit over time in the steady state is represented as

$V_0 = \lim_{t\to\infty} V_0(t) = \lim_{s\to 0} s \tilde{V}_0(s) = \frac{N_4(0)}{D_2'(0)}$
where,

$$N_4(0) = p_{02}p_{21}[p_{10}b_2 - p_{1,10}^{(9)}b_2 - p_{18}p_{89}p_{9,10}b_2 - p_{15}p_{59}^{(7)}p_{9,10}b_2 - p_{15}p_{56}p_{67}p_{9,10}b_2] + p_{03}p_{34}$$
$$[p_{13,1}p_{4,13}b_1 - p_{10}b_1 - p_{1,10}^{(9)}b_2 + p_{15}p_{51}b_1 - p_{18}p_{89}p_{9,10}b_2 - p_{15}p_{56}p_{67}p_{9,10}b_2 - p_{15}p_{59}^{(7)}$$
$$p_{9,10}b_2] + p_{18}p_{89}[p_{01}p_{9,10}b_2 - p_{03}p_{9,10}p_{31}^{(13)}b_2]p_{15}[-p_{01}p_{59}^{(7)}p_{9,10}b_2 - p_{01}p_{56}p_{67}p_{9,10}b_2] - p_{01}p_{10}b_1)$$

here,

$$b_1 = 1 - p_{10,9}p_{9,10} - p_{10,12}p_{12,10} - p_{11,9}p_{9,10}p_{10,11}$$

$$b_2 = 1 - p_{10,0} - p_{10,12}p_{12,10}$$

9. Profit Function Analysis

Having determined the reliability charateristics, the profit function P(t) can be calculated. Profit is defined as excess of revenue over the cost, hence the expected total profit made during(0,t] is expressed as :
P(t) = Expected total revenue in(0,t] - Expected total expenditure in(0,t]

= $K_0\mu_{up}(t) - K_1\mu_b(t) - K_2V_0(t)$

where,
K_0 = revenue per unit up time of the system.
K_1 = The cost per unit during which the repairman is engaged to fix the failed unit.
K_2 = Cost of repair of each unit.
The expected total gain per unit of time in steady state is provided by:

$$P = \lim_{t\to\infty} \frac{P(t)}{t} = \lim_{s\to 0} s^2 \overset{*}{P}(s)$$

Therefore, we have

$$P = K_0A_0 - K_1B_0 - K_2V_0 \tag{6}$$

10. Estimation of the Parameters, MTSF, And Profit Function

10.1. Classical Estimation

10.1.1 ML Estimation

Let us take
$\underset{\sim}{X_1} = (x_{11}, x_{12}, ..., x_{1n_1})$, $\underset{\sim}{X_2} = (x_{21}, x_{22}, ...x_{2n_2})$, $\underset{\sim}{X_3} = (x_{31}, x_{32}, ..., x_{3n_3})$,
$\underset{\sim}{X_4} = (x_{4_1}, x_{42}, ..., x_{4n_4})$, $\underset{\sim}{X_5} = (x_{51}, x_{52}, ..., x_{5n_5})$, $\underset{\sim}{X_6} = (x_{61}, x_{62}, ..., x_{6n_6})$,
$\underset{\sim}{X_7} = (x_{71}, x_{72}, ..., x_{7n_7})$ and $\underset{\sim}{X_8} = (x_{81}, x_{82}, ..., x_{8n_8})$
Therefore, Likelihood function of combined sample is :

$$L = (X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8 | \alpha_1, \alpha_2, \alpha_3, \lambda_1, \lambda_2, \lambda_3, \gamma_1, \gamma_2)$$

The pdf of exponential distribution is $f(x, \lambda) = \lambda \exp(-\lambda x)$, $x > 0$, λ>0

$$L = {\alpha_1}^{n_1}{\alpha_2}^{n_2}{\alpha_3}^{n_3}{\lambda_1}^{n_4}{\lambda_2}^{n_5}{\lambda_3}^{n_6}{\gamma_1}^{n_7}{\gamma_2}^{n_8} \exp -(\alpha_1W_1 + \alpha_2W_2 + \alpha_3W_3 + \lambda_1W_4 + \lambda_2W_5$$
$$+ \lambda_3W_6 + \gamma_1W_7 + \gamma_2W_8)$$

Here, $W_i = \sum_{n=1}^{n_i} x_{ij}$; i = 1,2,3,4,5,6,7,8
On solving, we get

$$logL = n_1 log\alpha_1 + n_2 log\alpha_2 + n_3 log\alpha_3 + n_4 log\lambda_1 + n_5 log\lambda_2 + n_6 log\lambda_3 + n_7 log\gamma_1 + n_8 log\gamma_2 - (\alpha_1 W_1 + \alpha_2 W_2 + \alpha_3 W_3 + \lambda_1 W_4 + \lambda_2 W_5 + \lambda_3 W_5 + \gamma_1 W_6 + \gamma_2 W_8) \quad (7)$$

The, MLE $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3\ , \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\gamma}_1\ , \hat{\gamma}_2)$ of the parameters $(\alpha_1, \alpha_2, \alpha_3, \lambda_1, \lambda_2, \lambda_3, \gamma_1, \gamma_2)$ are as under

$\hat{\alpha}_1 = \frac{n_1}{W_1}$, $\hat{\alpha}_2 = \frac{n_2}{W_2}$
$\hat{\alpha}_3 = \frac{n_3}{W_3}$, $\hat{\lambda}_1 = \frac{n_4}{W_4}$
$\hat{\lambda}_2 = \frac{n_5}{W_5}$, $\hat{\lambda}_3 = \frac{n_6}{W_6}$
$\hat{\gamma}_1 = \frac{n_7}{W_7}$, $\hat{\gamma}_2 = \frac{n_8}{W_8}$

The asymptotic distribution of$(\hat{\alpha_1} - \alpha_1,\ \hat{\alpha_2} - \alpha_2, \hat{\alpha_3} - \alpha_3, \hat{\lambda}_1\ - \lambda_1, \hat{\lambda}_2 - \lambda_2, \hat{\lambda}_3 - \lambda_3,\ \hat{\gamma_1} - \gamma_1, \hat{\gamma_2} - \gamma_2) \sim N_8(0, I^{-1})$, where I is the Fisher Information matrix with diagonal elements as

$I_{11} = \frac{n_1}{\alpha_1^2}, \quad I_{22} = \frac{n_2}{\alpha_2^2}, \quad I_{33} = \frac{n_3}{\alpha_3^2}, \quad I_{44} = \frac{n_4}{\lambda_1^2}, \quad I_{55} = \frac{n_5}{\lambda_2^2}, \quad I_{66} = \frac{n_6}{\lambda_3^2}, \quad I_{77} = \frac{n_7}{\gamma_1^2}, \quad I_{88} = \frac{n_8}{\gamma_2^2}$

and all non-diagonal elements are zero. Using MLE's invariance property, we can extract The MLE $\hat{M}$ & $\hat{P}$ of MTSF and Profit function. Also, asymptotic distribution of $(\hat{M} - M) is N(0, A' I^{-1} A)$ & that of $(\hat{P} - P) is N(0, B' I^{-1} B)$, where

$A' = (\frac{\delta M}{\delta \alpha_1}, \frac{\delta M}{\delta \alpha_2}, \frac{\delta M}{\delta \alpha_3}, \frac{\delta M}{\delta \alpha_2}, \frac{\delta M}{\delta \lambda_2}, \frac{\delta M}{\delta \lambda_3}, \frac{\delta M}{\delta \gamma_1}, \frac{\delta M}{\delta \gamma_2})$

$B' = (\frac{\delta P}{\delta \alpha_1}, \frac{\delta P}{\delta \alpha_2}, \frac{\delta P}{\delta \alpha_3}, \frac{\delta P}{\delta \lambda_1}, \frac{\delta P}{\delta \lambda_2}, \frac{\delta P}{\delta \lambda_3}, \frac{\delta P}{\delta \gamma_1}, \frac{\delta P}{\delta \gamma_2})$

10.2. Bayesian Estimation

Bayesian estimation is a statistical approach which is utilized to determine the impact of prior knowledge as well as the sample information on prior distributions of the parameters under study. The parameters involved in the model are taken to be random variables having independent Gamma prior distribution. Here, we estimate the unknown parameters taking into account the gamma prior distribution and the corresponding PDFs as

$$\alpha_1 \sim Gamma(a_1, b_1) \qquad (\alpha_1, a_1, b_1) > 0, \quad (8)$$

$$\alpha_2 \sim Gamma(a_2, b_2) \qquad (\alpha_2, a_2, b_2) > 0, \quad (9)$$

$$\alpha_3 \sim Gamma(a_3, b_3) \qquad (\alpha_3, a_3, b_3) > 0, \quad (10)$$

$$\lambda_1 \sim Gamma(a_4, b_4) \qquad (\lambda_1, a_4, b_4) > 0, \quad (11)$$

$$\lambda_2 \sim Gamma(a_5, b_5) \qquad (\lambda_2, a_5, b_5) > 0, \quad (12)$$

$$\lambda_3 \sim Gamma(a_6, b_6) \qquad (\lambda_3, a_6, b_6) > 0, \quad (13)$$

$$\gamma_1 \sim Gamma(a_7, b_7) \qquad (\gamma_1, a_7, b_7) > 0, \quad (14)$$

$$\gamma_2 \sim Gamma(a_8, b_8) \qquad (\gamma_2, a_8, b_8) > 0, \quad (15)$$

Here,$a_i\ and\ b_i$ (i = 1,2,3,4,5,6,7,8) denotes the shape and scale parameters
Now using likelihood function and taking prior distributions, the posterior distributions of these parameters are calculated as given below:

$$\alpha_1 | \underset{\sim}{X_1} \sim Gamma(n_1 + a_1, b_1 + W_1) \quad (16)$$

$$\alpha_2 | \underset{\sim}{X_2} \sim Gamma(n_2 + a_2, b_2 + W_2) \quad (17)$$

$$\alpha_3|\underset{\sim}{X_3} \sim Gamma(n_3 + a_3, b_3 + W_3) \tag{18}$$

$$\lambda_1|\underset{\sim}{X_4} \sim Gamma(n_4 + a_4, b_4 + W_4) \tag{19}$$

$$\lambda_2|\underset{\sim}{X_5} \sim Gamma(n_5 + a_5, b_5 + W_5) \tag{20}$$

$$\lambda_3|\underset{\sim}{X_6} \sim Gamma(n_6 + a_6, b_6 + W_6) \tag{21}$$

$$\gamma_1|\underset{\sim}{X_7} \sim Gamma(n_7 + a_7, b_7 + W_7) \tag{22}$$

$$\gamma_2|\underset{\sim}{X_8} \sim Gamma(n_8 + a_8, b_8 + W_8) \tag{23}$$

To derive width of HPD intervals and Bayes estimates for parameters, we generate observations from the posterior distributions listed above. To obtain Bayesian estimation of MTSF and profit function, the above draws are put directly into the equations [2] & [6]. Using a squared error loss function, Bayesian estimates of parameters and reliability characteristics are derived from the sample means of the relevant drawings.

11. Simulation Study

To explore the behaviour of parameters, estimates and reliability aspects, a simulation study is carried out. The values of the Standard Error (SE)/Posterior Standard Error (PSE) and the width of confidence/HPD intervals are shown in table 1-6. Samples of sizes $n_1 = n_2 = n_3 = n_4 = n_5 = n_6 = n_7 = n_8 = 100$ were taken from the six investigated distributions while presuming various parameter values as shown in Tables 1-6. The number of iterations used is 10000. R software is used for the computations purpose.

Table 1: *MTSF values for fixed $\lambda_1 = 0.05$ and varying α_1*

α_1	True MTSF	MLE.MTSF	SE	C.I	Bayes MTSF	PSE	HPD Interval
0.1	13.438	10.101	0.0107	0.0078	10.024	0.00070	0.00051
0.2	5.188	5.075	0.0099	0.0074	5.021	0.00063	0.00046
0.3	3.689	3.420	0.0100	0.0073	3.353	0.00062	0.00046
0.4	2.986	2.565	0.0099	0.0074	2.520	0.00061	0.00044
0.5	2.565	2.091	0.0102	0.0074	2.020	0.00061	0.00044
0.6	2.281	1.756	0.0104	0.0078	1.686	0.00061	0.00045
0.7	2.075	1.518	0.0106	0.0078	1.448	0.00059	0.00044
0.8	1.918	1.343	0.0109	0.0079	1.269	0.00061	0.00045
0.9	1.795	1.204	0.0111	0.0082	1.131	0.00062	0.00045
1	1.696	1.123	0.0111	0.0083	1.020	0.00046	0.00044

Table 2: *MTSF values for fixed λ_1=0.45 and varying α_1*

α_1	True MTSF	MLE.MTSF	SE	C.I	Bayes MTSF	PSE	HPD Interval
0.1	16.785	10.257	0.0023	0.017	10.063	0.0016	0.0012
0.2	5.934	5.178	0.0016	0.011	5.043	0.0011	0.00082
0.3	4.163	3.477	0.0013	0.010	3.368	0.0094	0.00069
0.4	3.341	2.626	0.0012	0.0093	2.531	0.00085	0.00062
0.5	2.846	2.111	0.0012	0.0089	2.028	0.00078	0.00058
0.6	2.511	1.76	0.0011	0.0086	1.693	0.00076	0.00056
0.7	2.268	1.528	0.0011	0.0086	1.454	0.00074	0.00053
0.8	1.082	1.354	0.0011	0.0086	1.275	0.00072	0.00053
0.9	1.936	1.230	0.0011	0.0081	1.135	0.00070	0.00052
1	1.817	1.114	0.0011	0.0086	1.024	0.00071	0.00051

Table 3: *MTSF values for fixed λ_1=0.85 and varying α_1*

α_1	True MTSF	MLE.MTSF	SE	C.I	Bayes MTSF	PSE	HPD Interval
0.1	19.686	10.359	0.032	0.024	10.111	0.0026	0.0019
0.2	6.500	5.219	0.020	0.015	5.064	0.0016	0.0011
0.3	4.524	3.479	0.016	0.012	3.382	0.0012	0.00091
0.4	3.612	2.661	0.014	0.010	2.541	0.0010	0.00078
0.5	3.063	2.135	0.013	0.0098	2.037	0.00095	0.00071
0.6	2.691	1.804	0.012	0.0093	1.700	0.00091	0.00066
0.7	2.419	1.548	0.012	0.0091	1.460	0.00085	0.00062
0.8	2.211	1.377	0.011	0.0088	1.280	0.00081	0.00060
0.9	2.047	1.219	0.011	0.0088	1.140	0.00080	0.00059
1	1.913	1.104	0.011	0.0087	1.028	0.00077	0.00057

Table 4: *Profit values for fixed λ_1=0.05 and varying α_1*

α_1	True profit	MLE.Profit	SE	C.I	Bayes Profit	PSE	HPD Interval
0.1	824.23	119.48	1.43	2.09	43.20	1.34	0.99
0.2	593.60	114.18	1.37	1.99	42.37	1.31	0.80
0.3	496.60	104.81	1.30	1.99	41.14	1.17	0.74
0.4	429.96	100.09	1.26	1.82	39.83	1.13	0.73
0.5	377.55	95.95	1.17	1.76	38.73	1.07	0.80
0.6	334.00	87.59	1.12	1.66	37.80	0.88	0.72
0.7	296.75	83.67	1.07	1.59	36.66	1.06	0.98
0.8	264.30	80.17	1.05	1.51	35.84	0.93	0.87
0.9	235.69	85.81	0.99	1.46	34.96	0.96	0.85
1	210.22	83.14	0.96	1.40	33.76	0.93	0.86

Table 5: *Profit values for fixed λ_1=0.45 and varying α_1*

α_1	True profit	MLE.Profit	SE	C.I	Bayes Profit	PSE	HPD Interval
0.1	852.97	121.30	1.47	2.12	43.42	1.41	1.02
0.2	638.57	115.93	1.40	2.04	42.23	1.34	1.09
0.3	539.29	110.87	1.33	1.98	41.41	1.09	0.96
0.4	469.04	106.09	1.29	1.84	39.97	1.19	0.95
0.5	413.21	101.62	1.21	1.76	39.12	1.03	0.82
0.6	366.64	97.00	1.16	1.69	37.91	0.93	0.98
0.7	326.78	92.87	1.11	1.62	36.84	0.99	0.70
0.8	292.10	89.00	1.06	1.54	35.87	1.01	0.96
0.9	261.57	84.73	1.01	1.50	34.84	0.77	0.85
1	210.22	81.18	0.97	1.44	33.90	0.79	0.83

Table 6: *Profit values for fixed $\lambda_1 = 0.85$ and varying α_1*

α_1	True profit	MLE.Profit	SE	C.I	Bayes Profit	PSE	HPD Interval
0.1	871.88	123.18	1.49	2.20	43.45	1.32	0.99
0.2	671.99	117.34	1.45	2.10	42.33	1.26	0.75
0.3	572.73	112.39	1.35	2.02	41.31	1.05	0.90
0.4	500.62	107.48	1.29	1.93	40.07	1.18	0.90
0.5	442.57	103.13	1.24	1.82	39.12	1.009	0.82
0.6	393.82	98.24	1.19	1.74	38.12	1.008	0.77
0.7	351.93	94.19	1.13	1.66	36.93	1.06	0.87
0.8	215.41	89.78	1.07	1.60	35.84	1.05	0.74
0.9	283.24	85.91	1.03	1.53	35.26	0.87	0.70
1	254.65	82.00	0.96	1.45	34.02	0.85	0.84

12. Graphical Study

A graphical analysis of the system model provides a more insightful and vivid representation of system behaviour. So for more concrete study, we plot MTSF and Profit function wrt α_1 failure rate of unit A for different values of λ_1 repair rate of unit A as 0.05, 0.45 and 0.85. Here all other parameters are fixed α_2= 0.9, α_3= 0.15, λ_2=0.35, λ_3=0.45, c=0.7, γ_1=0.6 and γ_2=0.8.

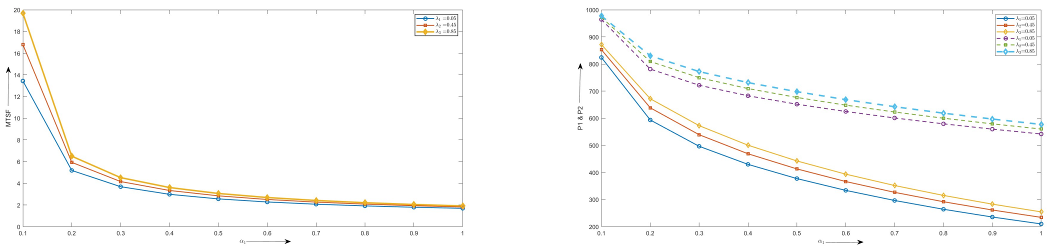

Figure 2: *(a)Behaviour of MTSF wrt to α_1 for different values of λ_1 and (b)Behaviour of P_1&P_2 wrt to α_1 for different values of λ_1*

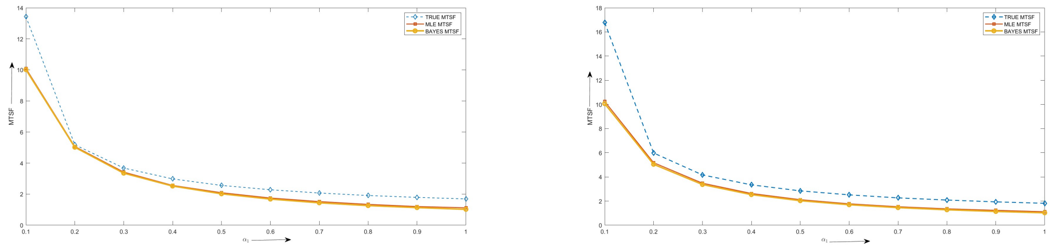

Figure 3: *(a)Behaviour of True MTSF, MLE MTSF & Bayes MTSF wrt to α_1 for λ_1= 0.05 and (b) Behaviour of True MTSF, MLE MTSF & Bayes MTSF wrt to α_1 for λ_1= 0.45*

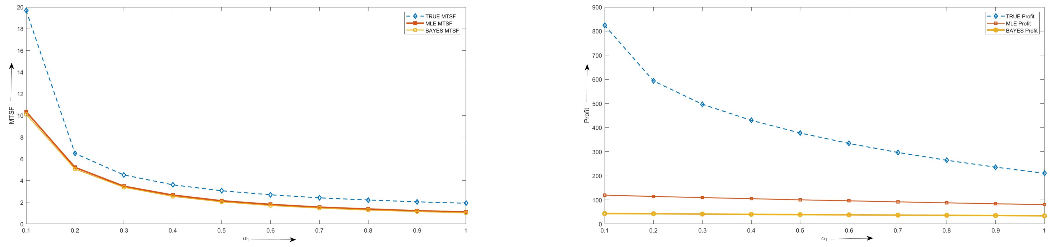

Figure 4: *(a) Behaviour of True MTSF, MLE MTSF & Bayes MTSF wrt to α_1 for λ_1= 0.85 and (b) Behaviour of True Profit, MLE Profit & Bayes Profit wrt to α_1 for λ_1= 0.05*

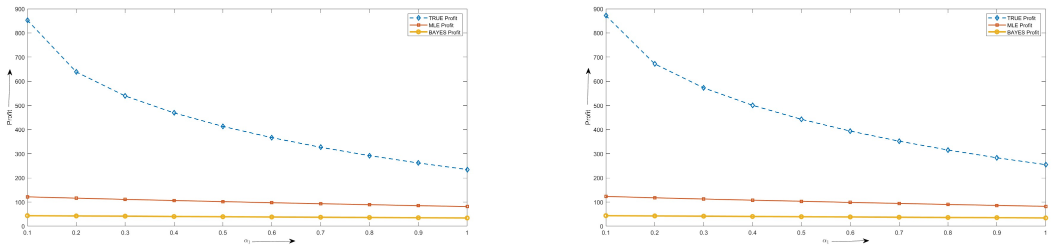

Figure 5: *(a)Behaviour of True Profit, MLE Profit & Bayes Profit wrt to α_1 for λ_1= 0.45 and (b) Behaviour of True Profit, MLE Profit & Bayes Profit wrt to α_1 for λ_1= 0.85*

13. Discussion and Conclusion

1. Tables and figures exhibits that MTSF decreases as the failure rate α_1 increases, but increases as the repair rate λ_1 increases. The same trend is followed for the profit function.
2. Tables 1-6 indicate that for fixed and variable parameters, Bayes estimates of the MTSF and profit function perform better than MLEs in terms of SE as well as in terms of the width of the confidence intervals as they have lower PSE and the width of HPD intervals.
3. Based on the above discussions, we conclude that for estimating the MTSF and Profit function of the analyzed model, Bayes approach outperforms the Classical approach.

14. Acknowledgement

A special thanks to Department of Science and Technology, Govt. of India for providing financial help in the form of INSPIRE fellowship.

References

[1] Sharma, Akshita and Kumar, Pawan (2019). Analysis of Reliability Measures of two Identical Unit System with One Switching Device and Imperfect Coverage *Reliability: Theory & Applications*, 14(1):44–52.

[2] Trivedi, Kishor S(2002). Probability & Statistics with Reliability, Queuing and Computer Science Applications *2nd Edition John Wiley & Sons, New York.*

[3] Gupta, Rakesh and Kishan, Ram and Kumar, Pawan (1999). A two Non-Identical-Unit Parallel System with Correlated Lifetimes *International Journal of Systems Science*, 30(10):1123–1129.

[4] Wang, Kuo-Hsiung and Chen, Yu-Ju(2009). Comparative Analysis of Availability Between three Systems with General Repair Times, Reboot Delay and Switching Failures *Applied Mathematics and Computation*, 215(1):384–394.

[5] Pham, Hoang(1992). Reliability Analysis of a High Voltage System with Dependent Failures and Imperfect Coverage*Reliability Engineering & System Safety*, 37(1):25–28.

[6] Ke, Jau-Chuan and Liu, Tzu-Hsin(2014). A Repairable System with Imperfect Coverage and Reboot *Applied Mathematics and Computation*, 246:148–158.

[7] Kumar, Pankaj and Jain, Madhu(2020). Reliability Analysis of a Multi-Component Machining System with Service Interruption, Imperfect Coverage, and Reboot *Reliability Engineering & System Safety*, 202:106991.

[8] Kadyan, MS and Malik, Suresh Chander and others(2020). Stochastic Analysis of a three Unit Non-Identical Repairable System with Simultaneous Working of Cold Standby Units *Journal of Reliability and Statistical Studies*, 385–400.

[9] Saxena, Vashali and Gupta, Rakesh and Singh, Bhupendra(2023). Classical and Bayesian Stochastic Analysis of a two Unit Parallel System with Working and Rest Time of Repairman *Reliability: Theory & Applications*, 18(1(72))43–55.

[10] Kishan, Ram and Jain, Divya(2012). A two Non-Identical Unit Standby System Model with Repair, Inspection and Post-Repair Under Classical and Bayesian Viewpoints *Journal of reliability and statistical studies*, 85–103.