ANALYSIS OF NON-MARKOVIAN BATCH ARRIVAL RETRIAL QUEUE WITH PRIORITY SERVICES, IMMEDIATE FEEDBACK, PUSH OUT, DIFFERENTIATED BREAKDOWNS, DELAYED REPAIR, RANDOMIZED VACATION

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Abstract

Priority and ordinary customers arrive according to Poisson processes, and their service time based on the general distribution. The server constantly offers a single service for both priority and ordinary customers. We compute the Laplace transforms of the time-dependent probabilities of system states using the probability generating function and supplementary variable technique. Numerical results are obtained which are also examined to facilitate the sensitivity analysis of system descriptions.

Keywords: Batch Arrivals; Priority Queues; Immediate Feedback; Push out; Differentiated Breakdowns; Delayed Repair; Randomized Vacation.
AMS Subject Classification (2010): 60K25, 68M30, 90B22.

1. Introduction

We see several queueing situations every day where customers must wait for service and there is a delay in providing it. Retrial queues in queueing theory have been the topic of a lot of exciting research over the last two decades. The concept of retrial queues has attracted the attention of numerous scholars and received important contributions from them. An $M/M/1$ retrial queueing system with Poisson arrival flows, impatient customers, breakdown, collisions was studied by Danilyuk et al. [9]. Nazarov et al. [16] investigated how, depending on whether the server is busy or idle, it is dependent on random failures and repairs in a retry queuing system with a finite number of sources and customer collision. For the aggregation of the customers and their group service, D'Arienzo et al. [10] created a single-server retrial queue with a MAP flow, PH service times, and a finite capacity. Ahuja et al. [1] explored the retrial queueing system with an optional service and finite population subject to balking. Pavai Madheswari et al. [17] analysed at an M/G/1 retrial queueing system with two service phases, the second of which is optional, and a server working on a Bernoulli vacation schedule. Innovative applications for performance study of various systems in telecommunications, data split networks, traffic management on high-speed networks, and production engineering make use of these queueing models.

The literature on retrial queueing has extensively researched retrial queues with various customer categories. An important component of priority discipline is preemptive and non-preemptive priorities. D'Apice et al. [11] considered a priority queueing model with many types of requests and restricted processor sharing. Ayyappan and Thilgavathy [3] determined priority queueing system with breakdown, repair, discouragement, single vacation, standby server, negative arrival and impatient customers. Li et al. [15] investigate equilibrium queueing strategies in an unobservable non-preemptive priority queue with homogeneous customers. Ammar and Rajadurai [2] introduced preemptive priority retrial queueing system with disaster under working breakdown services.

Most of the time, while discussing queueing, it is generally thought that the server is always accessible. However, server failure made the significant impact in queueing system. Therefore, in order to establish retrial queueing models, it is essential to carry out investigation on the retrial queue with breakdowns. Choudhury and Kalita [8] described a non-Markovian queueing system with breakdown, delayed repair and two general heterogeneous service, optional service. Krishna Kumar et al. [14] examined a Markovian retrial queue where the server is subject to breakdowns and repairs. Gao et al. [12] studied an M/G/1 retrial queue with two types of breakdowns. Ayyappan and Gowthami [5] researched the single server classical queueing system MAP/PH/1 with breakdown, repair, Bernoulli vacation and setup time. Begum and Choudhry[7] investigated a $M/(G1, G2)/1$ queue with service interruption consisting of a definite repairability.

Customers may be serviced more than once for particular reasons in several queueing situations. Customers have to re-join the queue and wait in queue after the service is completed. Optional re-service is a concept that can be considered as immediate feedback in this regard. The customer receives their service in the first step, and if there is any issue with it or they need it again, they will receive it immediately without having to wait in queue. Re-service has several practical applications in places like bank desks, functioning ATMs, large supermarkets, and medical facilities etc., The idea of immediate feedback (re-service) has been addressed by some authors, including Azhagappan and Deepa [6], Ayyappan and Deepa [4] and Jose and Deepthi [13]. According to the previously mentioned literature, a customer who wants to receive more service must visit the server once more at that moment.

The interesting parameter in this chapter is the randomized vacation policy. It is described as follows: After the vacation completion, if there is at least one unit present in the system, then the server immediately commence the service. Otherwise, the server will decide either remains idle or go for another vacation, if no units present in the system. The concept of variant vacation policy was proposed by Takagi [18], which is a generalization of the single and multiple vacation for the $M/G/1$ queueing system. Ke et al. [19] studied an $M^X/G/1$ queueing system with a randomized vacation policy and at most J vacations. Geo and Yao [20] developed this vacation policy for an $M^X/G/1$ queueing system, in which the server takes randomized vacation policy and at most J working vacations.

There are many papers dealing with unit's abandoned behaviour. Recently, Gao et al. [21] studied an $M/G/1$ retrial queue with abandoned customers and multi-optional vacations. Krishnamoorthy et al. [22] presented an $M_1, M_2/PH/1$ retrial queueing system with pre-emptive priority service, orbital search and abandoned units in which the retrial is failed, then the failed units abandoned the system with certain probabilities. In this model, the arriving ordinary unit may remove the ordinary unit, who is getting service from the system. Here, the interrupting ordinary unit is referred as the abandoned unit.

We consider non-Markovian batch arrival retrial queue with priority services, discouragement, re-service, differentiate breakdown, restoration and delayed vacation. Priority customers and ordinary customers arrive according to Poisson processes, and their service time based on the general distribution. The server consistently provides a single service for both priority customers and ordinary customers. In the event of the server being unavailable, ordinary customers may choose to balk the system. Server failure may happen at any time during normal engaged period. The two types of system failure are hard and soft failures. Hard failure can be characterised as an equipment breakdown which demands the availability of a skilled repair person, which is an extensive process. Soft failure is described as breakdown based on by circumstances as instead of mechanical components, and it can be generally resolved by restarting the system. Customer may re-enter the system as a feedback customer for receiving normal service due to inadequate quality of service after every priority service is completed. The server goes on vacation after priority services are completed; the time it takes the server to go on vacation is known as delay time.

The article's remaining content is formatted as follows: In Section 2, the mathematical model is presented. and the distribution of queue sizes is analysed in Section 3. Section 4 contains the exact expression for the governing equation. Section 5 of this article discusses steady state analysis. Section 6 lists stability condition. Section 7 provides an illustration of how system performance measures have an impact. Section 8 exhibits particular cases. In Sections 9 and 10, conclusions are drawn after deriving numerical and graphical results.

1.1. Integrating the Model into Real-life Situations

In the online food delivery network, independent contractors are responsible for delivering food orders to customers in their designated service areas. These contractors are self-employed individuals who have chosen to work with the food delivery platform, offering their transportation services to ensure that customers receive their meals promptly and efficiently. The nature of the food delivery business can sometimes lead to fluctuations in the availability of delivery orders within a specific area. This could be due to various factors, such as changes in customer demand, local events, or even the time of day. When independent contractors find that there are no orders available in their assigned service area, they may face a few challenges and considerations. To maintain their income and professional engagement during periods of low order availability, these contractors might choose to take a break from their work or explore other temporary job opportunities. By doing so, they can keep themselves occupied and ensure a steady income while waiting for orders to become available in their area again. This approach allows them to manage their workload and personal commitments more effectively.

Once the independent contractors have completed their temporary work or vacation, they can return to the online food delivery network and resume their services when orders are available in their area. This flexibility is crucial for contractors, as it enables them to balance their professional and personal lives while staying connected to the platform and being ready to serve customers when needed. In conclusion, the online food delivery networks independent contractors operate within separate service areas, and when delivery orders are scarce in their region, they may opt for vacations or other work to maintain their income and professional engagement. Upon the return of available orders, these contractors can resume their food delivery services, ensuring a balance between their work and personal lives. This flexibility helps them adapt to the dynamic nature of the food delivery business and maintain a sustainable work arrangement.

2. Mathematical Description

- **Arrival Process :**
  Two distinct customer arrive in batches through separate Poisson compound processes. $\lambda_p, \lambda_o > 0$ are used to indicate, for PC and OC, the respective arrival rates. Assume the initial order probability for both priority and ordinary customers $\lambda_p c_i dt \ (i = 1, 2, 3, ...)$ and $\lambda_o c_j dt \ (j = 1, 2, 3, ...)$respectively. The system has $i$and j batch size customers enters within a short period of time $(t, t + dt)$. Here, $0 \leq c_i, c_j \leq 1$, $\sum_{i/j=1}^{\infty} c_{i/j} = 1$.
- **Retrial Service Process :**
  Customers who are on retrial are treated the same as ordinary customers. Customers on a retrial are regular consumers. These customers will eventually return to orbit and seek their services again if the server is engaged or unavailable. Retrial service time is characterised by

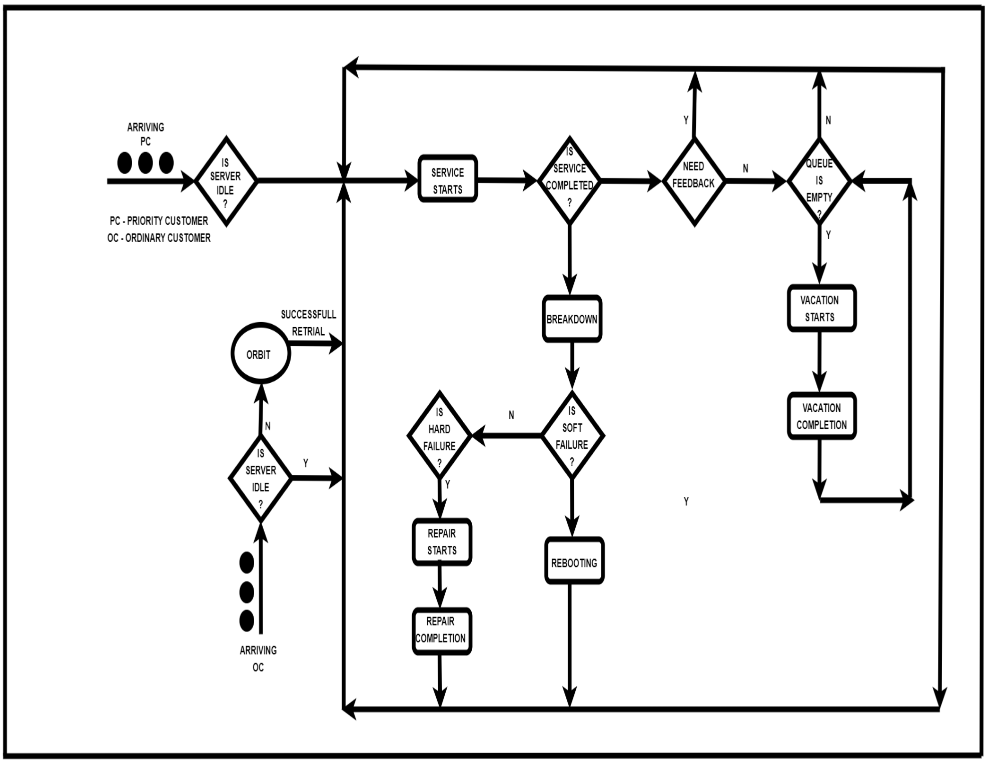

Figure 1: *Schematic representation*

a rate of $\beta(u)$, which is defined by the probability density function $m(u)$ and the probability distribution function $M(u)$. This rate follows a general distribution.

- **Regular Service Process :**
Customers with different queues are served in batches, including priority and ordinary customers. The server offers a single service at rates of $\mu_i(u)$, $i = 1, 2$ and service rates are distributed generally, characterized by probability distribution function $B_i(s)$ and probability density function $b_i(s)$, where $i = 1, 2$. If the priority queue is empty, ordinary customers can begin receiving service.
- **Immediate Feedback:**
Only customers with a priority are offered the feedback service. Once every customer in the priority queue have received their services, if customers are dissatisfied with the service they received. Customers could either abandon the system with probability $(1 - r)$ or they could get a re-service with probability 'r' without joining in queue.
- **Push Out:**
When the server is attending to an ordinary customers, a newly arriving ordinary customers has the potential to disrupt the ongoing service. It can either immediately take over the service area with probability 'q' or joins the orbit with probability '$\overline{q}$' $(= 1 - q)$.
- **Differentiate Breakdown:**
The service channel is susceptible to failure at any moment, even when the server is operating at its usual engaged pace during any phase of service. Consequently, the server will be inaccessible for a brief duration. Both hard and soft failure rates follow exponential distributions, with rates denoted by α_1 & α_2 respectively.
- **Delayed repair and Repair :**
The server is subject to hard failure. The breakdown server does not send for repair instantly. There exists a delay period before the repair process initiates. Following this delay, the repair process commences to restore functionality. The probability distribution function $D(s)$ and $R^{(2)}(s)$, along with the probability density function $d(s)$ and $r^{(2)}(s)$ are employed

to characterize the delay time and the duration of hard failure repairs, respectively. Let $\xi(u)$ and $\eta_2(u)$ be the completion rate for delay repair time and hard failure repair time. Soft failure repair time distributed exponentially with rate η_1.

- **Randomized Vacation:**
  When the server determines that the system is empty upon completing a service, it opts for a vacation. Upon concluding the vacation, if the server observes an empty system again it decides whether to embark on another vacation with prob. 'p' or to remain idle with a probability of '(1-p)'. The duration of the vacation adheres to a general distribution characterized by a rate of $\gamma(u)$. The time spent on vacation follows a probability distribution function $V(s)$, accompanied by its corresponding probability density function $v(s)$.

3. Analysis of queue size distribution

The formation of governing equations is the main focus of this section. This model has been solved using the probability generating function and supplementary variable technique with respect to the non-Markovian queueing system.

Assuming that $M(0) = 0$, $M(\infty) = 1$, $B_i(0) = 0$, $B_i(\infty) = 1$, $V(0) = 0$, $V(\infty) = 1$, $D(0) = 0$, $D(\infty) = 1$, and $R^{(2)}(0) = 0$, $R^{(2)}(\infty) = 1$ are continuous at $u = 0$ for $i = 1, 2$. So that the function $\beta(u)$, $\mu_1(u)$, $\mu_2(u)$, $\gamma(u)$, $\xi(u)$ and $\eta_2(u)$ are the conditional completion rates for retrial, priority and ordinary customers service rate, vacation and delay time to hard failure repair, hard failure repair respectively.

Also, $\beta(u) = \frac{dM(u)}{1-M(u)}$, $\mu_i(u) = \frac{dB_i(u)}{1-B_i(u)}$, $\gamma(u) = \frac{dV(u)}{1-V(u)}$ $\xi(u) = \frac{dD(u)}{1-D(u)}$ and $\eta_2(u) = \frac{dR^{(2)}(u)}{1-R^{(2)}(u)}$; $i = 1, 2$. are the hazard rate functions of $M(.)$, $Bi(.)$ for $i = 1, 2$, $V(.)$, $D(.)$ and $R^{(2)}(.)$ respectively.

Markov process for the given model is $\{N_p(t),\ N_o(t),\ Y(t),\ M^0(t),\ B_1^0(t),\ B_2^0(t),\ (V)^0(t),\ (D)^0(t),\ (R^{(2)})^0(t)\}$, where $N_p(t)$ and $N_o(t)$ denote the number of customers in the priority queue and ordinary queue respectively. $M^0(t)$, $B_1^0(t)$, $B_2^0(t)$, $(V)^0(t)$, $(D)^0(t)$, $(R^{(2)})^0(t)\}$ are the elapsed retrial, service, vacation, delay time to hard failure repair and hard failure repair time of the server at time 't'.

$Y(t)$ represents the server state. Here $Y(t) = (0, 1, 2, 3, 4, 5, 6)$, denotes: 0, the server is idle; 1, retrial state; 2, engaged with PC; 3, engaged with OC; 4, on vacation ; 5, delayed to hard failure repair, and 6, hard failure repair.

Let's represent the probability as $I_{0,n_2}(t)$, indicating the probability that at time t, $I_{0,n_2}(t)$ equals the event that $N_p(t) = 0$, $N_o(t) = 0$, and $Y(t) = 0$, where $t > 0$. We consider probability densities for this scenario.

$$
\begin{aligned}
I_{0,n_2}(u,t)du &= \Pr\{N_p(t) = 0, N_o(t) = n_2, Y(t) = 1; u \leq I^0(t) \leq u + du\}, n_2 \geq 1 \\
P^{(1)}_{n_1,n_2}(u,t)du &= \Pr\{N_p(t) = n_1, N_o(t) = n_2, Y(t) = 2; u \leq B_1^0(t) \leq u + du\}, \\
P^{(2)}_{n_1,n_2}(u,t)du &= \Pr\{N_p(t) = n_1, N_o(t) = n_2, Y(t) = 3; u \leq B_2^0(t) \leq u + du\}, \\
V_{n_1,n_2}(u,t)du &= \Pr\{N_p(t) = n_1, N_o(t) = n_2, Y(t) = 4; u \leq V^0(t) \leq u + du\}, \\
D_{n_1,n_2}(u,t)du &= \Pr\{N_p(t) = n_1, N_o(t) = n_2, Y(t) = 5; u \leq D^0(t) \leq u + du\}, \\
R^{(2)}_{n_1,n_2}(u,t)du &= \Pr\{N_p(t) = n_1, N_o(t) = n_2, Y(t) = 6; u \leq R^0(t) \leq u + du\}, \\
&\qquad \text{for } \ u \geq 0, t \geq 0, n_1 \geq 0 \ \text{ and} n_2 \geq 0.
\end{aligned}
$$

4. Equation Governing the System

$$\frac{d}{dt}I_{0,0}(t) = -(\lambda_p + \lambda_o)I_{0,0}(t) + (1-p)\int_0^\infty V_{0,0}(u,t)\gamma(u)du, \tag{1}$$

$$\frac{\partial}{\partial t}I_{0,n}(u,t) + \frac{\partial}{\partial u}I_{0,n}(u,t) = -(\lambda_p + \lambda_o + \beta(u))I_{0,n}(u,t), \tag{2}$$

$$\begin{aligned}\frac{\partial}{\partial t}P^{(1)}_{n_1,n_2}(u,t) + \frac{\partial}{\partial u}P^{(1)}_{n_1,n_2}(u,t) &= -(\lambda_p + \lambda_o + \alpha_1 + \alpha_2 + \mu_1(u))P^{(1)}_{n_1,n_2}(u,t)\\ &+ \lambda_p(1-\delta_{0n_1})\sum_{i=1}^{n_1} c_i P^{(1)}_{n_1-i,n_2}(u,t) + \lambda_o(1-\delta_{0n_2})\sum_{j=1}^{n_2} c_j P^{(1)}_{n_1,n_2-j}(u,t),\end{aligned} \tag{3}$$

$$\begin{aligned}\frac{\partial}{\partial t}P^{(2)}_{n_1,n_2}(u,t) + \frac{\partial}{\partial u}P^{(2)}_{n_1,n_2}(u,t) &= -(\lambda_p + b\lambda_o + \alpha_1 + \alpha_2 + \mu_1(u))P^{(2)}_{n_1,n_2}(u,t)\\ &+ \lambda_p(1-\delta_{0n_1})\sum_{i=1}^{n_1} c_i P^{(2)}_{n_1-i,n_2}(u,t) + \lambda_o(1-\delta_{0n_2})\sum_{j=1}^{n_2} c_j P^{(2)}_{n_1,n_2-j}(u,t),\end{aligned} \tag{4}$$

$$\begin{aligned}\frac{d}{dt}R^{(1)}_{n_1,n_2}(u,t) + \frac{d}{du}R^{(1)}_{n_1,n_2}(u,t) &= -(\lambda_p + b\lambda_o + \eta_1)R^{(1)}_{n_1,n_2}(t) + \lambda_p(1-\delta_{0n_1})\sum_{i=1}^{n_1} c_i R^{(1)}_{n_1-i,n_2}(t)\\ &+ \alpha_1\int_0^\infty (P^{(1)}_{n_1,n_2}(u,t) + P^{(2)}_{n_1,n_2}(u,t))du + \lambda_p(1-\delta_{0n_2})\sum_{j=1}^{n_2} c_j R^{(1)}_{n_1,n_2-i}(t),\end{aligned} \tag{5}$$

$$\begin{aligned}\frac{\partial}{\partial t}D_{n_1,n_2}(u,t) + \frac{\partial}{\partial u}D_{n_1,n_2}(u,t) &= -(\lambda_p + b\lambda_o + \xi(u))D_{n_1,n_2}(u,t)\\ &+ \lambda_p(1-\delta_{0n_1})\sum_{i=1}^{n_1} c_i D_{n_1-i,n_2}(u,t) + \lambda_o(1-\delta_{0n_2})\sum_{j=1}^{n_2} c_j D_{n_1,n_2-i}(u,t),\end{aligned} \tag{6}$$

$$\begin{aligned}\frac{\partial}{\partial t}R^{(2)}_{n_1,n_2}(u,t) + \frac{\partial}{\partial u}R^{(2)}_{n_1,n_2}(u,t) &= -(\lambda_p + b\lambda_o + \eta_2(u))R^{(2)}_{n_1,n_2}(u,t)\\ &+ \lambda_p(1-\delta_{0n_1})\sum_{i=1}^{n_1} c_i R^{(2)}_{n_1-i,n_2}(u,t) + \lambda_o(1-\delta_{0n_2})\sum_{j=1}^{n_2} c_j R^{(2)}_{n_1,n_2-i}(u,t),\end{aligned} \tag{7}$$

$$\begin{aligned}\frac{\partial}{\partial t}V_{n_1,n_2}(u,t) + \frac{\partial}{\partial u}V_{n_1,n_2}(u,t) &= -(\lambda_p + b\lambda_o + \gamma(u))V_{n_1,n_2}(u,t)\\ &+ \lambda_p(1-\delta_{0n_1})\sum_{i=1}^{n_1} c_i V_{n_1-i,n_2}(u,t) + \lambda_o(1-\delta_{0n_2})\sum_{j=1}^{n_2} c_j V_{n_1,n_2-j}(u,t).\end{aligned} \tag{8}$$

The preceding set of equations must be solved under the following boundary conditions at $u = 0$,

$$\begin{aligned}P^{(1)}_{n_1,n_2}(0,t) &= (1-r)\int_0^\infty P^{(1)}_{n_1+1,n_2}(u,t)\mu_1(u)du + r\int_0^\infty P^{(1)}_{n_1,n_2}(u,t)\mu_1(u)du\\ &+ \int_0^\infty P^{(2)}_{n_1+1,n_2}(u,t)\mu_2(u)du + R^{(1)}_{n_1+1,n_2}(t)\eta_1 + \lambda_p c_{n_1+1} I_{0,n_2}(t)\\ &+ \int_0^\infty V_{n_1+1,n_2}(u,t)\gamma(u)du + \int_0^\infty R^{(2)}_{n_1+1,n_2}(u,t)\eta(u)du,\end{aligned} \tag{9}$$

$$\begin{aligned}P^{(2)}_{0,n_2}(0,t) &= \lambda_o c_{n_2+1} I_{0,0}(t) + \int_0^\infty I_{0,n_2+1}(u,t)\beta(u)du + \sum_{i=1}^{n_2}\lambda_o C_i(u,t)\\ &+ \int_0^\infty I_{0,n_2+1-i}(u,t)du + \lambda_o q\int_0^\infty P^{(2)}_{0,n_2}(u,t)\mu_2(u)du,\end{aligned} \tag{10}$$

$$R^{(2)}_{n_1,n_2}(0,t) = \int_0^\infty D_{n_1,n_2}(u,t)du, \tag{11}$$

$$D_{n_1,n_2}(0,t) = \alpha_2 \int_0^\infty P^{(1)}_{n_1-1,n_2}(u,t)du + \alpha_2 \int_0^\infty P^{(2)}_{n_1,n_2}(u,t)du, \tag{12}$$

$$V_{0,0}(0,t) = (1-r)\int_0^\infty P^{(1)}_{0,0}(u,t)\mu_1(u)du + \int_0^\infty P^{(2)}_{0,0}\mu_2(u)(u,t)du + \int_0^\infty R^{(2)}_{0,0}(u,t)\eta_2(u)du + R^{(1)}_{0,n_2}(t)\eta_1 + p\int_0^\infty V_{0,0}(u,t)\gamma(u)du, \tag{13}$$

$$V_{n_1,n_2}(0,t) = 0, \tag{14}$$

$$I_{0,n_2}(0,t) = (1-r)\int_0^\infty P^{(1)}_{0,n_2}(u,t)\mu_1(u)du + \int_0^\infty P^{(2)}_{0,n_2}\mu_2(u)(u,t)du + \int_0^\infty R^{(2)}_{0,n_2}(u,t)\eta_2(u)du + R^{(1)}_{0,n_2}(t)\eta_1 \int_0^\infty V_{0,n_2}(u,t)\gamma(u)du. \tag{15}$$

$$P^{(1)}_{n_1,n_2}(0) = P^{(2)}_{n_1,n_2}(0) = D_{n_1,n_2}(0) = V_{n_1,n_2}(0) = R^{(1)}_{n_1,n_2}(0) = R^{(2)}_{n_1,n_2}(0) = 0,$$
$$I_{0,0} = 1, I_{0,n_2}(0) = 0, \quad \text{for } \ n_2 \geq 1 \ \text{ are the initial conditions.} \tag{16}$$

The PGF defined as,

$$I(u,t,z_o) = \sum_{n_2=1}^{\infty} z_p^{n_2} I_{0,n_2}(u,t); \qquad A(u,t,z_p,z_o) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} z_o^{n_1} z_p^{n} A_{n_1,n_2}(u,t);$$
$$A(u,t,z_p) = \sum_{n_1=0}^{\infty} z_p^{n_1} A_{n_1}(u,t); \qquad A(u,t,z_p) = \sum_{n_2=0}^{\infty} z_o^{n_2} A_{n_2}(u,t); \tag{17}$$

here $A = P^{(1)},\ P^{(2)},\ D,\ V,\ R^{(1)},\ R^{(2)}$.
We derive the following equations by applying Laplace transforms to equations (1) to (15) along with (16) and (17).

$$\overline{I}_0(u,s,z_o) = \overline{I}_0(0,s,z_o)e^{-(s+\lambda_p+\lambda_o)u - \int_0^u \beta(t)dt}, \tag{18}$$

$$\overline{P}^{(1)}(u,s,z_p,z_o) = \overline{P}^{(1)}(0,s,z_p,z_o)e^{-\phi_1(s,z)u - \int_0^u \mu_1(t)dt}, \tag{19}$$

$$\overline{P}^{(2)}(u,s,z_p,z_o) = \overline{P}^{(2)}(0,s,z_p,z_o)e^{-\phi_2(s,z)u - \int_0^u \mu_2(t)dt}, \tag{20}$$

$$\overline{V}(u,s,z_p,z_o) = \overline{V}(0,s,z_p,z_o)e^{-\phi_2(s,z)u - \int_0^u \gamma(t)dt}, \tag{21}$$

$$\overline{D}(u,s,z_p,z_o) = \overline{D}(0,s,z_p,z_o)e^{-\phi_2(s,z)u - \int_0^u \xi(t)dt}, \tag{22}$$

$$\overline{R}^{(2)}(u,s,z_p,z_o) = \overline{R}^{(2)}(0,s,z_p,z_o)e^{-\phi_2(s,z)u - \int_0^u \eta_2(t)dt}. \tag{23}$$

where,

$$\psi_1(s,z) = s + \lambda_p + \lambda_o(1 - C(z_o)) + \alpha_1 + \alpha_2,$$
$$\psi_2(s,z) = s + \lambda_p + \lambda_o(1 - C(z_o)),$$
$$\psi_3(s,z) = s + \lambda_p + \lambda_o(1 - C(z_o)) + \eta_1,$$
$$\phi_1(s,z) = s + \lambda_p(1 - C(z_p)) + \lambda_o(1 - C(z_o)) + \alpha_1 + \alpha_2,$$
$$\phi_2(s,z) = s + \lambda_p(1 - C(z_p)''0 + \lambda_o(1 - C(z_o)),$$
$$\phi_3(s,z) = s + \lambda_p(1 - C(z_p)) + \lambda_o(1 - C(z_o)) + \eta_1,$$
$$\sigma_1(s,z) = s + \lambda_p(1 - C(g(z_o))) + \lambda_o(1 - C(z_o)) + \alpha_1 + \alpha_2,$$
$$\sigma_2(s,z) = s + \lambda_p(1 - C(g(z_o))) + \lambda_o(1 - C(z_o)),$$
$$\sigma_3(s,z) = s + \lambda_p(1 - C(g(z_o))) + \lambda_o(1 - C(z_o)) + \eta_1.$$

$$\overline{I}_0(0,s,z_o) = \frac{\left\{\begin{array}{l}[(1-\overline{V}(\sigma_2(s,z)))\overline{V}_{0,0} + (s+\lambda_p+\lambda_o)\overline{I}_{0,0} - 1] \\ \overline{P}^{(2)}(0,s,z_o)[\overline{B}_2(\sigma_1(s,z)) + \Big[\alpha_2\overline{R}^{(2)}(\sigma_2(s,z))\overline{D}(\sigma_2(s,z)) \\ + \dfrac{\alpha_1\eta_1}{\sigma_3(s,z)}\Big]\Big[\dfrac{1-\overline{B}_2(\sigma_1(s,z))}{\sigma_1(s,z)}\Big]]\end{array}\right\}}{\left\{[(1-C(g(z_o)))\lambda_p\Big[\dfrac{1-\overline{M}(s+\lambda_p+\lambda_o)}{s+\lambda_p+\lambda_o}\Big]]\right\}},$$

$$\overline{P}^{(2)}(0,s,z_o) = \frac{\left\{\begin{array}{l}\lambda_o C(z_o)\overline{I}_{0,0}(s)1 - \lambda_p C(g(z_o))\Big[\dfrac{1-\overline{M}(s+\lambda_p+\lambda_o)}{s+\lambda_p+\lambda_o}\Big]] \\ -[\overline{I}_{0,0}(s)(s+\lambda_p+\lambda_o) - 1 + [(1-\overline{V}(\sigma_2(s,z)))\overline{V}_{0,0}]] \\ [\overline{M}(s+\lambda_p+\lambda_o) + C(z_o)\lambda_o\Big[\dfrac{1-\overline{M}(s+\lambda_p+\lambda_o)}{s+\lambda_p+\lambda_o}\Big]]\end{array}\right\}}{\left\{\begin{array}{l}(z_o-\lambda_p q z_o)\Big[\dfrac{1-\overline{B}_2(\psi_1(s,z))}{\psi_1(s,z)}\Big]][(1-C(g(z_o)))\lambda_p \\ \Big[\dfrac{1-\overline{M}(s+\lambda_p+\lambda_o)}{s+\lambda_p+\lambda_o}\Big]] - [\overline{M}(s+\lambda_p+\lambda_o) + C(z_o)\lambda_o \\ \Big[\dfrac{1-\overline{M}(s+\lambda_p+\lambda_o)}{s+\lambda_p+\lambda_o}\Big]][\overline{B}_2(\sigma_1(s,z))\Big[\alpha_2\overline{R}^{(2)}(\sigma_2(s,z))\overline{D}(\sigma_2(s,z)) \\ + \dfrac{\alpha_1\eta_1}{\sigma_3(s,z)}\Big]\Big[\dfrac{1-\overline{B}_2(\sigma_1(s,z))}{\sigma_1(s,z)}\Big]]\end{array}\right\}}, \tag{24}$$

$$\overline{P}^{(1)}(0,s,z_p,z_o) = \frac{\left\{\begin{array}{l}\lambda_p[C(z_p) - C(g(z_o))]\Big[\dfrac{1-\overline{M}(s+\lambda_p+\lambda_o)}{s+\lambda_p+\lambda_o}\Big]\overline{I}_0(0,s,z_o)(\overline{V}(\phi_2(s,z)) \\ -\overline{V}(\sigma_2(s,z)))\overline{V}_{0,0} + \overline{P}^{(2)}(0,s,z_o) + [\overline{B_2}(\phi_1(s,z)) - \overline{B_2}(\sigma_1(s,z)) \\ + \Big[\alpha_2\overline{R}^{(2)}(\phi_2(s,z))\overline{D}(\phi_2(s,z)) + \dfrac{\alpha_1\eta_1}{\phi_3(s,z)}\Big]\Big[\dfrac{1-\overline{B_2}(\sigma_1(s,z))}{\sigma_1(s,z)}\Big] \\ - \Big[\alpha_2\overline{R}^{(2)}(\sigma_2(s,z))\overline{D}(\sigma_2(s,z)) + \dfrac{\alpha_1\eta_1}{\sigma_3(s,z)}\Big]\Big[\dfrac{1-\overline{B_2}(\sigma_1(s,z))}{\sigma_1(s,z)}\Big]]\end{array}\right\}}{\left\{\begin{array}{l}[z_p - ((rz_p + (1-r))\overline{B_1}(\phi_1(s,z)) + (z_p\alpha_2\overline{R}^{(2)}(\phi_2(s,z)) \\ \overline{D}(\phi_2(s,z)) + \dfrac{\alpha_1\eta_1}{\phi_3(s,z)}\Big[\dfrac{1-\overline{B_1}(\phi_1(s,z))}{\phi_1(s,z)}\Big])]\end{array}\right\}}. \tag{25}$$

Theorem.1 When the system is operating normally, experiencing a breakdown, going on a randomized vacation, delay time to repair, or being repair, the probability generating function of the number of customers in the relevant queue will be provided using Laplace transforms.

$$\overline{I}_0(s,z_o) = \overline{I}_0(0,s,z_o)\Big[\frac{1-\overline{M}(s+\lambda_p+\lambda_o)}{s+\lambda_p+\lambda_o}\Big], \tag{26}$$

$$\overline{P}^{(1)}(s,z_p,z_o) = \overline{P}^{(1)}(0,s,z_p,z_o)\Big[\frac{1-\overline{B}_1(\phi_1(s,z))}{\phi_1(s,z)}\Big], \tag{27}$$

$$\overline{P}^{(2)}(s,z_p,z_o) = \overline{P}^{(2)}(0,s,z_o)\Big[\frac{1-\overline{B}_2(\phi_1(s,z))}{\phi_1(s,z)}\Big], \tag{28}$$

$$\overline{V}(s,z_p,z_o) = \overline{V}(0,s,z_o)\Big[\frac{1-\overline{V}(\phi_2(s,z))}{\phi_2(s,z)}\Big], \tag{29}$$

$$\overline{D}(s,z_p,z_o) = \overline{D}(0,s,z_o)\Big[\frac{1-\overline{D}(\phi_2(s,z))}{\phi_2(s,z)}\Big], \tag{30}$$

$$\overline{R}^{(2)}(s,z_p,z_o) = \overline{R}^{(2)}(0,s,z_p,z_o)\Big[\frac{1-\overline{R}^{(2)}(\phi_2(s,z))}{\phi_2(s,z)}\Big]. \tag{31}$$

Proof: The following result is reached by applying the renewal theory's solution and solving the previous equations (26) to (31) with respect to u through integration.

$$\int_0^\infty \big[1-H(u)\big]e^{-su}du = \frac{1-\overline{h(s)}}{s}. \tag{32}$$

The LST of the H(u) distribution function of a random variable is expressed by $\overline{h(s)}$. The following states' precise probability generating function results are as follows: $\overline{I}_0(s,z_o)$, $\overline{P}^{(1)}(s,z_p,z_o)$, $\overline{P}^{(2)}(s,z_p,z_o)$, $\overline{D}(s,z_p,z_o)$, $\overline{V}(s,z_p,z_o)$ and $\overline{R}^{(2)}(s,z_p,z_o)$ are obtained by using equation (26) to (31).

5. Steady State Analysis

Steady state analysis refers to the examination of a system's behavior once it has reached a stable condition where its key parameters remain relatively constant over time. In this state, the system's inputs and outputs balance out, resulting in a consistent and unchanging pattern of behavior. Steady state analysis is often used in various fields such as engineering, economics and physics to understand long-term behavior and performance characteristics of systems.

According to Tauberian property,

$$\lim_{s\to 0} s\overline{f}(s) = \lim_{t\to\infty} f(t).$$

The queue size's PGF is as follows, in spite of the system's current state:

$$W_q(z_p,z_o) = \frac{Nr(z_p,z_o)}{Dr(z_p,z_o)}, \tag{33}$$

$$\begin{aligned} Nr(z_p,z_o) &= N_1(z)D_2(z)D_3(z)\phi_1(z)\phi_2(z)\phi_3(z)\Big[\frac{1-\overline{M}(\lambda_1+\lambda_2)}{\lambda_1+\lambda_2}\Big] + N_2(z)D_1(z)D_3(z) \\ &\quad f_1(z)(1-\overline{B_2}\phi_1(z)) + \Big[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}\Big]N_3(z)D_1D_2(1-\overline{B_2}\phi_1(z))f_1(z), \\ Dr(z_p,z_o) &= D_1(z)D_2(z)D_3(z)D_4(z)\phi_1(z)\phi_2(z)\phi_3(z), \end{aligned}$$

where,

$$N_1(z) = \overline{V}_{00}[\overline{V}\sigma_2(z) - 1] - (\lambda_p + \lambda_o)\overline{I}_{00} + [\overline{B}_2\sigma_1(z) + (\alpha_2\overline{R}^{(2)}\sigma_2(z)\overline{D}\sigma_2(z) + \frac{\alpha_1}{\sigma_3(z)})\Big[\frac{1-\overline{B}_2\sigma_1(z)}{\sigma_1(z)}\Big]],$$

$$D_1(z) = 1 - C(g(z_o))\lambda_p\Big[\frac{1\overline{M}(\lambda_p+\lambda_o)}{\lambda_p+\lambda_o}\Big],$$

$$N_2(z) = \lambda_o C(z_o))\overline{I}_{00}[1 - C(g(z_o))\lambda_p\Big[\frac{1-\overline{M}(\lambda_p+\lambda_o)}{\lambda_p+\lambda_o}\Big]] + [1 - (\lambda_p+\lambda_o)\overline{I}_{00}$$
$$\overline{V}_{00}[\overline{V}\sigma_2(z) - 1]][\overline{M}(\lambda_p+\lambda_o) - \lambda_o C(z_o)\Big[\frac{1\overline{M}(\lambda_p+\lambda_o)}{\lambda_p+\lambda_o}\Big]],$$

$$D_2(z) = [z_o - \lambda_o q z_o\Big[\frac{1-\overline{B}_2\psi_1(z)}{\psi_1(z)}\Big]][1 - C(g(z_o))\lambda_p\Big[\frac{1\overline{M}(\lambda_p+\lambda_o)}{\lambda_p+\lambda_o}\Big]][[\overline{B}_2\sigma_1(z)$$
$$+ (\alpha_2\overline{R}^{(2)}\sigma_2(z)\overline{D}\sigma_2(z) + \frac{\alpha_1}{\sigma_3(z)})\Big[\frac{1-\overline{B}_2\sigma_1(z)}{\sigma_1(z)}\Big]]]$$
$$[\overline{M}(\lambda_p+\lambda_o) - \lambda_o C(z_o)\Big[\frac{1\overline{M}(\lambda_p+\lambda_o)}{\lambda_p+\lambda_o}\Big]],$$

$$N_3(z) = \lambda_p[C(z_p) - C(g(z_o))]\Big[\frac{1-\overline{M}(\lambda_p+\lambda_o)}{\lambda_p+\lambda_o}\Big]\overline{I}_0(0,z_o) + \overline{V}_{00}[\overline{V}\phi_2(z) - \overline{V}\sigma_2(z)]$$
$$[\overline{B}_2\phi_1(z) - \overline{B}_2\sigma_1(z)(\alpha_2\overline{R}^{(2)}\phi_2(z)\overline{D}\phi_2(z) + \frac{\alpha_1}{\phi_3(z)})\Big[\frac{1-\overline{B}_2\phi_1(z)}{\phi_1(z)}\Big]$$
$$- (\alpha_2\overline{R}^{(2)}\sigma_2(z)\overline{D}\sigma_2(z) + \frac{\alpha_1}{\sigma_3(z)})\Big[\frac{1-\overline{B}_2\sigma_1(z)}{\sigma_1(z)}\Big]]\overline{P}^{(2)}(0,z_0),$$

$$D_3(z) = z_p - [((1-r) + rz_p)\overline{B}_1\phi_1(z) + (\alpha_2\overline{R}^{(2)}\phi_2(z)\overline{D}\phi_2(z) + \frac{\alpha_1}{\phi_3(z)})\Big[\frac{1-\overline{B}_1\phi_1(z)}{\phi_1(z)}\Big]]$$

$$\psi_1(z) = \lambda_p + \lambda_o(1 - C(z_o)) + \alpha_1 + \alpha_2,$$
$$\psi_2(z) = \lambda_p + \lambda_o(1 - C(z_o)),$$
$$\psi_3(z) = \lambda_p + \lambda_o(1 - C(z_o)) + \eta_1,$$
$$\phi_1(z) = \lambda_p(1 - C(z_p)) + \lambda_o(1 - C(z_o)) + \alpha_1 + \alpha_2,$$
$$\phi_2(z) = \lambda_p(1 - C(z_p)) + \lambda_o(1 - C(z_o)),$$
$$\phi_3(z) = \lambda_p(1 - C(z_p)) + \lambda_o(1 - C(z_o)) + \eta_1,$$
$$\sigma_1(z) = \lambda_p(1 - C(g(z_o))) + \lambda_o(1 - C(z_o)) + \alpha_1 + \alpha_2,$$
$$\sigma_2(z) = \lambda_p(1 - C(g(z_o))) + \lambda_o(1 - C(z_o)),$$
$$\sigma_3(z) = \lambda_p(1 - C(g(z_o))) + \lambda_o(1 - C(z_o)) + \eta_1.$$

6. Stability condition

The stability requirement is a criterion that establishes whether a QS can manage incoming traffic without increasing indefinitely over time. A stable QS maintains consistent queue length and performance measurements over time, even with changing arrival rates.

We apply the normalising condition to determine $I_{0,0}$.

$$I_{0,0} + I_0 I + P^{(1)}(1,1) + P_{II}(1,1) + V(1,1) + R_I(1,1) + D(1,1) + R^{(2)}(1,1) = 1. \tag{34}$$

$$I_{0,0} = \frac{\left\{\begin{array}{l}[D_1(1,1)D_2(1,1)D_3^{'}(1,1)(\alpha_1+\alpha_2)\eta_1(\lambda_p+\lambda_o)(-E(X))] \\ \quad -[\Big[\dfrac{1-\overline{M}(\lambda_p+\lambda_o)}{\lambda_p+\lambda_o}\Big]N_1(1,1)D_3^{'}(1,1)D_2(1,1) \\ \quad (\alpha_1+\alpha_2)(\lambda_p+\lambda_o)(-E(X))\eta_1 \\ \quad (\lambda_p+\lambda_o)(-E(X))+N_2(1,1)D_3^{'}(1,1)D_1(1,1)f'_1(1,1) \\ \quad (1-B_2(\alpha_1+\alpha_2))+N_3^{'}(1,1)D_1(1,1)D_2(1,1) \\ \quad f_1^{'}(1,1)D_3(1,1)(1-B_1(\alpha_1+\alpha_2))]\end{array}\right\}}{\left\{D_1(1,1)D_2(1,1)D_3^{'}(1,1)(\alpha_1+\alpha_2)\eta_1(\lambda_p+\lambda_o)(-E(X))\right\}}, \tag{35}$$

and the utilization factor is given by

$$\rho = \frac{\left\{\begin{array}{l}[\Big[\dfrac{1-\overline{M}(\lambda_p+\lambda_o)}{\lambda_p+\lambda_o}\Big]N_1(1,1)D_3^{'}(1,1)D_2(1,1) \\ \quad (\alpha_1+\alpha_2)(\lambda_p+\lambda_o)(-E(X))\eta_1 \\ \quad (\lambda_p+\lambda_o)(-E(X))+N_2(1,1)D_3^{'}(1,1)D_1(1,1)f'_1(1,1) \\ \quad (1-B_2(\alpha_1+\alpha_2))+N_3^{'}(1,1)D_1(1,1)D_2(1,1) \\ \quad f_1^{'}(1,1)D_3(1,1)(1-B_1(\alpha_1+\alpha_2))]\end{array}\right\}}{\left\{D_1(1,1)D_2(1,1)D_3^{'}(1,1)(\alpha_1+\alpha_2)\eta_1(\lambda_p+\lambda_o)(-E(X))\right\}}. \tag{36}$$

The steady state stability requirement for the model is $\rho < 1$.

where,

$$N_3^{'}(1) = \lambda_p\Big[\frac{1\overline{M}(\lambda_p+\lambda_o)}{\lambda_p+\lambda_o}\Big]\bar{I}_0(0,1)[1-E(X_1)]E(X),$$

$$D_3^{'}(1) = 1-[r\overline{B}_2(\lambda_1+\lambda_2)+(\alpha_2(E(R^{(2)})+E(D))-\frac{\alpha_1}{\eta_1}-1)\Big[\frac{1-\overline{B}_1(\alpha_1+\alpha_2)}{(\alpha_1+\alpha_2)}\Big]],$$

$$f_1^{'}(1) = [\eta_1+\alpha_1-\alpha_2\eta_1(E(R^{(2)})+E(D))][-(\lambda_p+\lambda_o)E(X)],$$

7. Performance Assessments

Performance measures in QS are metrics used to evaluate and quantify various aspects of system behavior, efficiency and effectiveness. These measures help assess how well a QS is performing and provide insights into its operational characteristics.

The following is the expected queue size for PC and orbit size for OC

$$L_{q_1} = \frac{d}{dz_p}W_q(z_p,1)|_{z_p=1}, \tag{37}$$

$$L_{q_2} = \frac{d}{dz_o}W_q(1,z_o)|_{z_o=1}. \tag{38}$$

Where,

$$L_{q_1} = \frac{Dr_1^{''}(1)Nr_1^{'''}(1)-Dr_1^{'''}(1)Nr_1^{''}(1)}{3(Dr_1^{''}(1))^2}, \tag{39}$$

$$L_{q_2} = \frac{Dr_2^{''}(1)Nr_2^{'''}(1)-Dr_2^{'''}(1)Nr_2^{''}(1)}{3(Dr_2^{''}(1))^2}, \tag{40}$$

The following is the expected waiting time for priority queue:

$$W_{q_1} = \frac{L_{q_1}}{\lambda_p} \tag{41}$$

The following is the expected waiting time for orbit:

$$W_{q_2} = \frac{L_{q_2}}{\lambda_o}. \tag{42}$$

8. Particular Cases

Case 1: In the absence of priority queue, ordinary customers arrive individually without retrials, breakdowns, or push-out mechanisms. In this scenario, the model can be simplified of queue type $M/G/1$ with a general randomized vacation policy.

$$P_{II}(z) = \frac{I_{0,0}[1-\overline{B}_2(\lambda_o(1-z_o))]\{\overline{V}(\lambda_o(1-z_o)) - (1-z_o)(1-p)\overline{V}(\lambda_o) - 1\}}{(1-z_o)(1-p)\overline{V}(\lambda_o)\{z-\overline{B}_2(\lambda_o(1-z_o))\}},$$

$$V(z_o) = \frac{I_{0,0}[1-\overline{V}(\lambda_o(1-z_o))]}{(1-p)(1-z_o)\overline{V}(\lambda_o)}.$$

The aforementioned outcome bears resemblance to the findings of Chen et al. [23] albeit without the inclusion of a second optional service.

Case 2: In the absence of priority queue, ordinary customers arrive individually without any breakdowns or vacations. In such a scenario, this model can be simplified a RQ of type $M/G/1$ with abandoned customers.

$$I_0(z) = \frac{I_{0,0}z(1-z_o)[1-\overline{B}_2(\lambda_p(1-z_o)+\lambda_o q z_o)][1-\overline{M}(\lambda_o)]}{D(z_o)},$$

$$P_{II}(z) = \frac{I_{0,0}\overline{M}(\lambda_o)(1-z_o)[1-\overline{B}_2(\lambda_o(1-z_o)+\lambda_o q z_o)]}{D(z_o)},$$

where,

$$D(z) = \overline{B}_2(\lambda_o(1-z_o)+\lambda_o q z_o)\{(1-z_o+qz_o)(z_o+(1-z_o)\overline{M}(\lambda_o)) - z_o^2 q\} - z_o(1-z_o).$$

These findings align with the results reported by Krishna Kumar et al. [24].

9. Numerical Results

The numerical and graphical analyses of this model are covered in this section. We assumed that the distribution of service period, failure, repair and vacation period are all exponential.

Table 1: *The impact of the priority arrival rate (λ_p)*

λ_p	I_0	ρ	Lq_1	Wq_1	Lq_2	Wq_2
0.5	0.9096	0.0904	3.3151	4.2339	1.3073	0.6535
0.6	0.8464	0.1536	3.6657	4.4002	1.4388	0.7194
0.7	0.7722	0.2278	4.0216	4.5624	1.5698	0.7849
0.8	0.6847	0.3153	4.3764	4.7233	1.6996	0.8498
0.9	0.5809	0.4191	4.7243	4.8869	1.8275	0.9138
1.0	0.4573	0.5427	5.0592	5.0592	1.9526	0.9763
1.1	0.3090	0.6910	5.3756	5.2492	2.0737	1.0368
1.2	0.1299	0.8701	5.6679	5.4705	2.1890	1.0945

Table 1 exhibit that when an arrival rate (λ_p) for PQ escalates, then (L_{q_1}/L_{q_2}) and (W_{q_1}/W_{q_2}) also rises at $\lambda_o = 2$, $\alpha_1 = 2$, $\alpha_2 = 3$, $\mu = 4$, $\eta_1 = 5$, $\eta_2 = 8$, $\gamma = 20$, $p = 0.3$, $\xi = 15$, $\beta = 8$, $q = 0.4$, $r = 0.1$ and $\lambda_p = 0.5$ to 1.3.

Table 2: *Impact of hard failure repair rate (η_2)*

η_2	I_0	ρ	L_{q_1}	W_{q_1}	L_{q_2}	W_{q_2}
5.0	0.0435	0.9565	3.2910	3.2910	1.6708	0.8354
5.5	0.1807	0.8193	3.0328	3.0328	1.6612	0.8306
6.0	0.2742	0.7258	2.8290	2.8290	1.6507	0.8253
6.5	0.3421	0.6579	2.6642	2.6642	1.6392	0.8196
7.0	0.3937	0.6063	2.5283	2.5283	1.6267	0.8133
7.5	0.4344	0.5656	2.4145	2.4145	1.6128	0.8064
8.0	0.4673	0.5327	2.3177	2.3177	1.5975	0.7987
8.5	0.4944	0.5056	2.2346	2.2346	1.5804	0.7902
9.0	0.5173	0.4827	2.1623	2.1623	1.5612	0.7806

Table 2 exhibit that when an hard failure repair rate (η_2) escalates, then (L_{q_1}/L_{q_2}) and (W_{q_1}/W_{q_2}) also decreases at $\lambda_p = 1$, $\lambda_o = 2$, $\alpha_1 = 1$, $\alpha_2 = 5$, $\mu = 4$, $\eta_1 = 4$, $\gamma = 15$, $p = 0.4$, $\xi = 10$, $\beta = 12$, $q = 0.3$, $r = 0.1$ and $\eta_2 = 5.0$ to 10.0.

Table 3: *Impact of ordinary arrival rate (λ_o)*

λ_o	I_0	ρ	L_{q_1}	W_{q_1}	L_{q_2}	W_{q_2}
1.0	0.8423	0.1577	1.6669	3.3339	0.1541	0.1541
1.1	0.8402	0.1598	1.7448	3.4896	0.1954	0.1777
1.2	0.8371	0.1629	1.8228	3.6455	0.2440	0.2034
1.3	0.8329	0.1671	1.9008	3.8016	0.3005	0.2311
1.4	0.8275	0.1725	1.9790	3.9579	0.3655	0.2610
1.5	0.8204	0.1796	2.0572	4.1144	0.4396	0.2931
1.6	0.8116	0.1884	2.1355	4.2710	0.5237	0.3273
1.7	0.8004	0.1996	2.2139	4.4277	0.6182	0.3636

Table 3 exhibit that when an ordinary arrival rate (λ_o) for PQ escalates, then the (L_{q_1}/L_{q_2}) and the (W_{q_1}/W_{q_2}) also rises at $\lambda_p = 0.5$, $\alpha_1 = 0.8$, $\alpha_2 = 4$, $\mu = 3$, $\eta_1 = 3$, $\gamma = 17$, $p = 0.3$, $\xi = 10$, $\beta = 12$, $q = 0.2$, $r = 0.2$, $\eta_2 = 7$ and $\lambda_o = 1.0$ to 2.0.

We obviously follow the exponential distribution for service time, breakdown, repair and vacation time in graphical representations. Figures 2 - 4 illustrate the 2D graphs. (L_{q_1}, L_{q_2}) increases when the priority arrival rate (λ_p) increases, as demonstrates in Figure 2. Figure 3 demonstrates the behaviour of the queue sizes (L_{q_1}, L_{q_2}), which depends on the hard failure repair rate (η_2). The length of the queue grows as the soft failure rate improves. Figure 4 depicts the behaviour of the queue sizes (L_{q_1}, L_{q_2}), which is affected by the average customer arrival rate (λ_o).

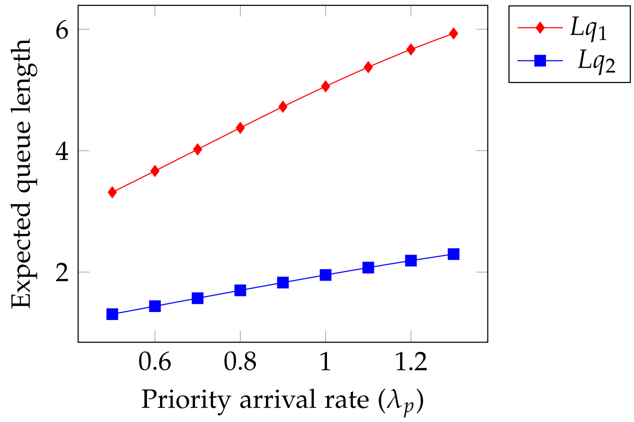

Figure 2: *Lq Vs λ_p*

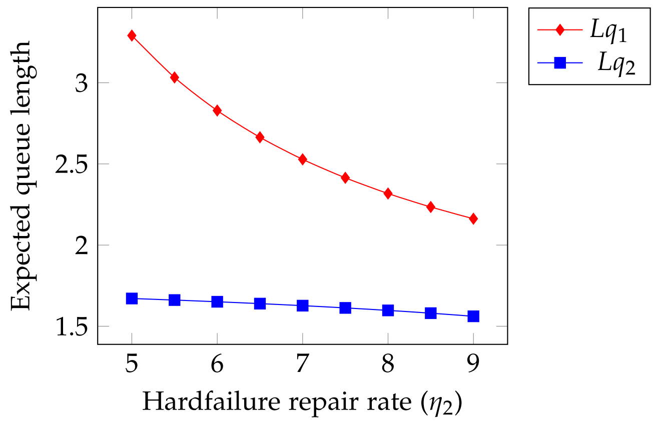

Figure 3: *Lq Vs η_2*

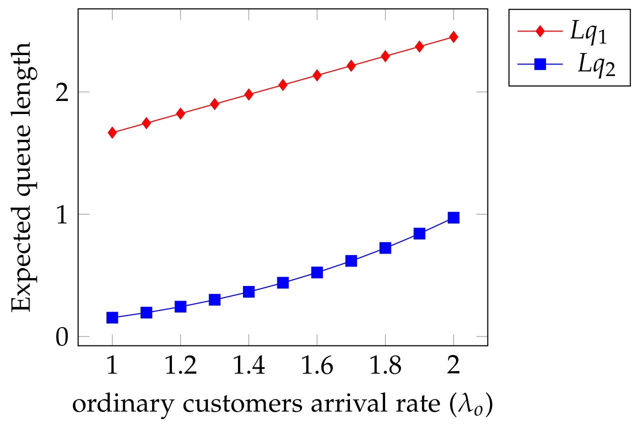

Figure 4: *Lq Vs λ_o*

10. Conclusion

In this research, we examined a single server retrial queueing system with non-preemptive priority service, immediate feedback, push out, differentiated breakdowns, delayed repair, randomized vacation. The analytical findings that are supported by numerical examples can be used to a wide range of real-world situations to produce results. The supplementary variable technique is used to determine the PGFs for the number of users in the system when it is free, busy, and under repair. The system's and orbit's average queue lengths contain many expressions. The mean busy period and other significant system performance measures are obtained. The conditional decomposition law is finally demonstrated to be effective for this retrial queueing system.In real-world queueing scenarios, our queueing system is more flexible when dealing with real-time systems used by many industries.

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