E-BAYESIAN ESTIMATION FOR BATHTUB-SHAPED LIFETIME DISTRIBUTION BASED ON UPPER RECORD VALUES

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Abstract

In this research paper, we presents the expected Bayesian (E-Bayesian) estimation of bathtub-shaped lifetime (BSL) distribution for scale parameter based on upper record values (URV) using a conjugate prior distribution. Also, we are considered different prior distributions for the E-Bayesian estimators. Some properties of the E-Bayesian estimators are discussed. A simulation study is given to compare the performance of the E-Bayesian estimators with Bayesian estimator. we notice that the E-Bayesian estimators are perform better than the Bayesian estimators. Moreover, the performance of the Bayesian estimators and E-Bayesian estimators for Prior II are better than Prior I. Also, we observe that if we increase the sample size n then the estimators are showing lesser mean square error (MSE).

Keywords: Bathtub-shaped lifetime distribution, Bayes estimation, E-Bayes estimation, Upper Record values.

1. Introduction

Suppose that $X_1, X_2, ...$ is a sequence of independent and identically distributed (*iid*) random variables with a cumulative distribution function (*cdf*) $F(x)$ and a probability density function (*pdf*) $f(x)$. Let $Y_m = max(min)\{X_1, X_2, ...X_m\}$ for $m \geq 1$. We say X_j is an URV or LRV (lower) of this sequence $\{X_m, m \geq 1\}$ if $Y_j > (<) Y_{j-1}, j \geq 2$. By definition, X_1 is URV as well as a LRV. One can transform the upper records to lower records by replacing the original sequence $\{X_j; j \geq 1\}$ by $\{-X_j; j \geq 1\}$ or $Pr(X_i > 0) = 1$ if for all i by $\left\{\frac{1}{X_i}, i \geq 1\right\}$, the LRV of this sequence will correspond to the URV of the original sequence, (Ahsanullah, 2004) [1].
The BSL distribution was introduced by Chen (2000) [2]. The probability density function (*pdf*) of the BSL distribution with parameters α and β ($Chen(\alpha, \beta)$) is

$$f(x; \alpha, \beta) = \alpha\beta x^{\beta-1} exp[\alpha(1 - e^{x^\beta}) + x^\beta], \quad x > 0, \quad \alpha, \beta > 0 \tag{1}$$

with corresponding cumulative distribution function (*cdf*) as

$$F(x; \alpha, \beta) = 1 - exp[\alpha(1 - e^{x^\beta})], \quad x > 0, \quad \alpha, \beta > 0. \tag{2}$$

The E-Bayesian method was introduced by Han (1997) [3]. In recent years, there has been a growing interest in the study of E-Bayesian estimation. E-Bayesian estimate and its properties were considered by Han (2007) [4], Han (2011) [6] and Han (2011) [7] for the case of one and two hyperparameters. E-Bayesian estimate for the parameters of the geometric distribution based on URV and their relations were obtained by Okasha and Wang (2016) [14]. Han (2009) [5] discussed

the properties of E-Bayes estimate with three different prior distributions of hyperparameters. Jaheen and Okasha (2011) [8] studied the E-Bayesian estimation for the Burr type XII distribution based on the type-II censoring. Okasha (2012) [11] discussed the E-Bayesian method for computing estimates for the unknown parameters of the Weibull distribution based on type-II-censored samples. Okasha (2014) [12] studied that E-Bayesian methods for estimating the parameters of Lomax distribution under the balanced squared error loss function based on type-II censored data. Okasha (2019) [13] used the Burr XII model for type-II censored data. He observed that the E-Bayes estimates performs better than the Bayes estimates. For more details, see, Kizilaslan (2017) [9], Kizilaslan (2019) [10] and Piriaei et al. (2020) [15].
The main object of this article is to discuss the E-Bayesian estimation of the BSL distribution based on URV. Bayesian estimators of the BSL distribution are given in Section 2. E-Bayesian estimation based on three different distributions to the hyperparameters is derived in Section 3. Properties of the E-Bayesian estimators based on squared error loss (SEL) function are discussed in Section 4. Simulation study is given in Section 5. Finally, the conclusion of this article is discussed in Section 6.

2. Bayesian Estimation

We consider m URV $X_{U(1)} = x_1, X_{U(2)} = x_2, ..., X_{U(m)} = x_m$, from $Chen(a, b)$, with pdf (1). In this case, Ahsanullah (2004) [1] gives the likelihood function as

$$l(\alpha, \beta|\underline{x}) = f(x_{U(m)}; \alpha, \beta) \prod_{i=1}^{m-1} \frac{f(x_{U(i)}; \alpha, \beta)}{1 - F(x_{U(i)}; \alpha, \beta)}. \tag{3}$$

Using (1), (2) and (3), we get

$$l(\alpha, \beta|\underline{x}) = \alpha^m \psi(\beta, \underline{x}) e^{-\alpha L}, \tag{4}$$

where

$$\underline{x} = (x_1, x_2, ...x_m), \qquad \psi(\beta, \underline{x}) = \beta^m \prod_{i=1}^{m} x_i^{\beta-1} e^{x_i^\beta}$$

and

$$L = e^{x_m^\beta} - 1.$$

When β is known in the two-parameter BSL distribution , the maximum likelihood estimator (MLE) of the scale parameter α, can be written as

$$\hat{\alpha}^{MLE} = \frac{m}{L}. \tag{5}$$

The gamma conjugate prior density of the parameter α can be expressed as

$$h(\alpha) = \frac{\gamma^\lambda}{\Gamma\lambda} \alpha^{\lambda-1} e^{-\alpha\gamma}, \quad \alpha > 0. \tag{6}$$

Using (4) and (6), we get posterior density of α, i.e.

$$q(\alpha|\underline{x}) = A^* \alpha^{m+\beta-1} e^{-\alpha(L+\gamma)}, \quad \alpha > 0, \beta > 0 \tag{7}$$

where

$$A^* = \frac{(L+\gamma)^{m+\lambda}}{\Gamma(m+\lambda)}.$$

The Bayes estimate of α based on the SEL function can be expressed as

$$\hat{\alpha}^{BE}(\lambda, \gamma) = \frac{(m+\lambda)}{(L+\gamma)}. \tag{8}$$

3. E-Bayesian Estimation

According to Han (1997) [3], the prior parameters λ and γ should be selected to guarantee that the prior $h(\alpha|\lambda,\gamma)$ in (6) is a decreasing function of α. The derivative of $h(\alpha|\lambda,\gamma)$ with respect to α is

$$\frac{dh(\alpha)}{\alpha} = \frac{\gamma^{\lambda}}{\Gamma\lambda}\alpha^{\lambda-2}e^{-\alpha\gamma}[(\lambda-1)-\alpha\gamma]. \tag{9}$$

Thus, for $0<\lambda<1$, $\gamma>0$ and $\alpha>0$, the prior $h(\alpha|\lambda,\gamma)$ is a decreasing function of α. Assuming that the prior parameters λ and γ are independent random variables and their density functions are $\pi_1(\lambda)$ and $\pi_2(\gamma)$ respectively. Then the joint density function of λ and γ is

$$\pi(\lambda,\gamma) = \pi_1(\lambda)\pi_2(\gamma). \tag{10}$$

The E-Bayesian estimate of α can be written as

$$\hat{\alpha}^{EBE} = E(\alpha|\underline{x}) = \int\int_D \hat{\alpha}^{BE}(\lambda,\gamma)\pi(\lambda,\gamma)d\lambda d\gamma, \tag{11}$$

where D is the domain of λ and γ for which the prior density is decreasing in α. $\hat{\alpha}^{BE}(\lambda,\gamma)$ is the Bayes estimate of α as given in (8).

3.1. E-Bayesian Estimation of α under SEL function

In order to obtain E-Bayesian estimation of α, the following distributions of the hyperparameters λ and γ are used

$$\begin{cases} \pi_1(\lambda,\gamma) = \frac{1}{kB(s,g)}\lambda^{s-1}(1-\lambda)^{d-1}, & 0<\lambda<1, \quad 0<\gamma<k, \\ \pi_2(\lambda,\gamma) = \frac{2}{k^2B(s,g)}(k-\gamma)\lambda^{s-1}(1-\lambda)^{d-1}, & 0<\lambda<1, \quad 0<\gamma<k, \\ \pi_3(\lambda,\gamma) = \frac{2\gamma}{k^2B(s,g)}\lambda^{s-1}(1-\lambda)^{d-1}, & 0<\lambda<1, \quad 0<\gamma<k. \end{cases} \tag{12}$$

The E-Bayesian estimates of the parameter α based on SEL function is obtained by using (8), (11) and (12) as

$$\hat{\alpha}^{EBE1} = \int\int_D \hat{\alpha}^{BE}(\lambda,\gamma)\pi_1(\lambda,\gamma)d\lambda d\gamma = \frac{1}{kB(s,g)}\int_0^1\int_0^c \left(\frac{m+\lambda}{L+\gamma}\right)\lambda^{s-1}(1-\lambda)^{d-1}d\gamma d\lambda,$$

$$= \frac{1}{k}\left(m+\frac{s}{s+g}\right) ln\left(\frac{L+k}{L}\right), \tag{13}$$

$$\hat{\alpha}^{EBE2} = \frac{2}{k}\left(m+\frac{s}{s+g}\right)\left[\frac{L+k}{k}ln\left(\frac{L+k}{L}\right)-1\right] \tag{14}$$

and

$$\hat{\alpha}^{EBE3} = \frac{2}{k}\left(m+\frac{s}{s+g}\right)\left[1-\frac{L}{k}ln\left(\frac{L+k}{L}\right)\right]. \tag{15}$$

4. Properties of E-Bayesian Estimation based on SEL function

In this section, we presents the relations among $\hat{\alpha}^{EBE1}$, $\hat{\alpha}^{EBE2}$ and $\hat{\alpha}^{EBE3}$.
Lemma. Let $0<k<L$, $s>0$, $g>0$ and $\hat{\alpha}^{EBEi}(i=1,2,3)$ be given by (13), (14) and (15). Then the following inequalities are:
(i) $\hat{\alpha}^{EBE2} < \hat{\alpha}^{EBE1} < \hat{\alpha}^{EBE3}$.
(ii) $lim_{L\to\infty}\hat{\alpha}^{EBE1} = lim_{L\to\infty}\hat{\alpha}^{EBE2} = lim_{L\to\infty}\hat{\alpha}^{EBE3}$.
Proof. (i) From (13), (14) and (15), we have

$$\hat{\alpha}^{EBE2} - \hat{\alpha}^{EBE1} = \hat{\alpha}^{EBE1} - \hat{\alpha}^{EBE3} = \frac{1}{k}\left(m+\frac{s}{s+g}\right)\left[\frac{k+2L}{k}ln\left(\frac{L+k}{L}\right)-2\right]. \tag{4.1}$$

For $-1 < t < 1$, we have: $ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + ... = \sum_{p=1}^{\infty}(-1)^{p-1}\frac{t^p}{p}$.
Let $t = \frac{k}{L}$, when $0 < k < L$ and $0 < \frac{k}{L} < 1$, we get:

$$\left[\frac{k+2L}{k} ln\left(\frac{L+k}{L}\right) - 2\right]$$

$$= \frac{k+2L}{k}\left[\left(\frac{k}{L}\right) - \frac{1}{2}\left(\frac{k}{L}\right)^2 + \frac{1}{3}\left(\frac{k}{L}\right)^3 - \frac{1}{4}\left(\frac{k}{L}\right)^4 + \frac{1}{5}\left(\frac{k}{L}\right)^5 - ...\right] - 2$$

$$= \left[\left(\frac{k}{L}\right) - \frac{1}{2}\left(\frac{k}{L}\right)^2 + \frac{1}{3}\left(\frac{k}{L}\right)^3 - \frac{1}{4}\left(\frac{k}{L}\right)^4 + \frac{1}{5}\left(\frac{k}{L}\right)^5 - ...\right] - 2$$

$$+ \left(2 - \left(\frac{k}{L}\right) + \frac{2}{3}\left(\frac{k}{L}\right)^2 - \frac{2}{4}\left(\frac{k}{L}\right)^3 + \frac{2}{5}\left(\frac{k}{L}\right)^4 - ...\right)$$

$$= \left(\frac{k^2}{6L^2} - \frac{k^3}{6L^3}\right) + \left(\frac{3k^4}{20L^4} - \frac{2k^5}{15L^5}\right) + ...$$

$$= \frac{k^2}{6L^2}\left(1 - \frac{k}{L}\right) + \frac{k^4}{60L^4}\left(9 - \frac{8k}{L}\right) + ... > 0. \tag{4.2}$$

According to (4.1) and (4.2), we have

$$\hat{\alpha}^{EBE2} - \hat{\alpha}^{EBE1} = \hat{\alpha}^{EBE1} - \hat{\alpha}^{EBE3} > 0,$$

that is

$$\hat{\alpha}^{EBE2} < \hat{\alpha}^{EBE1} < \hat{\alpha}^{EBE3}.$$

(ii) From (4.1) and (4.2), we get

$$lim_{L\to\infty}\left(\hat{\alpha}^{EBE2} - \hat{\alpha}^{EBE1}\right) = lim_{L\to\infty}\left(\hat{\alpha}^{EBE1} - \hat{\alpha}^{EBE3}\right)$$

$$= \frac{1}{c}\left(m + \frac{s}{s+g}\right) lim_{L\to\infty}\{\frac{c^2}{6L^2}\left(1 - \frac{c}{L}\right) + \frac{c^4}{60L^4}\left(9 - \frac{8c}{L}\right) + ...\}$$

$$= 0.$$

That is, $lim_{L\to\infty}\hat{\alpha}^{EBE1} = lim_{L\to\infty}\hat{\alpha}^{EBE2} = lim_{L\to\infty}\hat{\alpha}^{EBE3}$.
Thus, the proof is complete.

5. Simulation Study

This section presents, a simulation study for a comparison of Bayes and E-Bayes methods of estimation .
The steps of the simulation are:
1. Sample sizes $n = 20, 30, 40, 50$, no. of records $m = 8$ and for the each case the parameters $(\alpha, \beta) = (1, 2)$.
2. $(s, g) = (0.5, 0.5)$ and $k = 12$.
3. Estimates are calculated by two types of priors:

- for prior I, hyperparameter values, $(\lambda, \gamma) = (0.2, 2.5)$.
- for prior II, hyperparameter values, $(\lambda, \gamma) = (0.5, 3)$.

4. The URV are generated from $Chen(\alpha, \beta)$, by using $X = [log\{1 - \frac{log(1-U)}{\alpha}\}]^{\frac{1}{\beta}}$, where U is uniform $(0, 1)$.
5. The estimates $\hat{\alpha}^{BE}$ and $\hat{\alpha}^{EBEi}$, $i = 1, 2, 3$ under the SEL function are computed from (8) and (14)-(16).
6. Repeat the above steps for $10,000$ times. The average of all $10,000$ estimated values are, respectively, calculated and summarized.
7. The computational results are displayed in Table 1. All computations were performed using R Software.

Table 1: *Bayesian and E-Bayesian estimates of α (first row), average bias (second row) and mean square error (third row)*

		Prior I				Prior II			
n	Par	BE	EBE1	EBE2	EBE3	BE	EBE1	EBE2	EBE3
20	α	1.3940	1.0751	1.3433	0.8068	1.3331	1.0789	1.3491	0.8087
	AB	0.3940	0.0751	0.3433	−0.1932	0.3331	0.0789	0.3491	−0.1913
	MSE	0.2229	0.0389	0.1901	0.0467	0.1646	0.0404	0.1961	0.0461
30	α	1.3086	1.0151	1.2545	0.7757	1.2530	1.0148	1.2543	0.7753
	AB	0.3086	0.0151	0.2545	−0.2243	0.2530	0.0148	0.2543	−0.2247
	MSE	0.1478	0.0255	0.1178	0.0581	0.1075	0.0268	0.1204	0.0586
40	α	1.2505	0.9748	1.1958	0.7537	1.1995	0.9726	1.1927	0.7526
	AB	0.2505	−0.0252	0.1958	−0.2463	0.19951	−0.0274	0.1927	−0.2474
	MSE	0.1079	0.0222	0.0826	0.0677	0.0755	0.0219	0.0806	0.0682
50	α	1.2088	0.9459	1.1542	0.7376	1.1631	0.9445	1.1521	0.7367
	AB	0.2088	−0.0541	0.1542	−0.2625	0.1631	−0.0556	0.1521	−0.2632
	MSE	0.0839	0.0222	0.0626	0.0754	0.0591	0.0222	0.0617	0.0758

6. Conclusion

In this paper, Bayes and E-Bayes methods are considered for estimating the unknown scale parameter of the BSL distribution based on URV. Under SEL function and three distributions of the hyperparameters, the E-Bayesian estimators are introduced. Properties of E-Bayesian Estimation based on SEL function are derived. A comparison of Bayes and E-Bayes estimates of scale parameter is performed through a simulation study. From Table [1], we observed that the performance of the E-Bayesian estimators are better than the Bayesian estimators. Moreover, Bayesian estimators and E-Bayesian estimators for Prior II are better than Prior I. Also, from Table [1], we observe that estimators are showing lesser MSEs, as increase the sample size n.

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