PERFORMANCE ANALYSIS OF $M^{[X]}/G^B/1$ FEEDBACK RETRIAL QUEUE WITH VARIABLE SERVER MODEL

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Abstract

In this article, a working vacation policy-based on bulk arrival feedback retrial queueing system with variable server capacity has been analyzed. The server can serve a minimum of one customer and a maximum of $\mathcal{B}$ customers in a batch in accordance with the variable server capacity bulk service rule. As soon as the orbit becomes empty at the time of service completion, the server goes for a working vacation. The server works at a lower speed during a working vacation period. In addition, the steady state probability generating function for system size and orbit size is generated by incorporating the supplementary variables technique (SVT). Further, the conditional decomposition law is shown for this retrial queueing system. Moreover, system performance metrics, and significant special instances are discussed. Finally, the effects of various parameters on the system performance are analyzed numerically.

Keywords: Retrial queue, variable server capacity, working vacation, supplementary variable technique.

1. Introduction

The study of vacation queues (VQs) and retrial queues (RQs) in queueing theory has been going on for a while. When a consumer arrives and discovers the server is busy, they are directed to depart the service area and join a retry line called "orbit." In a RQ system, this is referred to as a RQ with repeated tries. The orbiting consumers may attempt their service request again when some time has passed. Additionally, the consumers in the orbit are allowed to request the same service repeatedly without affecting the other consumers. Modified models can be explored in RQs from Artalejo and Gomez Corral [1] and in VQs from Ke et al. [10]. The application of these queues in computer and communication systems is distinct.

In a VQ system, the server serves consumers at a slower rate speed during the working vacation (WV) period but fully discontinues service during the regular vacation period. Major uses for this queueing system (QS) include delivering network service, online service, file transfer service, and mail service, among others. Gautam Choudhury [7] probed a bulk arrival queue with a vacation period under single vacation strategy. Shan Gao [20] discussed a batch arrival queue with delayed single WV. A concise explanation of WV queueing systems was given in recent years by Chandrasekaran et al.[4]. Rajadurai [17] developed a unique form of the RQ model, which contained WV and breaks. An exponentially distributed multiple WV and a bulk arrival RQ with feedback were both studied by Pazhani Bala Murugan and Vijaykrishnaraj [16]. Madhu Jain and Anshul Kumar [12] analysed the $M^{[X]}/G/1$ model with WV, balking, unreliable server.

One of the most important aspects of communication systems is the feedback phenomena. If the service provided to a consumer is unsatisfactory, it may be tried again until it is. Maragathasundari and Balamurugan [14] studied the $M^{[X]}/G/1$ feedback queue with two stages of repair times, general delay times. Madhu Jain and Anshul Kumar [13] discovered the bulk arrival general service RQ subject to balking, feedback and vacation interruption under multiple WV policy.

In real-world circumstances like elevators, freight loading and unloading, big wheels, chemical industrial processes, communication networks, tourism, etc., bulk QS are frequently used. Bailey [2] invented batch service queueing techniques. Batch service queueing system have been researched by Sasikala and Indhira [18]. Jaiswal [9] is the source of the original research on the variable server capacity bulk service rule. Banerjee et al. [3] have thought about queueing models with a variable server capacity and bulk service rule. Recently, Sasikala et al. [19] discovered the bulk RQ system with Bernoulli vacation schedule and variable server capacity. In the WV queue for bulk arrival feedback, no work is being done. Therefore, we concentrated on batch arrival using a batch service feedback RQ system with variable server capacity while working on vacation.

The purpose of this research is to ascertain the queue length and orbit length distributions, which will be used to ascertain the system's other behaviour metrics. The structure of our article is as follows: We offer a detailed description of the queueing model in section 2 once the prerequisites have been met. In section 3, it has been clearly determined how the system behaves in steady-state (SS) conditions and what the probability generating function (PGF) of the queue size is at a random epoch. There are various important system behaviour indicators in section 4. Stochastic decomposition and some important specific occurrences are mentioned in section 5. There are both numerical and pictorial findings in section 6. Finally, the paper's key ideas are summarized in section 7.

2. Description of the model and its implementation in real world

Under WVs policy, we provide a $M^{[X]}/G^{\mathcal{B}}/1$ feedback RQ . The precise justification of our model is as follows:
The arrival process: According to the Poisson process, consumers are arriving for service at the rate α. where $\mathcal{F}$ is the batch size random variable with probability mass function $P\{\mathcal{F}=n\}=f_n$, $n=1,2,3,...$ probability generating function (PGF) $\mathcal{F}(\breve{\zeta})=\sum_{n=0}^{\infty}\breve{\zeta}^n f_n$ and mean batch size $\mathcal{E}(\mathcal{I})$.
The retrial process: If arriving consumers not getting service immediately due to some reasons, they gather together, and this is known as an "orbit". After a certain random amount of time, customers will retry for service. Inter retrial times have a random dist., $\mathcal{H}(\varpi)$ with corresponding "Laplace-Stieltijes Transform" (LST) $\mathcal{H}^*(\delta)$
The regular service process: Based on the variable server capacity bulk service regulation, the server will transmit the consumers. According to the variable server capacity batch service rule, the server will serve either fixed size, like a "$\mathcal{B}$", or all of the consumers from the orbit, depending on which is lower. If there are more than or equal to "$\mathcal{B}$" consumers in the orbit after the group of consumers has been transmitted, the server will proceed to transmit "$\mathcal{B}$" packets in a batch. If less than "$\mathcal{B}$" packets remain in the orbit after transmission, the server will take whole consumers to send in a batch. After the service has begun, late entrants are not permitted to participate in the ongoing service, even if the batch size is smaller than "$\mathcal{B}$". The service period represents a general dist. and it is marked by the arbitrary variable $\mathcal{D}$ with dist. function $\mathcal{D}(\varpi)$ having LST $\mathcal{D}^*(\delta)$.
Feedback rule: Unsatisfied consumers have the option to re-enter the orbit as feedback consumers once their normal service is complete in order to maybe receive another service with prob., β $(0\le\beta\le 1)$ will exit the system with prob., $\bar{\beta}=(1-\beta)$.
The working vacation policy: When the orbit is free, the server periodically takes a WV. The vacation period takes an exponential dist. with variable ω. If a consumer enters during a vacation time, the server keeps running at a reduced rate. During the WV time, tasks are carried out at a slower pace. If any consumers are in the orbit at a slower service completion moment during the vacation period, the server will end the vacation and return to the normal busy time, interrupting the vacation. If not, the vacation, keeps going. When the vacation gets over, the server restores

normal operations if there are still customers in the orbit. During the WV period, the service period is assessed by a random variable $\mathcal{D}_v$ with dist. function $\mathcal{D}_v(\varpi)$ and LST $\mathcal{D}_v^*(\delta)$.

The system's stochastic processes are considered to be independent of one another.

2.1. Practical application of the model

As an example of the proposed paradigm in action, consider that telephone consultations are a significant component of medical care delivery systems, which illustrates how the suggested paradigm is used in real-life scenarios. We take into consideration a telephone consultation system with a primary server (the chief physician) and a physician assistant (the working breakdown server). Patients attempt to schedule appointments for treatment over the phone, but there is a restriction on the no.of appointments (variable server capacity) that may be made each day for treatment. The physician assistant provides service only when the primary physician is unavailable, and it is noted that the assistant's service delivery is frequently slower than that of the primary physician. Furthermore, when each patient's regular service is completed (feedback), the dissatisfied patient may re-enter the orbit.

In order to schedule the appointments, a phone operator will be available, who usually manages the patients and doctors. If the phone line is busy when a patient calls, he must wait and try again later (retrial); if not, he will be given an appointment right away to see the head physician or the physician assistant for treatment. However, the phone operator will call (or search for) the FCFS customers in orbit as soon as the service is finished. It is predicted that the search time will be evenly divided, which is in line with the general retry time policy.

3. Overview of steady state probabilities

This division first develops the steady-state (SS) equations for the RQ system by considering the elapsed retrial period, the elapsed service time and the elapsed lower-speed service times as supplementary variable (SV). The PGF of the no. of consumers in the orbit and system, as well as the orbit length generating functions for numerous server states, are computed.

3.1. Probabilities and Notations

It is assumed in SS that $\mathcal{H}(0) = 0$, $\mathcal{H}(\infty) = 1$, $D(0) = 0$, $\mathcal{D}(\infty) = 1$ and $\mathcal{D}_v(0) = 0$, $\mathcal{D}_v(\infty) = 1$ are cont., at $\varpi = 0$. So that the func. $\chi(\varpi)$, $\eta(\varpi)$, $\eta_v(\varpi)$, are the hazard rates (HR) for retrial, service and slower pace service respectively.

Further, the subsequent notations and probabilities were defined:

$\chi(\varpi)$	-	HR for retrial (i.e.,) $\chi(\varpi)d\varpi = \frac{d\mathcal{H}(\varpi)}{1-\mathcal{H}(\varpi)}$
$\eta(\varpi)$	-	HR for service (i.e.,) $\eta(\varpi)d\varpi = \frac{d\mathcal{D}(\varpi)}{1-\mathcal{D}(\varpi)}$
$\eta_v(\varpi)$	-	HR for slower pace service (i.e.,) $\eta_v(\varpi)d\varpi = \frac{d\mathcal{D}_v(\varpi)}{1-\mathcal{D}_v(\varpi)}$
$\mathcal{Y}(\check{\tau})$	-	no.of consumers in the orbit
$\mathcal{H}^0(\check{\tau})$	-	elapsed retrial time
$\mathcal{D}^0(\check{\tau})$	-	elapsed service time
$\mathcal{D}_v^0(\check{\tau})$	-	elapsed WV times
$\Upsilon_n(\varpi, \check{\tau})$	-	Prob. that at time $\check{\tau}$ there are precisely n consumers in the orbit with the consumer going through a retrial having served their whole service period is ϖ.
$\Omega_n(\varpi, \check{\tau})$	-	Prob. that at time $\check{\tau}$ there are precisely n consumers in the orbit with the consumer going through normal service having served their whole service period is ϖ.

$\Psi_{v,n}(\varpi, \check{\tau})$ - Prob. that at time $\check{\tau}$ there are precisely n consumers in the orbit with the consumer going through slower pace service having served their whole service period is ϖ.

Apart from it, let $\mathcal{H}^0(\check{\tau})$,$\mathcal{D}^0(\check{\tau})$ and $\mathcal{D}^0_v(\check{\tau})$ be the elapsed retrial period, the elapsed period of normal service and the elapsed slower-rate service period respectively at time $\check{\tau}$. Additionally, generate the random variable,

$$\Theta(\check{\tau}) = \begin{cases} 0, & \text{if the server is available and in WV time} \\ 1, & \text{if the server is available and in normal service time} \\ 2, & \text{if the server is unavailable and in normal service at time } \check{\tau} \\ 3, & \text{if the server is unavailable and in lower speed rate at time } \check{\tau} \end{cases}$$

Here, we highlight the usage of bivariate Markov process to describe the system's state at time $\{\Theta(\check{\tau}), \mathcal{Y}(\check{\tau}); \check{\tau} \geq 0\}$, where $\Theta(\check{\tau})$ signifies the server state $(0, 1, 2, 3)$ depending on whether the server is free or busy on both normal service and WV periods. $\mathcal{Y}(\check{\tau})$ denotes the no. of consumers in the orbit. If $\Theta(\check{\tau}) = 1$ and $\mathcal{Y}(\check{\tau}) > 0$, then $\mathcal{H}^0(\check{\tau})$ is equivalent to the elapsed retrial time. If $\Theta(\check{\tau}) = 2$ and $\mathcal{Y}(\check{\tau}) \geq 0$, then $\mathcal{D}^0(\check{\tau})$ is equivalent to the elapsed time of the consumer served in normal busy period. If $\Theta(\check{\tau}) = 3$ and $\mathcal{Y}(\check{\tau}) \geq 0$, then $\mathcal{D}^0_v(\check{\tau})$ is equivalent to the elapsed time of the consumer being served in lower rate service period.

3.2. Ergodicity analysis of the model

We examine the embedded Markov chain's ergodicity during the departure and vacation epochs. Let $\{\check{\tau}_n; n = 1, 2, ...\}$ be the series of epochs where either a service period completion or a shorter service period happens. $G_n = \{\Theta(\check{\tau}_n+), \mathcal{Y}(\check{\tau}_n+)\}$ sequence of random vectors. The Markov chain formed by embedded in the RQ system. It follows from Appendix A that is the embedded Markov chain $\{\mathcal{G}_m; m\epsilon M\}$ is ergodic iff $\Lambda < \mathcal{B}$ for our system will be stable.

For the method $\{\mathcal{Y}(\check{\tau}), \check{\tau} \geq 0\}$, we specify the probabilities $\mathcal{Q}_0(\check{\tau}) = \mathcal{P}\{\Lambda(\check{\tau}) = 0, \mathcal{Y}(\check{\tau}) = 0\}$ and the prob. densities are
$\Upsilon_n(\varpi, \check{\tau})d\varpi = \mathcal{P}\{\Lambda(\check{\tau}) = 1, \mathcal{Y}(\check{\tau}) = n, \varpi \leq \mathcal{H}^0(\check{\tau}) < \varpi + d\varpi\}$,
for $\check{\tau} \geq 0$, $\varpi \geq 0$ and $n \geq 1$.
$\Omega_n(\varpi, \check{\tau})d\varpi = \mathcal{P}\{\Lambda(\check{\tau}) = 2, \mathcal{Y}(\check{\tau}) = n, \varpi \leq \mathcal{D}^0(\check{\tau}) < \varpi + d\varpi\}$,
for $\check{\tau} \geq 0$, $\varpi \geq 0$ and $n \geq 0$.
$\Psi_{v,n}(\varpi, \check{\tau})d\varpi = \mathcal{P}\{\Lambda(\check{\tau}) = 3, \mathcal{Y}(\check{\tau}) = n, \varpi \leq \mathcal{D}^0_v(\check{\tau}) < \varpi + d\varpi\}$,
for $\check{\tau} \geq 0$, $\varpi \geq 0$ and $n \geq 0$.
We presume that the stability requirement is satisfied in the sequel, so we may assign $\mathcal{Q}_0 = lim_{\check{\tau}\to\infty}\mathcal{Q}_0(\check{\tau})$ and limiting densities are
$\Upsilon_n(\varpi) = lim_{\check{\tau}\to\infty}\Upsilon_n(\varpi, \check{\tau})$; $\Omega_n(\varpi) = lim_{\check{\tau}\to\infty}\Omega_n(\varpi, \check{\tau})$;
$\Psi_{v,n}(\varpi) = lim_{\check{\tau}\to\infty}\Psi_{v,n}(\varpi, \check{\tau})$;
Using the supplementary variable method, we build the following system of equations.

$$\begin{aligned} \alpha\mathcal{Q}_0 = \bar{\beta}\int_0^\infty \Omega_0(\varpi)\eta(\varpi)d\varpi + \bar{\beta}\int_0^\infty \Psi_{v,0}(\varpi)\eta_v(\varpi)d\varpi \\ + \alpha\int_0^\infty \Omega_n(\varpi)d\varpi, n \geq 0 \end{aligned} \tag{1}$$

$$\frac{d}{d\varpi}\Upsilon_n(\varpi) + (\alpha + \chi(\varpi))\Upsilon_n(\varpi) = 0, n \geq 1 \tag{2}$$

$$\frac{d}{d\varpi}\Omega_0(\varpi) + (\alpha + \eta(\varpi))\Omega_0(\varpi) = 0, n = 0 \tag{3}$$

$$\frac{d}{d\varpi}\Omega_n(\varpi) + (\alpha + \eta(\varpi))\Omega_n(\varpi) = \alpha \sum_{k=1}^{n} \Omega_{n-k} f_k(\varpi), n \geq 1 \tag{4}$$

$$\frac{d}{d\varpi}\Psi_{v,0}(\varpi) + (\alpha + \omega + \eta_v(\varpi))\Psi_{v,0}(\varpi) = 0, n = 0 \tag{5}$$

$$\frac{d}{d\varpi}\Psi_{v,n}(\varpi) + (\alpha + \omega + \eta_v(\varpi))\Psi_{v,n}(\varpi) = \alpha \sum_{k=1}^{n} \Psi_{v,n-k} f_k(\varpi), n \geq 0 \tag{6}$$

At $\varpi = 0$ the SS boundary criteria are as follows:

$$\begin{aligned}\Upsilon_n(0) &= \beta \int_0^\infty \Omega_n(\varpi)\eta(\varpi)d\varpi + \bar{\beta} \int_0^\infty \Omega_{n-1}(\varpi)\eta(\varpi)d\varpi \\ &+ \beta \int_0^\infty \Psi_{v,n}(\varpi)\eta_v(\varpi)d\varpi + \bar{\beta} \int_0^\infty \Psi_{v,n-1}(\varpi)\eta_v(\varpi)d\varpi, n \geq 1\end{aligned} \tag{7}$$

$$\begin{aligned}\Omega_n(0) &= \int_0^\infty \Upsilon_{n+B}(\varpi)\chi(\varpi)d\varpi + \alpha \int_0^\infty \sum_{k=1}^{\infty} f_k \Upsilon_{n-k+B}(\varpi)d\varpi \\ &+ \omega \int_0^\infty \Psi_{v,n}(\varpi)d\varpi, n \geq 1\end{aligned} \tag{8}$$

$$\Omega_0(0) = \int_0^\infty \sum_{n=1}^{\mathcal{B}} \Upsilon_n(\varpi)\chi(\varpi)d\varpi + \alpha \sum_{k=1}^{\mathcal{B}} f_k \Upsilon_0 + \omega \int_0^\infty \Psi_{v,0}(\varpi)d\varpi, n = 0 \tag{9}$$

$$\Psi_{w,n}(0) = \begin{cases} \alpha \mathcal{Q}_0, & n = 0 \\ 0, & n \geq 1 \end{cases} \tag{10}$$

The normalizing criteria is

$$\mathcal{Q}_0 + \sum_{n=1}^{\infty} \int_0^\infty \Upsilon_n(\varpi)d\varpi + \sum_{n=0}^{\infty} \left(\int_0^\infty \Omega_n(\varpi)d\varpi + \int_0^\infty \Psi_{v,n}(\varpi)d\varpi \right) = 1 \tag{11}$$

3.3. The steady state solution

The GFs for $|\check{\zeta}| < 1$ in order to solve the aforementioned equations, are expressed in the form.

$$\Upsilon(\varpi, \check{\zeta}) = \sum_{n=1}^{\infty} \Upsilon_n(\varpi)\check{\zeta}^n; \Upsilon(0, \check{\zeta}) = \sum_{n=1}^{\infty} \Upsilon_n(0)\check{\zeta}^n;$$

$$\Omega(\varpi, \check{\zeta}) = \sum_{n=0}^{\infty} \Omega_n(\varpi)\check{\zeta}^n; \Omega(0, \check{\zeta}) = \sum_{n=0}^{\infty} \Omega_n(0)\check{\zeta}^n;$$

$$\Psi_v(\varpi, \check{\zeta}) = \sum_{n=0}^{\infty} \Psi_{v,n}(\varpi)\check{\zeta}^n; \Psi_v(0, \check{\zeta}) = \sum_{n=0}^{\infty} \Psi_{v,n}(0)\check{\zeta}^n;$$

Now multiply the SS equation and SS boundary criteria from (2) to (10) by $\check{\zeta}^n$ and summing over n, $(n = 0, 1, 2, ...)$

$$\frac{\partial}{\partial\varpi}\Upsilon(\varpi, \check{\zeta}) + (\alpha + \chi(\varpi))\Upsilon(\varpi, \check{\zeta}) = 0 \tag{12}$$

$$\frac{\partial}{\partial\varpi}\Omega(\varpi, \check{\zeta}) + (\alpha(1 - \mathcal{F}(\check{\zeta})) + \eta(\varpi))\Omega(\varpi, \check{\zeta}) = 0 \tag{13}$$

$$\frac{\partial}{\partial\varpi}\Psi_v(\varpi, \check{\zeta}) + (\alpha(1 - \mathcal{F}(\check{\zeta})) + \omega + \eta_v(\varpi))\Psi_v(\varpi, \check{\zeta}) = 0 \tag{14}$$

$$\Upsilon(0,\breve{\zeta}) = (\beta+\bar{\beta}\breve{\zeta})\int_0^\infty \Omega(\varpi,\breve{\zeta})\eta(\varpi)d\varpi + (\beta+\bar{\beta}\breve{\zeta})\int_0^\infty \Psi_v(\varpi,\breve{\zeta})\eta_v(\varpi)d\varpi - \alpha\mathcal{Q}_0 \tag{15}$$

$$\Omega(0,\breve{\zeta}) = \frac{1}{\breve{\zeta}^{\mathcal{B}}}\int_0^\infty \Upsilon(\varpi,\breve{\zeta})\chi(\varpi)d\varpi + \frac{\alpha\mathcal{F}(z)}{\breve{\zeta}^{\mathcal{B}}}\int_0^\infty \Upsilon(\varpi,\breve{\zeta})d\varpi + \omega\int_0^\infty \Psi_v(\varpi,\breve{\zeta})d\varpi \tag{16}$$

$$\Psi_v(0,\breve{\zeta}) = \alpha\mathcal{Q}_0 \tag{17}$$

Solving the partial differential eqns. (12) to (14), we obtain

$$\Upsilon(\varpi,\breve{\zeta}) = \Upsilon(0,\breve{\zeta})[1-\mathcal{H}(\varpi)]e^{-\alpha\varpi} \tag{18}$$

$$\Omega(\varpi,\breve{\zeta}) = \Omega(0,\breve{\zeta})[1-\mathcal{D}(\varpi)]e^{-\mathcal{S}(\breve{\zeta})\varpi} \tag{19}$$

$$\Psi_v(\varpi,\breve{\zeta}) = \Psi_v(0,\breve{\zeta})[1-\mathcal{A}_w(\varpi)]e^{-\mathcal{S}_v(\breve{\zeta})\varpi} \tag{20}$$

where $\mathcal{S}(\breve{\zeta}) = \alpha(1-\mathcal{F}(\breve{\zeta}))$, and $\mathcal{S}_v(\breve{\zeta}) = \omega + \alpha(1-\mathcal{F}(\breve{\zeta}))$
Inserting the eqns. (17) to (20) in (8) after some computation, we eventually arrive to,

$$\Omega(0,\breve{\zeta}) = \frac{\Upsilon(0,\breve{\zeta})}{\breve{\zeta}^{\mathcal{B}}}\{\mathcal{H}^*(\alpha) + \mathcal{F}(\breve{\zeta})[1-\mathcal{H}^*(\alpha)]\} + \alpha\mathcal{Q}_0\mathcal{V}(\breve{\zeta}) \tag{21}$$

where $\mathcal{V}(\breve{\zeta}) = \frac{\omega}{\omega+\alpha(1-\mathcal{F}(\breve{\zeta}))}\,(1-\mathcal{D}_v^*(\mathcal{S}_v(\breve{\zeta})))$,

$$\Upsilon(0,\breve{\zeta}) = (\beta+\bar{\beta}\breve{\zeta})\Omega(0,\breve{\zeta})\mathcal{D}^*(\mathcal{S}(\breve{\zeta})) + (\beta+\bar{\beta}\breve{\zeta})\Psi_v(0,\breve{\zeta})\mathcal{D}_v^*(\mathcal{S}_v(\breve{\zeta})) - \alpha\mathcal{Q}_0 \tag{22}$$

Combining (10) and (21) in (22), we get

$$\begin{aligned}&\Omega(0,\breve{\zeta})\{\breve{\zeta}^{\mathcal{B}} - (\beta+\bar{\beta}\breve{\zeta})[\mathcal{H}^*(\alpha) + \mathcal{F}(\breve{\zeta})(1-\mathcal{H}^*(\alpha))]\mathcal{D}^*(\mathcal{S}(\breve{\zeta}))\}\\ &= \alpha\mathcal{Q}_0\{\breve{\zeta}^{\mathcal{B}}\mathcal{V}(\breve{\zeta}) + [(\beta+\bar{\beta}\breve{\zeta})\mathcal{D}_v^*(S_v(\breve{\zeta}))-1][\mathcal{H}^*(\alpha) + \mathcal{F}(\breve{\zeta})(1-\mathcal{H}^*(\alpha))]\}\end{aligned} \tag{23}$$

In the following theorem, we are willing to exploring the marginal orbit size distributions caused by the server's system state.

Theorem 1. Under the stability requirement, $\Lambda < B$ provides the stationary dist., of the no. of customers in the orbit when the server is available, busy, reduced rate service, and the prob., that the server is available given by,

$$\Upsilon(\breve{\zeta}) = \frac{Ne(\breve{\zeta})}{De(\breve{\zeta})} \tag{24}$$

$$\begin{aligned}Ne(\breve{\zeta}) &= \breve{\zeta}^{\mathcal{B}}\mathcal{Q}_0(1-\mathcal{H}^*(\alpha))\{(\beta+\bar{\beta}\breve{\zeta})[\mathcal{D}^*(\mathcal{S}(\breve{\zeta}))\mathcal{V}(\breve{\zeta}) + \mathcal{D}_v^*(\mathcal{S}_v(z))]-1\}\\ De(\breve{\zeta}) &= \breve{\zeta}^{\mathcal{B}} - (\beta+\bar{\beta}\breve{\zeta})\{\mathcal{H}^*(\alpha) + \mathcal{F}(\breve{\zeta})[1-\mathcal{H}^*(\alpha)]\}\mathcal{D}^*(\mathcal{S}(\breve{\zeta}))\end{aligned}$$

$$\Omega(\breve{\zeta}) = \frac{\alpha\mathcal{Q}_0(1-\mathcal{D}^*(\mathcal{S}(\breve{\zeta})))}{\mathcal{S}(\breve{\zeta})De(\breve{\zeta})}\{\breve{\zeta}^{\mathcal{B}}\mathcal{V}(\breve{\zeta}) + [(\beta+\bar{\beta}\breve{\zeta})\mathcal{D}_v^*(\mathcal{S}_v(\breve{\zeta}))-1][\mathcal{H}^*(\alpha) + \mathcal{F}(\breve{\zeta})[1-\mathcal{H}^*(\alpha)]]\} \tag{25}$$

$$\Psi_v(\breve{\zeta}) = \frac{\alpha\mathcal{Q}_0}{\omega}\mathcal{V}(\breve{\zeta}) \tag{26}$$

where

$$\mathcal{Q}_0 = \frac{\mathcal{B} - \{\bar{\beta} - \alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D}) + \mathcal{E}(\mathcal{I})(1-\mathcal{H}^*(\alpha))\}}{\begin{aligned}&\alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D})\{\tfrac{2\alpha}{\omega}(1-\mathcal{D}_v^*(\omega)) - 2\mathcal{D}_v^*(\omega)\mathcal{H}^*(\alpha) + \mathcal{D}_v^*(\omega) + \mathcal{H}^*(\alpha) + 1\}\\ &- \mathcal{E}(\mathcal{I})(1-\mathcal{H}^*(\alpha))[1+\alpha\mathcal{E}(\mathcal{D}_v)] - \alpha\mathcal{E}(\mathcal{D})[1+\mathcal{B}(1-\mathcal{D}_v^*(\omega))] + \mathcal{B}(1+\tfrac{\alpha}{\omega}(1-\mathcal{D}_v^*(\omega)))\end{aligned}} \tag{27}$$

Proof. Taking the eqns. (18)-(20) and integrate with respect to ϖ and compute the PG $\Upsilon(\check{\zeta}) = \int_0^\infty \Upsilon(\varpi, \check{\zeta}) d\varpi$, $\Omega(\check{\zeta}) = \int_0^\infty \Omega(\varpi, \check{\zeta}) d\varpi$, $\Psi_w(\check{\zeta}) = \int_0^\infty \Psi_w(\varpi, \check{\zeta}) d\varpi$. We calculate the prob. that the server is empty using the normalization condition (Q_0) by establishing functions as, when there is no consumer in the orbit $\check{\zeta} = 1$ in (24)-(26) and whenever the condition of L'Hospital is needed, we get $\mathcal{Q}_0 + \Upsilon(1) + \Omega(1) + \Psi_w(1) = 1$. ■

Theorem 2. Utilizing the PGF function, the no. of consumers in the system and the orbit size dist. at a stationary point of period are calculated under the stability constraint $\Lambda < \mathcal{B}$,

$$\mathcal{K}_s(\check{\zeta}) = \frac{Ne_s(\check{\zeta})}{De_s(\check{\zeta})} \tag{28}$$

$$\begin{aligned}
Ne_s(\check{\zeta}) =& \mathcal{Q}_0\{\mathcal{S}(\check{\zeta})\{\check{\zeta}^{\mathcal{B}} - (\beta + \bar{\beta}\check{\zeta})\{\mathcal{H}^*(\alpha) + \mathcal{F}(\check{\zeta})[1 - \mathcal{H}^*(\alpha)]\}\mathcal{D}^*(\mathcal{S}(\check{\zeta}))\} \\
& [1 + \frac{\alpha}{\omega}\check{\zeta}\mathcal{V}(\check{\zeta})]\} + \check{\zeta}^{\mathcal{B}}\mathcal{S}(\check{\zeta})(1 - \mathcal{H}^*(\alpha))\{(\beta + \bar{\beta}\check{\zeta})[\mathcal{D}^*(\mathcal{S}(\check{\zeta}))\mathcal{V}(\check{\zeta}) + \mathcal{D}_v^*(\mathcal{S}_v(z))] - 1\} \\
& + \check{\zeta}\alpha(1 - \mathcal{D}^*(\alpha(1 - \mathcal{F}(\check{\zeta}))))\{\check{\zeta}^{\mathcal{B}}\mathcal{V}(\check{\zeta}) + [(\beta + \bar{\beta}\check{\zeta})\mathcal{D}_v^*(\mathcal{S}_v(\check{\zeta})) - 1] \\
& [\mathcal{H}^*(\alpha) + \mathcal{F}(\check{\zeta})[1 - \mathcal{H}^*(\alpha)]]\}\} \\
De_s(\check{\zeta}) =& \mathcal{S}(\check{\zeta})\{\check{\zeta}^{\mathcal{B}} - (\beta + \bar{\beta}\check{\zeta})\{\mathcal{H}^*(\alpha) + \mathcal{F}(\check{\zeta})[1 - \mathcal{H}^*(\alpha)]\}\mathcal{D}^*(\mathcal{S}(\check{\zeta}))\}
\end{aligned}$$

$$\mathcal{K}_0(\check{\zeta}) = \frac{Ne_0(\check{\zeta})}{De_s(\check{\zeta})} \tag{29}$$

$$\begin{aligned}
Ne_0(\check{\zeta}) =& \mathcal{Q}_0\{\mathcal{S}(\check{\zeta})\{\check{\zeta}^{\mathcal{B}} - (\beta + \bar{\beta}\check{\zeta})\{\mathcal{H}^*(\alpha) + \mathcal{F}(\check{\zeta})[1 - \mathcal{H}^*(\alpha)]\}\mathcal{D}^*(\mathcal{S}(\check{\zeta}))\} \\
& [1 + \frac{\alpha}{\omega}\mathcal{V}(\check{\zeta})]\} + \check{\zeta}^{\mathcal{B}}\mathcal{S}(\check{\zeta})(1 - \mathcal{H}^*(\alpha))\{(\beta + \bar{\beta}\check{\zeta})[\mathcal{D}^*(\mathcal{S}(\check{\zeta}))\mathcal{V}(\check{\zeta}) + \mathcal{D}_v^*(\mathcal{S}_v(z))] - 1\} \\
& + \alpha(1 - \mathcal{D}^*(\alpha(1 - \mathcal{F}(\check{\zeta}))))\{\check{\zeta}^{\mathcal{B}}\mathcal{V}(\check{\zeta}) + [(\beta + \bar{\beta}\check{\zeta})\mathcal{D}_v^*(\mathcal{S}_v(\check{\zeta})) - 1] \\
& [\mathcal{H}^*(\alpha) + \mathcal{F}(\check{\zeta})[1 - \mathcal{H}^*(\alpha)]]\}
\end{aligned}$$

where $\mathcal{Q}_0$ is denoted by eqn. (27). **Proof.** The PGF of the no. of consumer in the system $(\mathcal{K}_s(\check{\zeta}))$ and in the orbit $(\mathcal{K}_0(\check{\zeta}))$ is calculated by using $\mathcal{K}_s(\check{\zeta}) = \mathcal{Q}_0 + \Upsilon(\check{\zeta}) + \Omega(\check{\zeta}) + \Psi_w(\check{\zeta})$. The eqns. (28) and (29) may be derived directly when the eqns. (24)-(27) are substituted in the earlier results. ■

4. System performance measures

In this section, different system states are used to derive a number of pertinent system probabilities, system efficiency metrics, and the model's mean busy time and mean busy cycle.

4.1. Probabilities of system states

Utilizing eqns, (24)-(26) we obtain the findings shown below, giving $\check{\zeta} \to 1$ and, if feasible, using L'Hospital's rule.
(i) Let Υ be the SS prob. of the server is available during the retrial,

$$\Upsilon = \Upsilon(1) = \mathcal{Q}_0(1 - \mathcal{H}^*(\alpha))\left\{\frac{\bar{\beta} + \alpha\mathcal{E}(\mathcal{I})[\mathcal{E}(\mathcal{D})\mathcal{D}_v^*(\omega) + \frac{1}{\omega}(1 - \mathcal{D}_v^*(\omega)) - \mathcal{E}(\mathcal{D}_v)]}{\mathcal{B} - \{\bar{\beta} - \alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D}) + \mathcal{E}(\mathcal{I})(1 - \mathcal{H}^*(\alpha))\}}\right\} \tag{30}$$

(ii) Let Ω be the SS prob. that the server is full,

$$\Omega = \Omega(1) = \alpha\mathcal{E}(\mathcal{D})\mathcal{Q}_0\left\{\frac{\mathcal{E}(\mathcal{I})(1 - \mathcal{D}_v^*(\omega))[\mathcal{H}^*(\alpha) + \frac{\alpha}{\omega}] + (\bar{\beta} - \mathcal{B})\mathcal{D}_v^*(\omega) + \mathcal{B} - 1}{\mathcal{B} - \{\bar{\beta} - \alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D}) + \mathcal{E}(\mathcal{I})(1 - \mathcal{H}^*(\alpha))\}}\right\} \tag{31}$$

(iii) Let Ψ_w be the SS prob. that the server is on WV,

$$\Psi_v = \Psi_v(1) = \frac{\alpha \mathcal{Q}_0}{\omega}[1 - \mathcal{D}_v^*(\omega)] \tag{32}$$

(iv) Let Υ_f be the SS prob. that the server is failure,

$$\Upsilon_f = \alpha \times \Omega(1) = \alpha^2 \mathcal{E}(\mathcal{D})\mathcal{Q}_0 \left\{ \frac{\mathcal{E}(\mathcal{I})(1-\mathcal{D}_v^*(\omega))[\mathcal{H}^*(\alpha) + \frac{\alpha}{\omega}] + (\bar{\beta} - \mathcal{B})\mathcal{D}_v^*(\omega) + \mathcal{B} - 1}{\mathcal{B} - \{\bar{\beta} - \alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D}) + \mathcal{E}(\mathcal{I})(1 - \mathcal{H}^*(\alpha))\}} \right\} \tag{33}$$

4.2. Mean size of a system and orbit

When the system is in a steady state,
(i) With respect to $\check{\zeta}$, (29) and providing $\check{\zeta} = 1$ yields the mean no. of consumers in the orbit $(\mathcal{L}_q)$

$$\mathcal{L}_q = \mathcal{K}_0'(1) = \lim_{\check{\zeta} \to 1} \frac{d}{d\check{\zeta}} \mathcal{K}_0(\check{\zeta}) = \mathcal{Q}_0 \left[\frac{\mathcal{N}_q'''(1)\mathcal{D}_q''(1) - \mathcal{D}_q'''(1)\mathcal{N}_q''(1)}{3(\mathcal{D}_q''(1))^2} \right] \tag{34}$$

$$\begin{aligned}
\mathcal{N}_q''(1) =& -2\alpha\mathcal{E}(\mathcal{I})\{[1 + \frac{\alpha}{\omega}(1 - \mathcal{D}_v^*(\omega))][\mathcal{B} - \bar{\beta} + \alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D}) - \mathcal{E}(\mathcal{I})(1 - \mathcal{H}^*(\alpha))] \\
&+ (1 - \mathcal{H}^*(\alpha))\{\bar{\beta} + \alpha\mathcal{E}(\mathcal{I})[\mathcal{E}(\mathcal{D})\mathcal{D}_v^*(\omega) + \frac{1}{\omega}(1 - \mathcal{D}_v^*(\omega)) - E(\mathcal{D}_v)]\} \\
&- \alpha\mathcal{E}(\mathcal{D})\{\mathcal{E}(\mathcal{I})(1 - \mathcal{D}_v^*(\omega))[\mathcal{H}^*(\alpha) + \frac{\alpha}{\omega}] + (\bar{\beta} - \mathcal{B})\mathcal{D}_v^*(\omega) + \mathcal{B} - 1\}\} \\
\mathcal{D}_q''(1) =& -2\alpha\mathcal{E}(\mathcal{I})\{\mathcal{B} + \bar{\beta} - \mathcal{E}(\mathcal{I})(1 - \mathcal{H}^*(\alpha)) + \alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D})\} \\
\mathcal{N}_q'''(1) =& -6\alpha\mathcal{E}(\mathcal{I})[\mathcal{B} - \bar{\beta} + \alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D}) - \mathcal{E}(\mathcal{I})(1 - \mathcal{H}^*(\alpha))]\{\frac{\alpha}{\omega}\mathcal{E}(\mathcal{I})(1 + \omega\mathcal{E}(\mathcal{D}) \\
&- \mathcal{D}_v^*(\omega))\} + \mathcal{D}_q'''(1)[1 + \frac{\alpha}{\omega}(1 - \mathcal{D}_v^*(\omega))] - 3\alpha\mathcal{E}(\mathcal{I})(1 - \mathcal{H}^*(\alpha)) \\
&\{\bar{\beta} + \alpha\mathcal{E}(\mathcal{I})(1 + \mathcal{B})[\frac{1}{\omega}(1 - \mathcal{D}_v^*(\omega)) + \mathcal{E}(\mathcal{D})\mathcal{D}_v^*(\omega) - \mathcal{E}(\mathcal{D}_v)] \\
&+ 2\bar{\beta}\{[\frac{\alpha}{\omega}\mathcal{E}(\mathcal{I})(1 + \omega\mathcal{E}(\mathcal{D}_v) - \mathcal{D}_v^*(\omega))] - \alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D})(1 - \mathcal{D}_v^*(\omega)) \\
&- \alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D}_v)\} - 2\alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D})[\frac{\alpha}{\omega}\mathcal{E}(\mathcal{I})(1 + \omega\mathcal{E}(\mathcal{D}) - \mathcal{D}_v^*(\omega))] + (1 - \mathcal{D}_v^*(\omega)) \\
&[\alpha^2\mathcal{E}(\mathcal{I})\mathcal{E}^2(\mathcal{D}) - \alpha\mathcal{E}(\mathcal{I}(\mathcal{I} - 1))\mathcal{E}(\mathcal{D})] - \alpha\mathcal{E}(\mathcal{I}(\mathcal{I} - 1))\mathcal{E}(\mathcal{D}_v) - \alpha^2\mathcal{E}(\mathcal{I})\mathcal{E}^2(\mathcal{D}_v) \\
&+ \mathcal{V}''(1) + 3\alpha\{\alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D})\{\mathcal{B}(\mathcal{B} - 1)(1 - \mathcal{D}_v^*(\omega)) + (\mathcal{B} + 1)[\frac{\alpha}{\omega}\mathcal{E}(\mathcal{I}) \\
&(1 + \omega\mathcal{E}(\mathcal{D}) - \mathcal{D}_v^*(\omega))] + 2\alpha\mathcal{E}(\mathcal{I})(1 - \mathcal{H}^*(\alpha))[\bar{\beta} - \alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D}_v)] \\
&- 2\bar{\beta}\alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D}_v) + \alpha^2\mathcal{E}(\mathcal{I})\mathcal{E}^2(\mathcal{D}_v) - \alpha\mathcal{E}(\mathcal{I}(\mathcal{I} - 1))\mathcal{E}(\mathcal{D}_v) + \mathcal{V}''(1)\} \\
&+ \alpha[\mathcal{E}(\mathcal{I}(\mathcal{I} - 1))\mathcal{E}(D) - \alpha\mathcal{E}(\mathcal{I})\mathcal{E}^2(\mathcal{D})]\{\bar{\beta} + \alpha\mathcal{E}(\mathcal{I})[\mathcal{E}(\mathcal{D})\mathcal{D}_v^*(\omega) \\
&+ \frac{1}{\omega}(1 - \mathcal{D}_v^*(\omega)) - \mathcal{E}(\mathcal{D}_v)]\}\}\} \\
\mathcal{D}_q'''(1) =& -3\alpha\mathcal{E}(\mathcal{I})\{\mathcal{B}(\mathcal{B} - 1) - \mathcal{E}(\mathcal{I}(\mathcal{I} - 1))(1 - \mathcal{H}^*(\alpha)) + 2\{\alpha\bar{\beta}\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D}) \\
&+ \alpha\mathcal{E}(\mathcal{I}(\mathcal{I} - 1))\mathcal{E}(\mathcal{D}) + \mathcal{E}(\mathcal{I})(1 - \mathcal{H}^*(\alpha))[\alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D}) + \bar{\beta}]\}\}
\end{aligned}$$

where $\mathcal{V}''(1) = \frac{\alpha}{\omega}\mathcal{E}(\mathcal{I}(\mathcal{I}-1))[1 + \omega\mathcal{E}(\mathcal{D}_v) - \mathcal{D}_v^*(\omega)] + \frac{\mathcal{E}(\mathcal{I})}{\omega^3}\{\omega^2\mathcal{E}^2(\mathcal{D}_v) - 2\alpha\omega\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D}_v) + \alpha\mathcal{E}(\mathcal{I})E(\mathcal{D}_v)\} + \alpha\mathcal{E}(\mathcal{I})(1 - \mathcal{D}_v^*(\omega))$
(ii) With regard to $\check{\zeta}$, (28) and providing $\check{\zeta} = 1$ yields the mean no. of consumers in the system (L_s)

$$\mathcal{L}_s = K_s'(1) = \lim_{\check{\zeta} \to 1} \frac{d}{d\check{\zeta}} \mathcal{K}_s(\check{\zeta}) = \mathcal{Q}_0 \left[\frac{\mathcal{N}_s'''(1)\mathcal{D}_q''(1) - \mathcal{D}_q'''(1)\mathcal{N}_q''(1)}{3(\mathcal{D}_q''(1))^2} \right] \tag{35}$$

$$\mathcal{N}_s'''(1) = \mathcal{N}_q'''(1) + 6\alpha\mathcal{E}(\mathcal{I})\{\mathcal{E}(\mathcal{D})\{\mathcal{E}(\mathcal{I})(1-\mathcal{D}_v^*(\omega))[\mathcal{H}^*(\alpha)+\frac{\alpha}{\omega}] + (\bar{\beta}-\mathcal{B})\mathcal{D}_v^*(\omega) + \mathcal{B} - 1\} - \frac{\alpha}{\omega}[1-\mathcal{D}_v^*(\omega)]\{\mathcal{B} - \bar{\beta} + \alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D}) - (1-\mathcal{H}^*(\alpha))\}\}$$

(iii) The mean period of the consumers in the system ($\mathcal{W}_s$) and the mean period of the consumers in the queue ($\mathcal{W}_q$) are estimated utilizing Little's method. $\mathcal{W}_s = \frac{\mathcal{L}_s}{\alpha\mathcal{E}(\mathcal{I})}$ and $\mathcal{W}_q = \frac{L_q}{\alpha\mathcal{E}(\mathcal{I})}$, respectively.

4.3. Mean busy period and the busy cycle

Under SS circumstances, let the projected lengths of the busy period and busy cycle be $\mathcal{A}(\mathcal{T}_y)$ and $\mathcal{A}(\mathcal{T}_{\check{\zeta}})$, respectively. The conclusions are directly obtained from the analysis of an alternate renewal process [6], which leads to

$$\mathcal{Q}_0 = \frac{A(\mathcal{T}_0)}{\mathcal{A}(\mathcal{T}_y) + \mathcal{A}(\mathcal{T}_0)}; \mathcal{A}(\mathcal{T}_y) = \frac{1}{\alpha}\left(\frac{1}{\mathcal{Q}_0} - 1\right); \mathcal{A}(\mathcal{T}_{\check{\zeta}}) = \frac{1}{\alpha\mathcal{Q}_0} = \mathcal{A}(\mathcal{T}_0) + \mathcal{A}(\mathcal{T}_y). \tag{36}$$

where $\mathcal{T}_0$ amount of time the system was in its empty condition. As the duration between the arrivals of two consumers differs exponentially. We have the equation $\mathcal{A}(\mathcal{T}_0) = (1/\alpha)$. with variable α. We may recover (27) by applying (36) the previously discovered results,

$$\mathcal{A}(\mathcal{T}_y) = \frac{1}{\alpha} \times \left\{ \frac{\begin{array}{r} \alpha\mathcal{E}(\mathcal{I})E(\mathcal{D})\{\frac{2\alpha}{\omega}(1-\mathcal{D}_v^*(\omega)) - 2\mathcal{D}_v^*(\omega)\mathcal{H}^*(\alpha) + \mathcal{D}_v^*(\omega) + \mathcal{H}^*(\alpha) + 1\} \\ -\mathcal{E}(I)(1-\mathcal{H}^*(\alpha))[1+\alpha\mathcal{E}(\mathcal{D}_v)] - \alpha\mathcal{E}(\mathcal{D})[1+\mathcal{B}(1-\mathcal{D}_v^*(\omega))] \\ +\mathcal{B}(1+\frac{\alpha}{\omega}(1-\mathcal{D}_v^*(\omega))) \end{array}}{\mathcal{B} - \{\bar{\beta} - \alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D}) + \mathcal{E}(\mathcal{I})(1-\mathcal{H}^*(\alpha))\}} - 1 \right\} \tag{37}$$

$$\mathcal{A}(\mathcal{T}_{\check{z}}) = \frac{1}{\alpha} \times \left\{ \frac{\begin{array}{r} \alpha\mathcal{E}(\mathcal{I})E(\mathcal{D})\{\frac{2\alpha}{\omega}(1-\mathcal{D}_v^*(\omega)) - 2\mathcal{D}_v^*(\omega)\mathcal{H}^*(\alpha) + \mathcal{D}_v^*(\omega) + \mathcal{H}^*(\alpha) + 1\} \\ -\mathcal{E}(I)(1-\mathcal{H}^*(\alpha))[1+\alpha\mathcal{E}(\mathcal{D}_v)] - \alpha\mathcal{E}(\mathcal{D})[1+\mathcal{B}(1-\mathcal{D}_v^*(\omega))] \\ +\mathcal{B}(1+\frac{\alpha}{\omega}(1-\mathcal{D}_v^*(\omega))) \end{array}}{\mathcal{B} - \{\bar{\beta} - \alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D}) + \mathcal{E}(\mathcal{I})(1-\mathcal{H}^*(\alpha))\}} \right\} \tag{38}$$

5. Stochastic Decomposition and Special cases

Here, the stochastic decomposition aspect of the system size distribution is examined. In $M/G/1$ queueing models with server vacations, stochastic decomposition has been extensively explored by Fuhrman and Cooper [5]. The no. of consumers in the system at SS at a random point in period is distributed as the sum of two independent RVs, one of which is the no. of consumers in the corresponding standard QS at a random point in time without vacations, and the other of which may have different probabilistic interpretations depending on the scheduling of the vacations. Furthermore, it has been discovered by Krishnakumar and Arivudainambi[11] that stochastic decomposition is valid for a no. of $M/G/1$ RQ models.

Theorem 3. The PGF of no.of consumers in the system $(\mathcal{K}_s(\check{\zeta}))$ can be executed as convolution of two independent RVs like, i.e.,$\mathcal{K}_s(\check{\zeta}) = \mathcal{M}_a(\check{\zeta}).\mathcal{M}_b(\check{\zeta})$,
(i) The PGF of no.of consumers $\psi(\check{\zeta})$ in the $M^{[X]}/G^{\mathcal{B}}/1$ feedback RQ with variable server capacity under WV policy.
(ii) The PGF of no.of consumers in the orbit given that the server is idle $\mathcal{M}_b(\check{\zeta})$. **Proof.** The PGF

of the system length can be decomposed as derived:
The stochastic decomposition law formulation is $K_s(\check{\zeta}) = \mathcal{M}_a(\check{\zeta}).\mathcal{M}_b(\check{\zeta})$,
(i) The system size distribution of the $M^{[X]}/G^{\mathcal{B}}/1$ feedback RQ with variable server capacity under WV is $\psi(\check{\zeta})$ and its distribution can be assigned by $\mathcal{H}^*(\alpha) \to 1$ in the eqn., (28).

$$\psi(\check{\zeta}) = \mathcal{Q}_0 \left\{ \frac{\begin{array}{r}(1-\mathcal{F}(\check{\zeta}))\{\check{\zeta}^{\mathcal{B}} - (\beta+\bar{\beta}\check{\zeta})\mathcal{D}^*(\mathcal{S}(\check{\zeta}))\}[1+\frac{\alpha}{\omega}\check{\zeta}\mathcal{V}(\check{\zeta})] + \check{\zeta}(1-\mathcal{D}^*(\alpha(1-\mathcal{F}(\check{\zeta}))))\\ \{\check{\zeta}^{\mathcal{B}}\mathcal{V}(\check{\zeta}) + [(\beta+\bar{\beta}\check{\zeta})\mathcal{D}^*_v(\mathcal{S}_v(\check{\zeta}))-1]\}\end{array}}{(1-\mathcal{F}(\check{\zeta}))\{\check{\zeta}^{\mathcal{B}} - (\beta+\bar{\beta}\check{\zeta})\mathcal{D}^*(\mathcal{S}(\check{\zeta}))\}} \right\}$$

(ii) The conditional distribution of the no.of consumers in the system at random point in period given the server is empty $\mathcal{M}_b(\check{\zeta})$.

$$\mathcal{M}_b(\check{\zeta}) = \frac{\mathcal{Q}_0 + \mathcal{Q}(\check{\zeta}) + \Psi_v(\check{\zeta})}{\mathcal{Q}_0 + \mathcal{Q}(1) + \Psi_v(1)}$$

$$\begin{aligned}\mathcal{M}_b(\check{\zeta}) = &\left\{ \frac{\{\mathcal{B} - \{\bar{\beta} - \alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D}) + \mathcal{E}(\mathcal{I})(1-\mathcal{H}^*(\alpha))\}}{\begin{array}{l}De(\check{\zeta})\{\mathcal{B} - \{\bar{\beta} - \alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D}) + \mathcal{E}(\mathcal{I})(1-\mathcal{H}^*(\alpha))\} + (1-\mathcal{H}^*(\alpha))\\ \quad\{\bar{\beta} + \alpha\mathcal{E}(\mathcal{I})[\mathcal{E}(\mathcal{D})\mathcal{D}^*_v(\omega) + \frac{1}{\omega}(1-\mathcal{D}^*_v(\omega)) - \mathcal{E}(\mathcal{D}_v)]\} + \frac{\alpha}{\omega}(1-\mathcal{D}^*_v(\omega))\}\end{array}} \right\}\\ &\times\{[\check{\zeta}^{\mathcal{B}} - (\beta+\bar{\beta}\check{\zeta})\{\mathcal{H}^*(\alpha) + \mathcal{F}(\check{\zeta})[1-\mathcal{H}^*(\alpha)]\}\mathcal{D}^*(\mathcal{S}(\check{\zeta}))] + \check{\zeta}^{\mathcal{B}}\mathcal{Q}_0(1-\mathcal{H}^*(\alpha))\\ &\qquad\{(\beta+\bar{\beta}\check{\zeta})[\mathcal{D}^*(\mathcal{S}(\check{\zeta}))\mathcal{V}(\check{\zeta}) + \mathcal{D}^*_v(\mathcal{S}_v(z))] - 1\} + \frac{\alpha}{\omega}\mathcal{V}(\check{\zeta})\}\end{aligned}$$

From the aforementioned stochastic decomposition law, we see that $\mathcal{K}_s(\check{\zeta}) = \mathcal{M}_a(\check{\zeta}).\mathcal{M}_b(\check{\zeta})$, which is consistent with the decomposition results of Geo et al. [6], are also applicable for this particular vacation system. ■

5.1. Special cases

In this section, we examine a few real-world examples of our strategy that are consistent with recent literature.

Case (i):
Let $\mathcal{P}r[\mathcal{F} = 1] = 1$, $\mathcal{B} = 1$, $\omega, \bar{\beta} = 0$ and $\mathcal{H}^*(\alpha) \to 1$. Our model can be simplified to a $M/G/1$ queue. The results agree with Takagi [21].

$$\mathcal{K}_s(\check{\zeta}) = \mathcal{Q}_0 \left\{ \frac{\mathcal{N}e_s(\check{\zeta})}{\mathcal{D}e_s(\check{\zeta})} \right\} \tag{39}$$

$$\begin{aligned}\mathcal{N}e_s(\check{\zeta}) &= (1-\check{\zeta})\{\check{\zeta} - \mathcal{D}^*(\alpha(1-\check{\zeta}))\} + \check{\zeta}(1-\mathcal{D}^*(\alpha(1-\check{\zeta})))\{\mathcal{D}^*_v(\alpha(1-\check{\zeta}))\}\\ \mathcal{D}e_s(\check{\zeta}) &= (1-\check{\zeta})\{\check{\zeta} - \mathcal{D}^*(\alpha(1-\check{\zeta}))\}\\ \text{where,}\quad \mathcal{Q}_0 &= \frac{1+\alpha\mathcal{E}(I)\mathcal{E}(\mathcal{D})}{\alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D}) - \alpha\mathcal{E}(\mathcal{D})}\end{aligned}$$

Case (ii):
Let $\mathcal{P}r[\mathcal{F} = 1] = 1$, $\mathcal{B} = 1$, and $\omega, \bar{\beta} = 0$. Our model simplified to an $M/G/1$ RQ. Here are the results agree with Gao and Wang [6].

$$\mathcal{K}_s(\check{z}) = \mathcal{Q}_0 \left\{ \frac{\mathcal{N}e_s(\check{z})}{\mathcal{D}e_s(\check{z})} \right\} \tag{40}$$

$$
\begin{aligned}
\mathcal{N}e_s(\check{z}) =&(1-\check{z})\{\check{z}-[\mathcal{H}^*(\alpha)+\check{z}(1-\mathcal{H}^*(\alpha))]\mathcal{D}^*(\alpha(1-\check{z}))\}+\check{z}\alpha(1-\check{z})[1-\mathcal{H}^*(\alpha)]\\
&(\mathcal{D}_v^*(\alpha(1-\check{z})-1))+\alpha\check{z}[1-\mathcal{D}^*(\alpha(1-\check{z}))]\{\mathcal{D}_v^*(\alpha(1-\check{z})-1)\\
&[\mathcal{H}^*(\alpha)+\check{z}(1-\mathcal{H}^*(\alpha))]\}\\
\mathcal{D}e_s(\check{z}) =&\alpha(1-\check{z})\{\check{z}-[\mathcal{H}^*(\alpha)+\check{z}(1-\mathcal{H}^*(\alpha))]\mathcal{D}^*(\alpha(1-\check{z}))\}\\
\text{where,}\quad \mathcal{Q}_0 =&\frac{1+\alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D})+\mathcal{E}(\mathcal{I})(1-\mathcal{H}^*(\alpha))}{\alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D})\{1-\mathcal{H}^*(\alpha)\}-\alpha\mathcal{E}(\mathcal{D})-\mathcal{E}(\mathcal{I})(1-\mathcal{H}^*(\alpha))[1+\alpha\mathcal{E}(\mathcal{D}_v)]+\mathcal{B}}
\end{aligned}
$$

Case (iii):
Let $\mathcal{P}r[\mathcal{F}=1]=1$, $\mathcal{B}=1$, and $\bar{\beta}=0$. our model simplified to an $M/G/1$ queue with WVs. Here are the results agree with Zhang and Hou [23].

$$
\mathcal{K}_s(\check{z}) = \mathcal{Q}_0\left\{\frac{\mathcal{N}e_s(\check{z})}{\mathcal{D}e_s(\check{z})}\right\} \tag{41}
$$

$$
\begin{aligned}
\mathcal{N}e_s(\check{z}) =&(1-\check{z})\{\check{z}-[\mathcal{H}^*(\alpha)+\check{z}(1-\mathcal{H}^*(\alpha))]\mathcal{D}^*(\alpha(1-\check{z}))\}+\check{z}\alpha(1-\check{z})[1-\mathcal{H}^*(\alpha)]\\
&(\mathcal{D}_v^*(\alpha(1-\check{z})-1))+\alpha\check{z}[1-\mathcal{D}^*(\alpha(1-\check{z}))]\{\mathcal{D}_v^*(\alpha(1-\check{z})-1)\\
&[\mathcal{H}^*(\alpha)+\check{z}(1-\mathcal{H}^*(\alpha))]\}\\
\mathcal{D}e_s(\check{z}) =&\alpha(1-\check{z})\{\check{z}-[\mathcal{H}^*(\alpha)+\check{z}(1-\mathcal{H}^*(\alpha))]\mathcal{D}^*(\alpha(1-\check{z}))\}
\end{aligned}
$$

6. Numerical results

The various effects on system performance measurements are demonstrated using MATLAB in this section. We examine exponentially distributed retrial times, service times, and slower service times. The numerical measurements that satisfy the stability condition are chosen at random.

Table 2 clearly displays that arrival rate (α) escalates, $\mathcal{L}_q$, $\mathcal{L}_s$, Ψ_v are increases. Table 3 displays that feedback rate β escalates, $\mathcal{L}_q$, $\mathcal{L}_s$, are increases and Q_0 decreases. Table 4 displays that lower service rate η_v escalates, $\mathcal{L}_q$, $\mathcal{L}_s$, Ψ_v and $\mathcal{Q}_0$ decreases.

With the impact of the parameters B, α, β, ω, $\chi(\varpi)$, $\eta(\varpi)$, $\eta_v(\varpi)$, Fig. 1 illustrate the

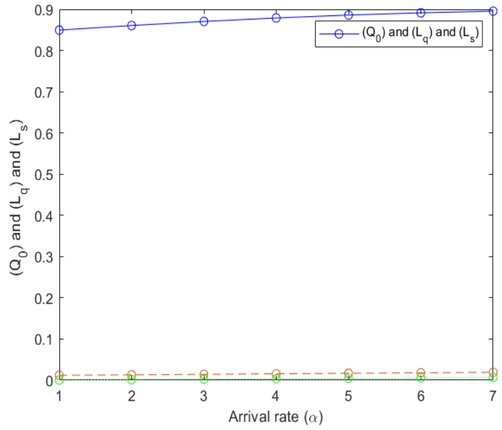
Table 2: *$\mathcal{Q}_0$ and $\mathcal{L}_q$ for different arrival rate (α) for the values of $\mathcal{B}=30$, $\beta=0.5$, $\omega=2$, $\chi(\varpi)=6$, $\eta(\varpi)=0.6$, $\eta_v(\varpi)=0.7$*

Arrival rate (α)	$\mathcal{Q}_0$	$\mathcal{L}_q$	$\mathcal{L}_s$	Ψ_v	$\mathcal{W}_q$
1	0.8495	0.0120	0.0005	0.1274	0.0840
2	0.8606	0.0131	0.0015	0.2582	0.0459
3	0.8704	0.0142	0.0026	0.3917	0.3341
4	0.8789	0.0156	0.0036	0.5273	0.0272
5	0.8859	0.0168	0.0047	0.6644	0.0235
6	0.8914	0.0180	0.0058	0.8023	0.0210
7	0.8954	0.0192	0.0068	0.9402	0.0192

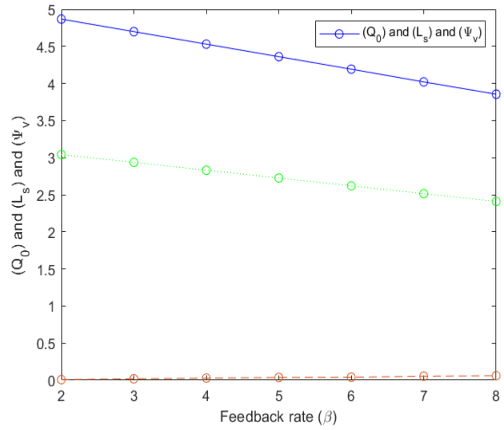
two-dimensional plot that depict the system performance measures. In Fig. $1(a)$, displays the escalation of the arrival rate (α), $(\mathcal{L}_q)$ and $(\mathcal{W}_q)$ increases. In Fig. $1(b)$, we found that $(\mathcal{L}_s)$ increases while diminishing the feedback rate β and (Ψ_v).

The three-dimensional graph representing the system performance metrics is shown in Fig. 2. In Fig. $2(a)$, the surface displays the elevation the (b), we found that $(\mathcal{W}_q)$ diminishes while increasing the feedback rate (β), $(\mathcal{L}_s)$. In Fig. $2(c)$, we found that $(\mathcal{Q}_0)$ and (Ψ_v) diminishes while increasing the lower service rate η_v.

The numerical findings above may be used to determine the impact of attributes on the system's assessment criteria, and we can be sure that the results are representative of actual conditions.

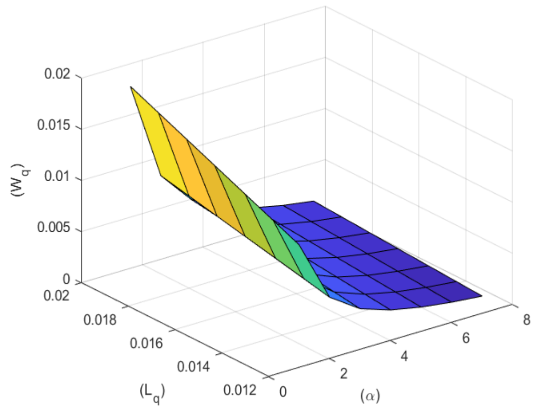


(a) $\mathcal{Q}_0$, $\mathcal{L}_q$, $\mathcal{W}_q$ vs arrival rate α

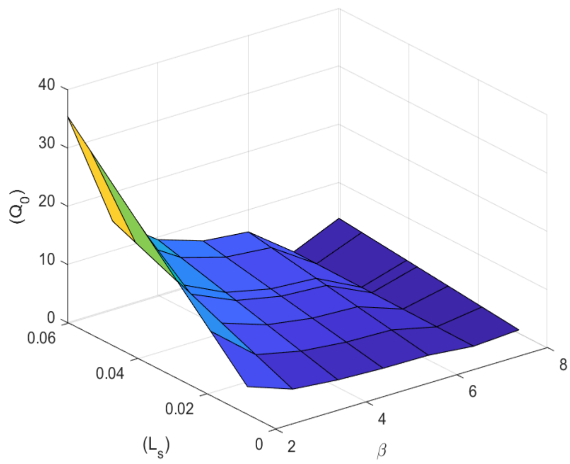


(b) $\mathcal{Q}_0$, $\mathcal{L}_s$, Ψ_v vs feedback rate β

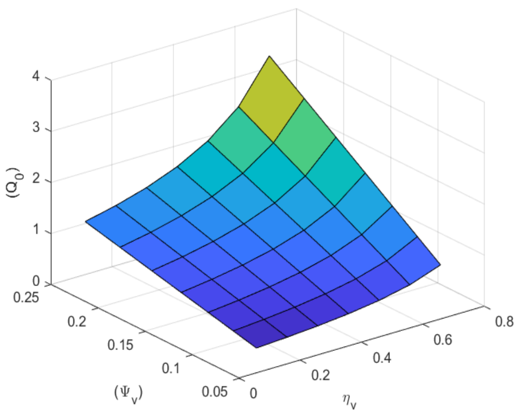
Figure 1: *2D visualization of α and β*



(a) $\mathcal{L}_q$, $\mathcal{W}_q$ vs arrival rate α



(b) $\mathcal{Q}_0$, $\mathcal{L}_s$ vs feedback rate β



(c) $\mathcal{Q}_0$, Ψ_v vs lower service rate η_v

Figure 2: *3D visualization of α, β, and η_v*

Table 3: *$\mathcal{Q}_0$ and $\mathcal{L}_q$ for different arrival rate (α) for the values of $\mathcal{B} = 30$, $\alpha = 5$, $\omega = 4$, $\chi(\varpi) = 6$, $\eta(\varpi) = 0.6$, $\eta_v(\varpi) = 0.5$*

Feedback rate (β)	$\mathcal{Q}_0$	$\mathcal{L}_q$	$\mathcal{L}_s$	Ψ_v	$\mathcal{W}_q$
2	4.8675	0.0205	0.0082	3.0421	0.0286
3	4.6988	0.0322	0.0189	2.9367	0.0450
4	4.5301	0.0431	0.0287	2.8313	0.0603
5	4.3614	0.0531	0.0377	2.7259	0.0743
6	4.1928	0.0623	0.0407	2.6205	0.0872
7	4.0209	0.0706	0.0536	2.5150	0.0989
8	3.8554	0.0781	0.0603	2.4096	0.0109

Table 4: *$\mathcal{Q}_0$ and $\mathcal{L}_q$ for different lower service rate (η_v) for the values of $\mathcal{B} = 30$, $\alpha = 1$, $\omega = 4$, $\chi(\varpi) = 4$, $\eta(\varpi) = 0.6$, $\beta = 0.7$*

Lower service rate (η_v)	$\mathcal{Q}_0$	$\mathcal{L}_q$	$\mathcal{L}_s$	Ψ_v	$\mathcal{W}_q$
0.1	1.0937	0.0430	0.0030	0.2461	0.0998
0.2	1.0457	0.0138	0.0026	0.2091	0.0968
0.3	1.0018	0.0134	0.0021	0.1753	0.0941
0.4	0.9614	0.0131	0.0018	0.1442	0.0916
0.5	0.9242	0.0128	0.0015	0.1155	0.0893
0.6	0.8897	0.0124	0.0012	0.0889	0.0871
0.7	0.8577	0.0122	0.0009	0.0643	0.0851

7. Conclusion

We examined the $M^{[X]}/G^{\mathcal{B}}/1$ feedback retrial queueing system with variable server capacity under working vacation in this article. If all essential and appropriate conditions are met, the system can be stabilized. When it is ideal, normally busy, and on lower rate service, the PGF of the no. of system consumers and its orbit are calculated using the PGF approach and the supplementary variable technique. Eventually, a wide range of numerical findings are presented to examine the impact of system parameters. The results of this study may be used in the design of different computer communication systems, packet switching networks, manufacturing lines, and postal systems by network and software engineers. Lastly, a study of a bulk service queueing system with priority consumers under working vacation could enhance this work. Additionally, it may be worthwhile to investigate in the future of transient solution for bulk service under a working vacation.

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Appendix A

Theorem 4. The embedded Markov chain $\{\mathcal{G}_m; m\epsilon M\}$ is ergodic iff $\Lambda < \mathcal{B}$ for our system will be stable, where $\Lambda = \bar{\beta} - \alpha\mathcal{E}(\mathcal{I})E(\mathcal{D}) + E(\mathcal{I})(1 - \mathcal{H}^*(\alpha))$.

Proof. Foster's [15] criteria, which claim that the chain $\{\mathcal{G}_m; m \in M\}$ is an irreducible and aperiodic chain, may be used to easily confirm the required condition of ergodicity. Assuming a non-negative measure $e(r)$, $r \in M$ and $\delta > 0$, the Markov chain is ergodic, and mean drift $\nu_r = \mathcal{E}[e(u_{m+1}) - e(u_m)/v_m = r]$ with a limited exception r's, $r \in M$ and $\nu_r \leq -\delta \ \forall \ r \in M$,. In this case, we're focusing on the function $e(r) = r$. Next, we obtain

$$\nu_r = \begin{cases} \bar{\beta} - \alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D}) - \mathcal{B}, & \text{if } r\text{=0} \\ \bar{\beta} - \alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D}) + \mathcal{E}(\mathcal{I})(1 - \mathcal{H}^*(\alpha)) - \mathcal{B}, & \text{if } r\text{=1,2,...} \end{cases}$$

In this case, $\bar{\beta} - \alpha\mathcal{E}(\mathcal{I})\mathcal{E}(\mathcal{D}) + \mathcal{E}(\mathcal{I})(1 - \mathcal{H}^*(\alpha)) < \mathcal{B}$ is undoubtedly a prerequisite for ergodicity.

As said by Humblett et al. [8], if the Markov chain $\{\mathcal{G}_m; m\epsilon M\}$ matches Kaplan's status, specifically $\nu_r < \infty \ \forall \ r \geq 0$ and $\exists \ r_0 \in M$ such that $\nu_r \geq 0$ for $r \geq r_0$, the necessary condition is satisfied. $\mathcal{V} = (v_{qr})$ is the the unit-step transition matrix of $\{\mathcal{G}_m; m \in M\}$ for $r < q - j$ and $q > 0$. The Markov chain's non-ergodicity is suggested by $\Lambda \geq B$. ■

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