RT&A, No 3 (79)
Volume 19, September 2024

SINE-TOPP-LEONE EXPONENTIATED G FAMILY OF DISTRIBUTIONS: PROPERTIES, SURVIVAL REGRESSION AND APPLICATION

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Abstract

In this research article, we have introduced a new class of continuous probability distributions known as the Sine Topp-Leone Exponentiated-G family of distributions. This newly proposed family exhibits a higher degree of flexibility compared to some of the established distribution families. The various models within this family find wide-ranging applications in fields such as physics, engineering, and medicine. Some statistical properties of the Sine Topp-Leone Exponentiated-G family of distributions such as moments, moment generating function, quantile function and order statistics are derived. Two special models were also presented and studies. Maximum likelihood estimation method was used to estimate parameters of the models. The consistency of the proposed family was determine using simulation studies. Two real life datasets were analyzed to show the flexibility of the proposed model and the results of the analysis showed that, the proposed model was more efficient and best fit the data sets than its competitors.

Keywords: Sine-G Family, Topp-Leone Exponentiated G, Survival Analysis, Survival Regression, Maximum Likelihood Estimate

1. Introduction

The Topp-Leone Distribution is a statistical concept that finds its roots in probability theory and data analysis. It was developed by [1] is 2015 and it is a powerful tool in the field of statistics for modeling and understanding random variables with various applications across different domains. It is a relatively recent addition to the family of probability distributions in statistics and has gained prominence for its adaptability in modeling various types of data with flexibility and precision. This distribution offers a valuable tool for statisticians, data scientists, and researchers in diverse fields, enabling them to capture the underlying characteristics of data sets that may not conform to traditional distribution assumptions. The Topp Leone Distribution is particularly well-suited for modeling data with heavy tails, which means it can effectively describe observations that exhibit extreme values or outliers. This characteristic is especially important in fields like finance, where extreme events can have significant consequences, and in environmental science, where rare but impactful events need to be accounted for in risk assessment.

Recently, some researchers have developed numerous families of Topp-Leone Distribution which include: Topp-Leone G Family of Distribution by [2], Topp-leone Marshal Olkin G by [3], Transmuted Topp-Leone G by [4], Topp-Leone Exponentiated G Family by [5], Topp-Leone Odd Lindley G by [6], Odd Log Logistic Topp-Leone by [7], Frechet Topp-Leone G Family by [8], Topp-Leone Odd Log Logistic Family by [9], Type II generalized Topp-Leone G Family by [10], Type II Exponentiated Half-Logistic Topp-Leone G by [11], Topp-Leone Marshal Olkin G by [12], Type II Topp-Leone G by [13], the Weibull Topp-Leone G by [14], Odd Weibull Topp-Leone G by [15], Topp Leone Odd Burr III G by [16], The Burr III Topp-Leone G by [17], Topp-Leone Gompertz G by [18], Topp-Leone Exponential G by [19] and Topp-Leone Generalized Half-Log Logistic G by [20].

In order hand, the Sine-G family of probability distributions is a class of continuous probability distributions that is often used in statistical modeling and data analysis. This family is characterized by its flexibility and ability to capture a wide range of data patterns, making it a valuable tool for statisticians and data scientists. The PDF of a Sine-G distribution is defined in terms of the sine function, which introduces oscillatory behavior into the distribution. This oscillatory behavior can be adjusted by varying the distribution's parameters, allowing it to fit data with different shapes and characteristics. One of the notable features of the Sine-G family is its ability to model data with heavy tails, which means it can effectively describe extreme or outlier values in a dataset. This makes it useful in fields such as finance, where extreme events can have a significant impact on investment portfolios and risk assessment. The Sine-G family is also capable of modeling data with various degrees of skewness and kurtosis, providing a versatile tool for capturing complex data patterns that may not conform to traditional distribution assumptions like the normal distribution.

Some of the recent development of the Sine G family include: Sine Topp-Leone G by Al-[21], the New Sine G Family by [22], the Sine Kumaraswamy G by [23], Exponentiated Sine G by [24], Transmuted Sine G by [25], Sine Marshall–Olkin G by [26] and Sine Inverse Lomax G by [27]. These developments of flexible families of distributions through innovative transformations, as mentioned in this research article, reflects the dynamic nature of statistical research. Such advancements hold promise for improving the accuracy and applicability of statistical models in diverse domains and addressing the complexities of real-world data.

2. Methods

2.1 The Sine-G Family of Probability Distribution

Let $h(x;\xi)$and H(x; ξ) be the pdf and cdf of a Univariate continuous distribution, then, the Sine-G family of probability distribution according to [28] is defined by:

$$F(x;\xi)=\int_0^{\frac{\pi}{2}H(x;\xi)}\cos t dt=\sin\left\{\frac{\pi}{2}H(x;\xi)\right\} \tag{1}$$

with corresponding pdf given by:

$$f(x;\xi)=\frac{\pi}{2}h(x;\xi)\cos\left\{\frac{\pi}{2}H(x;\xi)\right\} \tag{2}$$

where $H(x;\xi)$ and $h(x;\xi)$ are the cdf and the pdf of any baseline distribution with vector parameter ξ.

2.2 Topp-Leone Exponentiated-G Family of Distributions

The cdf of the Topp-Leone Exponentiated G family of distribution according to [4] is given by:

$$F(x;\alpha,\theta,\xi)=\left\{1-\left[1-G(x;\xi)^{\alpha}\right]^{2}\right\}^{\theta} \tag{3}$$

with corresponding pdf defined by:

$$f(x;\alpha,\theta,\xi)=2\alpha\theta g(x;\xi)G(x;\xi)^{\alpha-1}\left[1-G(x;\xi)^{\alpha}\right]\left\{1-\left[1-G(x;\xi)^{\alpha}\right]^{2}\right\}^{\theta-1} \tag{4}$$

where α is a shape parameter, $g(x;\xi)$ and $G(x;\xi)$ are pdf and cdf of any baseline distribution respectively and ξ is a vector parameter of the baseline distribution.

2.3 The Proposed Sine Topp-Leone Exponentiated G Family of Distributions

The cdf of the new Sine Topp-Leone Exponentiated G Family is given by:

$$F(x;\alpha,\theta,\xi)=\sin\left\{\frac{\pi}{2}\left[1-\left(1-G(x;\xi)^{\alpha}\right)^{2}\right]^{\theta}\right\} \tag{5}$$

with corresponding pdf given by:

$$f(x;\alpha,\theta,\xi)=\frac{\pi}{2}2\alpha\theta g(x;\xi)G(x;\xi)^{\alpha-1}\left[1-G(x;\xi)^{\alpha}\right]\left\{1-\left[1-G(x;\xi)^{\alpha}\right]^{2}\right\}^{\theta-1} \times\cos\left\{\frac{\pi}{2}\left[1-\left(1-G(x;\xi)^{\alpha}\right)^{2}\right]^{\theta}\right\} \tag{6}$$

The survival function S(x), hazard function h(x), reversed hazard function r(x) and the quantile functions Q(x) of the STLE-G are presented in equation (7) to (10).

$$S(x)=1-\sin\left\{\frac{\pi}{2}\left[1-\left(1-\left[1-\left(1+\frac{x}{\lambda}\right)^{-\gamma}\right]^{\alpha}\right)^{2}\right]^{\theta}\right\} \tag{7}$$

$$h(x)=\frac{\frac{\pi}{2}2\alpha\theta g(x;\xi)G(x;\xi)^{\alpha-1}\left[1-G(x;\xi)^{\alpha}\right]\left\{1-\left[1-G(x;\xi)^{\alpha}\right]^{2}\right\}^{\theta-1}\cos\left\{\frac{\pi}{2}\left[1-\left(1-G(x;\xi)^{\alpha}\right)^{2}\right]^{\theta}\right\}}{1-\sin\left\{\frac{\pi}{2}\left[1-\left(1-G(x;\xi)^{\alpha}\right)^{2}\right]^{\theta}\right\}} \tag{8}$$

$$r(x)=\frac{\frac{\pi}{2}2\alpha\theta g(x;\xi)G(x;\xi)^{\alpha-1}\left[1-G(x;\xi)^{\alpha}\right]\left\{1-\left[1-G(x;\xi)^{\alpha}\right]^{2}\right\}^{\theta-1}\cos\left\{\frac{\pi}{2}\left[1-\left(1-G(x;\xi)^{\alpha}\right)^{2}\right]^{\theta}\right\}}{\sin\left\{\frac{\pi}{2}\left[1-\left[1-G(x;\xi)^{\alpha}\right]^{2}\right]^{\theta}\right\}} \tag{9}$$

$$x=\Phi(u)=G^{-1}\left\{1-\left[1-\left(1-\frac{\sin^{-1}(u)}{\pi/2}\right)^{\frac{1}{\theta}}\right]^{\frac{1}{2}}\right\}^{\frac{1}{\alpha}} \tag{10}$$

where $G^{-1}(x,\xi)$ is the quantile function of the baseline distribution $G(x;\xi)$.

2.4 Expansion of Density

The pdf and the cdf of the Sine Topp-Leone Exponentiated G family can be expanded using power series expansion as follows:

$$f(x;\alpha,\theta,\xi)=\frac{\pi}{2}2\alpha\theta g(x;\xi)G(x;\xi)^{\alpha-1}\left[1-G(x;\xi)^{\alpha}\right]\left\{1-\left[1-G(x;\xi)^{\alpha}\right]^{2}\right\}^{\theta-1}\cos\left\{\frac{\pi}{2}\left[1-\left(1-G(x;\xi)^{\alpha}\right)^{2}\right]^{\theta}\right\}$$

$$\cos\left\{\frac{\pi}{2}\left[1-\left[1-G(x)^{\alpha}\right]^{2}\right]^{\theta}\right\}=\sum_{i=0}^{\infty}\frac{(-1)^{i}}{(2i)!}\frac{\pi^{2i}}{2^{2i}}\left[1-\left[1-G(x)^{\alpha}\right]^{2}\right]^{\theta i}$$

$$\left[1-\left[1-G(x)^{\alpha}\right]^{2}\right]^{\theta i+\theta-1}=\sum_{i=0}^{\infty}(-1)^{j}\binom{\theta i+\theta-1}{j}\left[1-G(x)^{\alpha}\right]^{2j}$$

$$\left[1-G(x)^{\alpha}\right]^{2j+1}=\sum_{i=0}^{\infty}(-1)^{k}\binom{2j+1}{k}G(x)^{\alpha}$$

$$f(x)=\sum_{i,j,k=0}^{\infty}\frac{2\alpha\theta(-1)^{i+j+k}}{(2i)!}\frac{\pi^{2i+1}}{2^{2i+1}}\binom{\theta i+\theta-1}{j}\binom{2j+1}{k}g(x)G(x)^{\alpha(k+1)-1}$$

Let $\Psi=\frac{2\alpha\theta(-1)^{i+j+k}}{(2i)!}\frac{\pi^{2i+1}}{2^{2i+1}}\binom{\theta i+\theta-1}{j}\binom{2j+1}{k}$

Therefore, the reduced form of the pdf is given by:

$$f(x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Psi g(x)G(x)^{\alpha(k+1)-1} \tag{11}$$

The cdf can also be expanded as follows:

$$\sin\left\{\frac{\pi}{2}\left[1-\left[1-G(x)^{\alpha}\right]^{2}\right]^{\theta}\right\}=\sum_{l=0}^{\infty}\frac{(-1)^{l}}{(2l+1)!}\frac{\pi^{2l+1}}{2^{2l+1}}\left[1-\left[1-G(x)^{\alpha}\right]^{2}\right]^{2l+1}$$

$$\left[1-\left[1-G(x)^{\alpha}\right]^{2}\right]^{2l+1}=\sum_{i=0}^{\infty}(-1)^{m}\binom{2l+1}{m}\left[1-G(x)^{\alpha}\right]^{2m}$$

$$\left[1-G(x)^{\alpha}\right]^{2m}=\sum_{n=0}^{\infty}(-1)^{n}\binom{2m}{n}G(x)^{\alpha n}$$

$$F(x)=\sum_{l,m,n=0}^{\infty}\frac{(-1)^{l+m+n}}{(2l+1)!}\frac{\pi^{2l+1}}{2^{2l+1}}\binom{2l+1}{m}\binom{2m}{n}G(x)^{\alpha n}$$

Let $\Phi=\frac{(-1)^{l+m+n}}{(2l+1)!}\frac{\pi^{2l+1}}{2^{2l+1}}\binom{2l+1}{m}\binom{2m}{n}$

Therefore, the reduced form of the cdf is given as:

$$F(x)=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Phi G(x)^{\alpha n} \tag{12}$$

2.5 Mathematical Properties

2.5.1 The Moment

Moments is used to study many important properties of distribution such as dispersion, tendency, skewness and kurtosis. The r^{th} moments of the Sine Type II Topp Leone G family of distribution is obtained as follow:

$$\mu_r^{'} = \int_{-\infty}^{\infty} x^r f(x)dx$$

Therefore, the moment of the Sine Topp-Leone Exponentiated G is obtained as follows:

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \Psi \int_0^{\infty} g(x)G(x)^{\alpha(k+1)-1}dx$$

Let $\Theta = \int_0^{\infty} g(x)G(x)^{\alpha(k+1)-1}dx$,

Therefore, the rth moment of the Sine Topp-Leone Exponentiated is given by:

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \Psi\Theta \tag{13}$$

2.5.2 Moment Generating Function

The moment generating function of a random variable X is defined as $E(e^{tx})$.

$$M_x(t) = E\left(e^{tx}\right) = \int_{-\infty}^{\infty} e^{tx} f(x;\alpha,\xi)dx$$

We say that mgf of X exists, if there exists a positive constant a such that $M_t(x)$ is finite for all s $\in [-a, a]$. The moment generating function of the Sine Topp-Leone Exponentiated G family of Distributions is given by:

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} g(x)G(x)^{\alpha(k+1)-1}dx$$

$$M_x(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \Psi \int_0^{\infty} e^{tx} g(x)G(x)^{\alpha(k+1)-1}dx$$

Let $\Upsilon = \int_0^{\infty} e^{tx} g(x)G(x)^{\alpha(k+1)-1}dx$

Therefore, the moment generating function of the Sine Topp-Leone Exponentiated G is given by:

$$M_x(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \Psi\Upsilon \tag{14}$$

2.5.3 Entropy

Entropy is used as a measure of information or uncertainty, which present in a random observation of its actual population. There will be the greater uncertainty in the data if the value of entropy is large. For some probability distributions expression, the differential entropy is considered mostly effective. It can be derived using the formula:

$$I_\theta(x) = \frac{1}{1-\theta} \log \int_{-\infty}^{\infty} f(x)^\theta \, dx$$

The entropy for the Sine Topp-Leone Exponentiated G family of distributions is given by:

$$f(x)^\theta = \left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Psi g(x) G(x)^{\alpha(k+1)-1} \right)^\theta$$

$$f(x)^\theta = \left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Psi \right)^\theta \left(g(x) G(x)^{\alpha(k+1)-1} \right)^\theta$$

Let $\Omega = g(x) G(x)^{\alpha(k+1)-1}$

Therefore, the entropy is given as:

$$I_\theta(x) = \frac{1}{1-\theta} \left[\left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Psi \right)^\theta \int_0^\infty \Omega^\theta \, dx \right] \tag{15}$$

2.5.6 Order Statistics

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from a continuous population having a pdf f(x) and cdf F(x), Let $X_{1:n} \leq X \leq X_{n:n}$ be the corresponding order statistics (OS). [29] defined the pdf of $X_{1:n}$ that is the i^{th} order statistics by:

$$f_{p,q}(x) = \frac{f(x)}{B(p, q-p+1)} \left(F(x) \right)^{p-1} \left(1 - F(x) \right)^{q-p}$$

The order statistics of the Sine Topp-Leone Exponentiated G is given by:

$$f_{p,q}(x) = \frac{\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Psi g(x) G(x)^{\alpha(k+1)-1}}{B(p, q-p+1)} \left(\sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Phi G(x)^{\alpha n} \right)^{p-1} \left(1 - \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Phi G(x)^{\alpha n} \right)^{q-p} \tag{16}$$

2.6 Parameter Estimation

2.6.1 Maximum Likelihood Estimate

Let $x_1, x_2, x_3, \ldots, x_n$ be a random sample of size n from the Sine Type II Topp-Leone G family of distribution. Then, the likelihood function of the Sine Type II Topp-Leone G family is derived as follows:

$$\ell = n\log(\pi) + n\log(\alpha) + n\log(\theta) + \sum_{i=1}^{n}\log g(x;\alpha,\theta,\xi) + (\alpha-1)\sum_{i=1}^{n}\log G(x;\alpha,\theta,\xi) + \sum_{i=1}^{n}\log\left(1-G(x;\alpha,\theta,\xi)^{\alpha}\right)$$
$$(\theta-1)\sum_{i=0}^{n}\log\left[1-\left(1-G(x;\alpha,\theta,\xi)^{\alpha}\right)^{2}\right] + \sum_{i=0}^{n}\log\cos\left\{\frac{\pi}{2}\left[1-\left(1-G(x;\alpha,\theta,\xi)^{\alpha}\right)^{2}\right]^{\theta}\right\} \quad (17)$$

Differentiating the likelihood function in equation (17) with respect to α gives the following expression:

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log G(x) - \sum_{i=1}^{n}\frac{G(x)^{\alpha}\ln G(x)}{\left[1-G(x)^{\alpha}\right]} + \sum_{i=1}^{n}\pi\theta\left(1-G(x)^{\alpha}\right)\left[1-\left[1-G(x)^{\alpha}\right]^{2}\right]^{\theta-1}$$
$$\times G(x)^{\alpha}\ln G(x)\tan\left\{\frac{\pi}{2}\left[1-\left[1-G(x)^{\alpha}\right]^{2}\right]^{\theta}\right\} \quad (18)$$

Differentiating the likelihood function in equation (17) with respect to θ gives:

$$\frac{\partial\ell}{\partial\theta} = \frac{n}{\theta} + \sum_{i=0}^{n}\log\left[1-\left(1-G(x)^{\alpha}\right)^{2}\right] - \sum_{i=0}^{n}\frac{\pi}{2}\tan\left\{\frac{\pi}{2}\left[1-\left(1-G(x)^{\alpha}\right)^{2}\right]^{\theta}\right\}\left[1-\left(1-G(x)^{\alpha}\right)^{2}\right]^{\theta}\ln\left[1-\left(1-G(x)^{\alpha}\right)^{2}\right] \quad (19)$$

Differentiating the likelihood function in equation (17) with respect to ξ gives:

$$\frac{\partial\ell}{\partial\xi} = \sum_{i=0}^{n}\frac{g'(x)}{g(x)} + (\alpha-1)\sum_{i=0}^{n}\frac{g(x)}{G(x)} - \sum_{i=0}^{n}\frac{\alpha g(x)G(x)^{\alpha-1}}{\left(1-G(x)^{\alpha}\right)} + (\theta-1)\sum_{i=0}^{n}\frac{2\alpha g(x)G(x)^{\alpha-1}\left(1-G(x)^{\alpha}\right)}{\left[1-\left(1-G(x)^{\alpha}\right)^{2}\right]}$$
$$-\sum_{i=0}^{n}\pi\alpha\theta g(x)G(x)^{\alpha-1}\left(1-G(x)^{\alpha}\right)\left[1-\left(1-G(x)^{\alpha}\right)^{2}\right]^{\theta-1}\tan\left\{\frac{\pi}{2}\left[1-\left(1-G(x)^{\alpha}\right)^{2}\right]^{\theta}\right\} \quad (20)$$

The expression in equation (18), (19) and (20) are the maximum likelihood estimates of the parameters α, θ and the vector parameter ξ.

2.7 Special Models of STLE-G Family

Here, we consider two special models of the STLE-G family along with the plots of their density and hazard rate function.

2.7.1 Sine Topp-Leone Exponentiated Lomax (STLE-L) Distribution

Let the Lomax distribution be the baseline distribution with cdf and pdf defined by:

$$F(x) = 1-\left(1+\frac{x}{\lambda}\right)^{-\gamma} \quad (21)$$

And

$$f(x)=\frac{\gamma}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\gamma+1)} \tag{22}$$

where γ is a shape parameter and λ is a scale parameter, the cdf and pdf of the STLE-L distribution are respectively given by:

$$F(x;\alpha,\theta,\gamma,\lambda)=\sin\left\{\frac{\pi}{2}\left[1-\left(1-\left[1-\left(1+\frac{x}{\lambda}\right)^{-\gamma}\right]^{\alpha}\right)^{2}\right]^{\theta}\right\} \tag{23}$$

$$f(x;\alpha,\theta,\gamma,\lambda)=\frac{\pi}{2}2\alpha\theta\left(\frac{\gamma}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\gamma+1)}\right)\left[1-\left(1+\frac{x}{\lambda}\right)^{-\gamma}\right]^{\alpha-1}\left[1-\left[1-\left(1+\frac{x}{\lambda}\right)^{-\gamma}\right]^{\alpha}\right]$$

$$\times\left\{1-\left[1-\left[1-\left(1+\frac{x}{\lambda}\right)^{-\gamma}\right]^{\alpha}\right]^{2}\right\}^{\theta-1}\cos\left\{\frac{\pi}{2}\left[1-\left(1-\left[1-\left(1+\frac{x}{\lambda}\right)^{-\gamma}\right]^{\alpha}\right)^{2}\right]^{\theta}\right\} \tag{24}$$

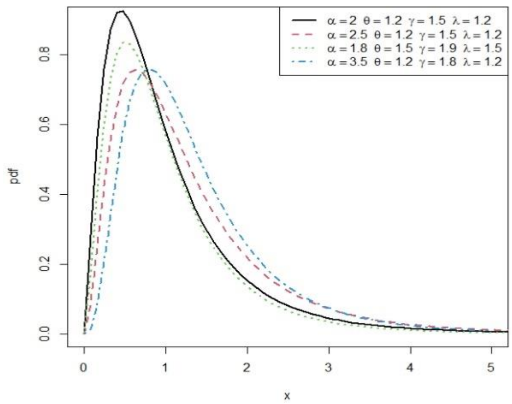

Figure 1: *pdf plot of STLE-Lomax distribution*

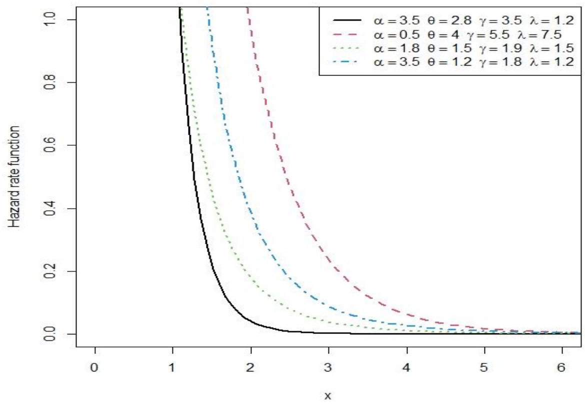

Figure 2: *plot of the hrf of the STLE-L distribution*

The survival function, hazard function, reverse hazard function and quantile function of the proposed STLE-Lomax distribution is presented below:

$$S(x)=1-\sin\left\{\frac{\pi}{2}\left[1-\left[1-\left(1-\left(1-\frac{x}{\lambda}\right)^{\gamma}\right)^{\alpha}\right]^{2}\right]^{\theta}\right\}$$

$$h(x)=\frac{\frac{\pi}{2}2\alpha\theta\left(\frac{\gamma}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\gamma+1)}\right)\left[1-\left(1+\frac{x}{\lambda}\right)^{-\gamma}\right]^{\alpha-1}\left[1-\left[1-\left(1+\frac{x}{\lambda}\right)^{-\gamma}\right]^{\alpha}\right]\left\{1-\left[1-\left[1-\left(1+\frac{x}{\lambda}\right)^{-\gamma}\right]^{\alpha}\right]^{2}\right\}^{\theta-1}\cos\left\{\frac{\pi}{2}\left[1-\left(1-\left[1-\left(1+\frac{x}{\lambda}\right)^{-\gamma}\right]^{\alpha}\right)^{2}\right]^{\theta}\right\}}{1-\sin\left\{\frac{\pi}{2}\left[1-\left[1-\left(1-\left(1-\frac{x}{\lambda}\right)^{\gamma}\right)^{\alpha}\right]^{2}\right]^{\theta}\right\}}$$

$$r(x)=\frac{\pi}{2}2\alpha\theta\left(\frac{\gamma}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\gamma+1)}\right)\left[1-\left(1+\frac{x}{\lambda}\right)^{-\gamma}\right]^{\alpha-1}\left[1-\left[1-\left(1+\frac{x}{\lambda}\right)^{-\gamma}\right]^{\alpha}\right]$$
$$\left\{1-\left[1-\left[1-\left(1+\frac{x}{\lambda}\right)^{-\gamma}\right]^{\alpha}\right]^{2}\right\}^{\theta-1}\cot\left\{\frac{\pi}{2}\left[1-\left(1-\left[1-\left(1+\frac{x}{\lambda}\right)^{-\gamma}\right]^{\alpha}\right)^{2}\right]^{\theta}\right\}$$

$$Q(u)=\lambda\left\{\left[1-\left[1-\left(1-\left(\frac{\sin^{-1}(u)}{\pi/2}\right)^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right]^{\frac{1}{\alpha}}\right]^{-\frac{1}{\gamma}}-1\right\}$$

2.7.2 Sine Topp-Leone Exponentiated Weibull (STLE-W) Distribution

Let the Weibull distribution be the baseline distribution with cdf and pdf defined by:

$$f(x)=\frac{\gamma}{\lambda}\left(\frac{x}{\lambda}\right)^{\gamma-1}e^{-\left(\frac{x}{\lambda}\right)^{\gamma}} \tag{25}$$

$$F(x)=1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}} \tag{26}$$

where γ is a shape parameter and λ is a scale parameter, the cdf and pdf of the STLE-L distribution are respectively given by:

$$F(x;\alpha,\theta,\gamma,\lambda)=\sin\left\{\frac{\pi}{2}\left[1-\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{\alpha}\right]^{2}\right]^{\theta}\right\} \tag{27}$$

$$f(x;\alpha,\theta,\gamma,\lambda)=\frac{\pi}{2}2\alpha\theta\left[\frac{\gamma}{\lambda}\left(\frac{x}{\lambda}\right)^{\gamma-1}e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right]\left[1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right]^{\alpha-1}\left\{1-\left[1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right]^{\alpha}\right\}$$
$$\left\{1-\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{\alpha}\right]^{2}\right\}^{\theta-1}\cos\left\{\frac{\pi}{2}\left[1-\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{\alpha}\right]^{2}\right]^{\theta}\right\} \tag{28}$$

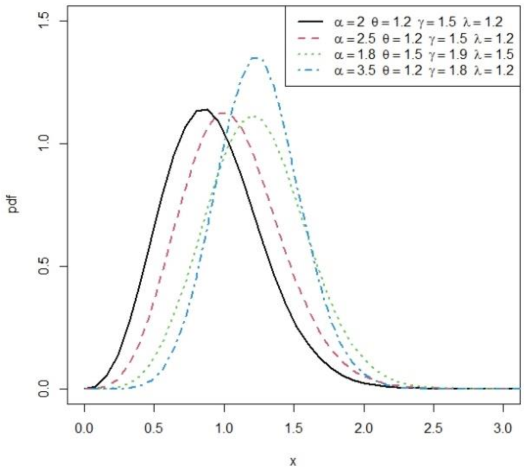

Figure 3: *pdf plot of STLE-W distribution*

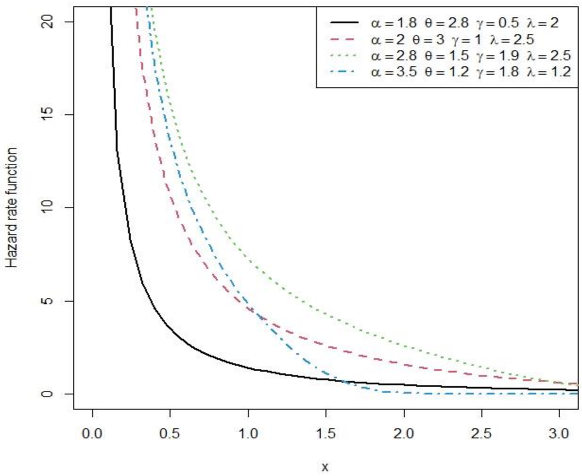

Figure 4: *Plot of the hrf of the STLE-W Distribution*

The survival function, hazard function, reverse hazard function and quantile function of the proposed STLE-Weibull distribution is presented below.

$$S(x)=1-\sin\left\{\frac{\pi}{2}\left[1-\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{\alpha}\right]^{2}\right]^{\theta}\right\}$$

$$h(x)=\frac{\frac{\pi}{2}2\alpha\theta\left[\frac{\gamma}{\lambda}\left(\frac{x}{\lambda}\right)^{\gamma-1}e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right]\left[1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right]^{\alpha-1}\left\{1-\left[1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right]^{\alpha}\right\}\left\{1-\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{\alpha}\right]^{2}\right\}^{\theta-1}\cos\left\{\frac{\pi}{2}\left[1-\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{\alpha}\right]^{2}\right]^{\theta}\right\}}{1-\sin\left\{\frac{\pi}{2}\left[1-\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{\alpha}\right]^{2}\right]^{\theta}\right\}}$$

$$r(x)=\frac{\pi}{2}2\alpha\theta\left[\frac{\gamma}{\lambda}\left(\frac{x}{\lambda}\right)^{\gamma-1}e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right]\left[1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right]^{\alpha-1}\left\{1-\left[1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right]^{\alpha}\right\}\left\{1-\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{\alpha}\right]^{2}\right\}^{\theta-1}\cot\left\{\frac{\pi}{2}\left[1-\left[1-\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}\right)^{\alpha}\right]^{2}\right]^{\theta}\right\}$$

$$Q(u)=\lambda\left\{-\log\left[1-\left[1-\left(1-\left(\frac{\sin^{-1}(u)}{\pi/2}\right)^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{\gamma}}\right\}$$

3. Results

3.1 Assessing the Consistency of the Parameter Estimates of the New Family

To evaluate the performance of the recently introduced Sine Tope Leone Exponentiated Lomax distribution, we conducted a Monte Carlo Simulation method. The aim of the simulation is to compute the mean, bias, and root mean square error of the estimated parameters obtained through maximum likelihood estimation. The simulated data was generated using the quantile function of STLE-Lomax distribution for various sample sizes, specifically: n=20, 50, 100, 150, 200, and 250, with 1000 replications for each sample size. Throughout these simulations, we set the parameters to $\alpha = 1.72$, θ=1.2, $\theta = 0.05$ and $\gamma = 0.99$. The results of the parameter estimates, bias, and root mean square error from the new distribution are summarized in Table 1

Table 1: *Estimate, Bias and RMSE of the new STLE-Lomax Distribution*

N	Properties	$\alpha = 1.72$	$\lambda = 1.2$	$\theta = 0.05$	$\gamma = 0.99$
	Est.	1.8431	1.3277	0.0562	1.1256
20	Bias	0.1231	0.1277	0.0062	0.1356
	RMSE	0.3805	0.4730	0.0164	0.5804
	Est.	1.7846	1.2416	0.0525	1.0996
50	Bias	0.0646	0.0416	0.0025	0.1096
	RMSE	0.2661	0.2967	0.0093	0.5034
	Est.	1.7541	1.2241	0.0512	1.0692
100	Bias	0.0341	0.0241	0.0012	0.0792
	RMSE	0.2148	0.2022	0.0063	0.4446
	Est.	1.7488	1.2086	0.0507	1.0646
150	Bias	0.0288	0.0086	0.0007	0.0746
	RMSE	0.1836	0.1693	0.0051	0.4603
	Est.	1.7480	1.2024	0.0506	1.0535
200	Bias	0.0280	0.0024	0.0006	0.0635
	RMSE	0.1561	0.1397	0.0044	0.4229
	Est.	1.7488	1.1957	0.0504	1.0575
250	Bias	0.0288	-0.0043	0.0004	0.0675
	RMSE	0.1521	0.1250	0.0040	0.4147

The results of the Monte Carlo Simulations are shown in table 1 above. The values of biases and RMSEs tend to zero as shown in Table 1 and the estimates tend to the true parameter values as the sample size increases, indicating that the estimates are efficient and consistent.

3.2 Application

Two datasets were considered for illustrative purposes and comparison with the baseline distribution [30] and other extensions of the Lomax distribution such as: Tope-Leone Exponentiated Lomax distribution by [31], Type II Topp-Leone Lomax by [32], Half Logistic-Lomax distribution developed by [33], Weibull Lomax distribution by [34] and Gompertz Lomax distribution by [35]. For each data set, we estimated the unknown parameters of each distribution by the maximum-likelihood method and also obtained the values of the Akaike information criterion (AIC) for the proposed model and the competitors.

The pdf of Lomax developed by [30] is given by:

$$f(x)=\frac{\alpha}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\alpha+1)} \tag{34}$$

The pdf of TLE-Lm as developed by [31] has pdf defined by:

$$f(x)=2\alpha\beta\gamma\lambda(1+\lambda x)^{-(\alpha+1)}(1-(1+\lambda x)^{-\beta})^{\gamma-1}[1-(1-(1+\lambda x)^{-\alpha})^{\beta}]\left\{1-[1-(1-(1+\lambda x)^{-\alpha})^{\beta}]\right\}^{\gamma-1} \tag{33}$$

The pdf of TIITL-Lm developed by [32] is defined by:

$$f(x)=\frac{2\alpha\gamma}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\alpha+1)}\left[1-\left(1+\frac{x}{\lambda}\right)^{-\alpha}\right]\left\{1-\left[1-\left(1+\frac{x}{\lambda}\right)^{-\alpha}\right]^{2}\right\}^{\gamma-1} \tag{35}$$

The pdf of HL-Lm developed by [33] is defined by:

$$f(x)=\frac{2\alpha\lambda(1+\lambda x)^{-(\alpha+1)}}{[1+(1+\lambda x)^{-\alpha}]^{2}} \tag{36}$$

The pdf of W-Lm distribution by [34] is defined by:

$$f(x)=\frac{2\gamma\beta\alpha}{\lambda}\left(1+\frac{x}{\lambda}\right)^{\beta\alpha-1}\left[1-\left(1+\frac{x}{\lambda}\right)^{-\alpha}\right]^{\beta-1}\exp\left\{-\gamma\left\{\left(1+\frac{x}{\lambda}\right)^{-\alpha}-1\right\}^{\beta}\right\} \tag{37}$$

The pdf of GoLm distribution by [35] is defined by:

$$f(x)=\theta\gamma\lambda(1+\lambda x)^{\gamma\alpha-1}\exp\left\{\frac{\theta}{\alpha}\left[1-(1+\lambda x)^{-\gamma\alpha}\right]\right\} \tag{38}$$

3.2.1 First Data Set

The first data set as listed below represents the COVID-19 positive cases record in Pakistan from March 24 to April 28, 2020, previously used by [36] and [37]: 2, 2, 3, 4, 26, 24, 25, 19, 4, 40, 87, 172, 38, 105, 155, 35, 264, 69, 283, 68, 199, 120, 67, 36, 102, 96, 90, 181, 190, 228, 111, 163, 204, 192, 627, 263.

3.2.2 Second Dataset

The second data set represents the failure times of the air conditioning system of an airplane. The data set was given by [38], it has been used by [39], and also by [4]. The data set is presented below:

23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95.

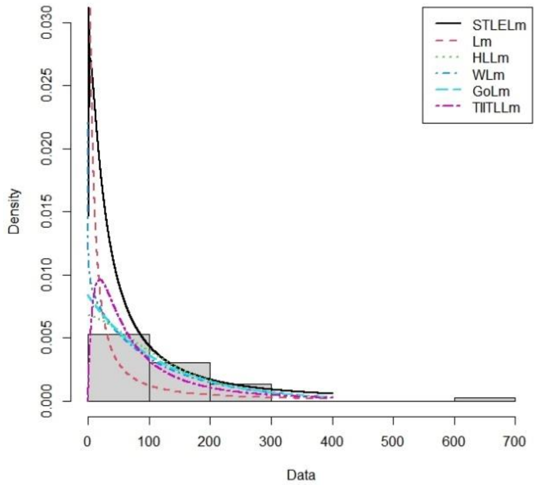

Figure 5: *Density plot of the STLE-Lomax distribution for the first data sets*

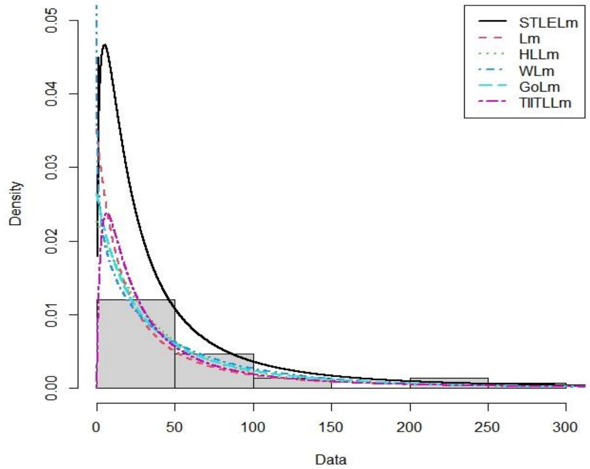

Figure 6: *Density plot for the STLE-Lomax distribution for the second data set*

Table 2 and table 3 gives the summary statistics of the two data sets such as the mean, the median, the first and third quartile, the minimum and the maximum values.

Table 2: *Summary Statistics of the two data sets*

Data	Minimum	Q_1	Median	Mean	Q_3	Maximum
Dataset I	2.00	32.75	93.00	119.28	183.25	627.0
Dataset II	1.00	12.50	22.00	59.60	83.00	261.0

Table 3: *Estimate, Bias and RMSE of the new STLE-Lomax Distribution*

Data set I	α	λ	θ	γ	LL	AIC
STLE-Lm	1.03323	0.11602	0.40035	2.56160	-189.4279	386.8558
TIITL-Lm	0.18430	24.5957	14.4732	-	-213.8879	433.7758
Lm	0.41228	6.73500	-	-	-224.5627	453.1253
HL-Lm	7.14437	0.00191	-	-	-208.3690	420.7379
W-Lm	0.01813	0.84166	1.12087	1.67046	-207.9697	423.9394
Go-Lm	7.96169	2.14265	47.014355	-	-209.7675	425.5349
Dataset II	α	λ	θ	γ	LL	AIC
STLE-Lm	1.77323	0.13691	0.64458	1.70984	-131.7712	271.5425
TIITL-Lm	34.2579	2.67719	0.06294	-	-167.6184	341.2368
Lm	17.1694	27.7386	-	-	-153.0699	310.1398
HL-Lm	0.05118	0.00178	-	-	-134.2354	272.4707
W-Lm	5.54118	5.62194	0.23070	0.18368	-183.2427	274.4856
Go-Lm	0.03744	0.01936	0.00819	-	-186.4186	378.8372

Table 3 presents the results of the two datasets. The analysis compared the performance of the Sine-Topp-Leone Exponentiated Lomax distribution against several other distributions, namely the Type II Topp Leone Lomax distribution, Lomax distribution, Half-Logistic-Lomax distribution, Weibull-Lomax distribution, and Gompertz-Lomax distribution. The results indicated that the proposed Sine Topp-Leone Exponentiated Lomax distribution outperformed some competing distributions, as it exhibits the lowest AIC value.
The visual assessment of the goodness of fit, as depicted in Figures 5 and 6, further validates the superiority of the proposed distribution when compared to other competing distributions. Therefore, it can be concluded that the proposed family of distributions is the most suitable choice for modeling both the COVID-19 and failure times of the air conditioning system of an airplane datasets.

4. Discussion

In this research, we introduced a new family of lifetime distributions by applying a Sine Transformation. Two special distributions were derived from the family by considering Lomax and Weibull distributions. Numerical analysis of fitting two real live data sets was presented using a maximum likelihood technique and density plots were provided to visually assess the outcomes.

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