ON A DISCRETE TIME SHOCK MODEL IN CRITICAL SITUATION

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Abstract

In this paper, we study a discrete time shock model which is defined based on the length of the time between successive shocks. For a system that is exposed to a sequence of random shocks over time under this model, if the interarrival time between two successive shocks is equal to a prefixed critical time point such as δ, the system fails, and the system is not damaged otherwise. We have considered two situations for the system, which are regular situation and critical situation, then we investigate the statistical behavior of the systemŠs lifetime under these situations. More precisely, we obtain the probability generating function of the system's lifetime, the mean time to failure, the variance of the system's lifetime, the Laplace transform of the system's lifetime, and some other related results. We end the paper with an example including numerical comparisons of the results.

Keywords: discrete time shock model, intershock time, critical time point

1. Introduction

Studying systems in various sciences is very important in maintaining, improving and reducing their errors. Shock models are useful models for studying the systems which are subject to random shocks at random times. Since the probabilistic behavior of a system under the influence of factors such as shocks is important to prevent system failure, therefore, shock models have attracted a great deal of interest in applied probability, reliability theory, and engineering. There are three basic types of shock models in the literature, which are cumulative shock models, extreme shock models, and run shock models. In a cumulative shock model, a system break down because of a cumulative effect; in an extreme shock model, a system break down because of one single large shock; in a run shock model, the range of a certain number of consecutive shocks is considered a failure criterion. For more on these, see, e.g., [1], [8], [18], and [23]. Moreover, models that can be produced by combining these traditional models can be found in [9].

In addition to the above traditional shock models, there are other shock models that have been introduced and developed in recent years. The so-called *δ-shock model* is among these new models that have received more attention. According to the δ-shock model, the system fails when the interarrival time between two successive shocks (intershock time) falls below a prefixed threshold $\delta > 0$. Therefore, the lifetime of the system is defined by $T = \sum_{i=1}^{N} X_i$, where $X_1, X_2, \ldots$ represent the intershock times, and the random variable N is the wating time for the first intershock time which is less than δ, i.e., $\{N = n\}$ iff $\{X_1 > \delta, \ldots, X_{n-1} > \delta, X_n \leq \delta\}$. The δ-shock model was first introduced by Li et al. in [13], after which it was widely studied by many scientists and researchers, for example, [14], [27], [15], [16], and [26]. Some extensions and generalizations were provided for the δ-shock model, see, e.g., [3], [19], and [20]. Moreover, a mixed shock model is defined by Wang and Zhang in [26] in which the system fails when an extreme shock occurs or a δ-shock. Ma and Li [17] have introduced and studied a censored δ-shock model. Eryilmaz [4] studied the lifetime behavior of a discrete time δ-shock model. Tuncel and Eryilmaz [24] investigate the survival function and the mean lifetime of the system failure considering the proportional hazard rate model. Goyal et al. [6] studied a general class of shock models with dependent intershock times.

In this paper, we aim to study a new discrete time shock model, which can be reduced to the classical δ-shock model in a special case. In general, in the new model, we do not have a critical threshold as was the case for δ-shock models, however, we have a critical time point so that if the intershock time is equal to the critical time point, then the system fails. To our knowledge, there is no such a shock model as this in the literature. Furthermore, we investigate the system under the assumption that the intershock times $X_1, X_2, \ldots$ are inflated at a particular point of time. This means that the frequency of some points of the times between successive shocks may occur more than expected in regular models. We will study the lifetime of the system under the new model and new assumptions. To this end, we obtain the probability generating function of the system's lifetime, the mean time to failure of the system, the variance of the system's lifetime, the Laplace transform of the system's lifetime, and some other related results. Furthermore, we provide an illustrative example for the new model.

The paper organized as follows. Section 2 introduces the model and assumptions. The statistical properties of the intershock times under the inflation property are investigate in Section 3. The life properties of the system under the new model are derived in Section 4. The paper ends with an example in Section 5.

2. Model Description

2.1. Regular Situation

We consider a system that is subject to a sequence of external shocks occur randomly over time. Let X_i denote the time between ith and $(i+1)$th shocks for $i = 1, 2, \ldots$. We assume that the intershock times $X_1, X_2, \ldots$ take positive integer values and are independent identically distributed (i.i.d.) by an arbitrary discrete distribution with probability mass function (pmf) $P(X = x)$ (with $P(X = 0) = 0$). The performance of the system is such that when the intershock time is not equal to a prefixed positive integer δ (i.e. $X_n \neq \delta$), the system does not fail, and if $X_n = \delta$, the system fails. Under these assumptions the lifetime of the system is defined as follows:

$$T_\delta = \sum_{i=1}^{N} X_i, \tag{1}$$

where the stopping random variable N is defined as

$$\{N = n\} \quad \Leftrightarrow \quad \{X_1 \neq \delta, X_2 \neq \delta, \ldots, X_{n-1} \neq \delta, X_n = \delta\}. \tag{2}$$

Hence, the pmf of N is obtained as follows:

$$P(N = n) = \big(1 - P(X_1 = \delta)\big)^{n-1} P(X_1 = \delta), \qquad n = 1, 2, \ldots,$$

that is, the random variable N has the geometric distribution with mean $\frac{1}{P(X_1=\delta)}$.

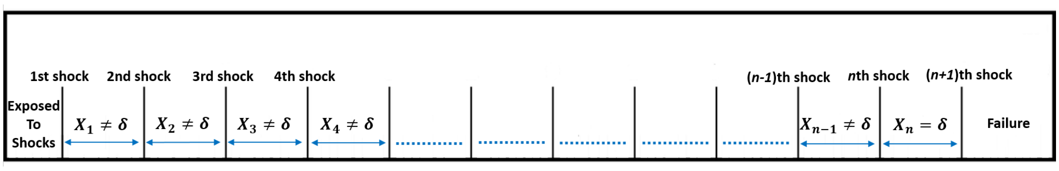

Figure 1: Visual understanding of the system under the model.

Note that in the case where $\delta = \min\{X_i : i = 1, 2, \dots\}$ we have

$$\begin{aligned}\{N = n\} &= \{X_1 \neq \delta, X_2 \neq \delta, \dots, X_{n-1} \neq \delta, X_n = \delta\} \\ &= \{X_1 > \delta, X_2 > \delta, \dots, X_{n-1} > \delta, X_n = \delta\}, \\ &= \{X_1 > \delta, X_2 > \delta, \dots, X_{n-1} > \delta, X_n \leq \delta\},\end{aligned}$$

thus, the model reduces to the classical δ-shock model.

Note also that when the intershock times X_i's follow a regular distribution, we say that the system is in a regular situation.

2.2. Critical Situation

We are interested in studying the condistions in which the system is in a critical situation. Since the system fails whenever the intershock time equals the critical time point δ, so one of the conditions that makes the system to go from a regular situation to a critical situation is that the intershock times have an overdispersion at the point δ. This means that the frequency of intershock times at point δ may occur more than expected in a regular model. In this case, the intershock times X_i's follow an inflated distribution (with inflation point δ), and we denote the lifetime of the system by T_δ^+.

3. Properties of Intershock Times Under Inflation

According to the descriptions in Section 2, the intershock time X takes positive integer values with a pmf $P(X = x)$. Based on the assumptions, the distribution of intershock times X is either regular (not inflated) or inflated. Thus, as useful notations, henceforth, instead of $P(X = x)$ we will use $P_{reg}(X = x)$ and $P_+(X = x)$, respectively, if X is distributed by a regular distribution or if X is distributed by an inflated distribution.

In the context of inflated distributions, a discrete random variable X with support $\mathcal{X}$ is said to be inflated at a particular point k ($k \in \mathcal{X}$), if its pmf is given by

$$P_+(X = x) = \begin{cases} \alpha + (1-\alpha)P_{reg}(X = x) & \text{if } x = k, \\ \\ (1-\alpha)P_{reg}(X = x) & \text{if } x \in \mathcal{X} - \{k\}, \end{cases} \tag{3}$$

where $\alpha \in [0, 1]$ is an inflation parameter.

For example, if $X_1, X_2, \dots$ are i.i.d. 3-inflated geometric distribution, we have $P_{reg}(X = x) = p(1-p)^{x-1}$ for $x = 1, 2, \dots$, and $0 < p \leq 1$, thus

$$P_+(X = x) = \begin{cases} \alpha + (1-\alpha)p(1-p)^2 & \text{if } x = 3, \\ (1-\alpha)p(1-p)^{x-1} & \text{if } x \in \{1, 2, 4, 5, 6, \dots\}. \end{cases}$$

For some references on inflated distributions, see, for example, [5], [10], [12], and [21].

In following, we investigate some distributional properties of a general k-inflated distribution. First, the main property of an inflated distribution is proved in the following theorem.

Theorem 1. Let the intershock time X follows the k-inflated distribution in (3). Then $P_+(X = k) \geq P_{reg}(X = k)$.

Proof. We have $0 \leq P_{reg}(X = x) \leq 1$ for any x in its support. On the other hand, for $\alpha = 0$ we have $P_+(X = k) = P_{reg}(X = k)$, and for $\alpha = 1$, $P_+(X = k)$ is degenerated pmf. Now, let us consider $\alpha \in (0, 1)$. Therefore multiplying both sides of $P_{reg}(X = k) < 1$ by α and adding $-\alpha P_{reg}(X = k)$ to both sides gives

$$\alpha - \alpha P_{reg}(X = k) > 0.$$

Finally, adding both sides by $P_{reg}(X = k)$, we get $\alpha + (1 - \alpha)P_{reg}(X = k) > P_{reg}(X = k)$, that is, $P_+(X = k) > P_{reg}(X = k)$. This completes the proof. ■

Therefore, by Theorem 1, in a k-inflated distribution the probability of occurrence of k is higher than in a distribution with regular pmf $P_{reg}(X = x)$.

In the next theorem, we obtain the cumulative distribution function (cdf) of a k-inflated random variable distributed by (3).

Theorem 2. Let the intershock time X follows the k-inflated distribution in (3). If the cdf of X in regular mode is denoted by $F_{reg}(x)$, then the cdf of X in inflated mode is given by

$$F_+(x) = \begin{cases} (1-\alpha)F_{reg}(x) & \text{if } x < k, \\ \\ \alpha + (1-\alpha)F_{reg}(x) & \text{if } x \geq k. \end{cases}$$

Proof. Using (3), we have for $x < k$,

$$F_+(x) = \sum_{j=0}^{x} P_+(X = j) = (1-\alpha)\sum_{j=0}^{x} P_{reg}(X = j) = (1-\alpha)F_{reg}(x),$$

and if $x \geq k$, we have

$$\begin{aligned} F_+(x) = \sum_{j=0}^{x} P_+(X = j) &= \left(\left((1-\alpha)\sum_{j=0}^{x} P_{reg}(X = j)\right) - (1-\alpha)P_{reg}(X = k)\right) \\ &+ \alpha + (1-\alpha)P_{reg}(X = k) \\ &= \alpha + (1-\alpha)F_{reg}(x). \end{aligned}$$

This completes the proof. ■

Theorem 3. Let the intershock time X follows the k-inflated distribution in (3). If the reliability function of X in regular mode is denoted by $\bar{F}_{reg}(x)$, then the reliability function of X in inflated mode is given by

$$\bar{F}_+(x) = \begin{cases} \alpha + (1-\alpha)\bar{F}_{reg}(x) & \text{if } x < k, \\ \\ (1-\alpha)\bar{F}_{reg}(x) & \text{if } x \geq k. \end{cases}$$

Proof. By using the definition of reliability function ($\bar{F}_+(x) = P_+(X > x)$) and using Theorem 2 the proof is straightforward. ■

In the following theorem, we obtain the moments related to interarrival times between successive shocks under k-inflated distribution in (3).

Theorem 4. Let $E_{reg}[X^r]$ be the rth moment of the intershock time X in its regular mode. The rth moment of the k-inflated version of X distributed by (3) is

$$E_+[X^r] = \alpha k^r + (1-\alpha)E_{reg}[X^r].$$

In particular, for $r = 1$ and $r = 2$, we have

$$E_+[X] = \alpha k + (1-\alpha)E_{reg}[X], \tag{4}$$
$$E_+[X^2] = \alpha k^2 + (1-\alpha)E_{reg}[X^2]. \tag{5}$$

Proof. We have

$$\begin{aligned} E_+[X^r] = \sum_x x^r P_+(X=x) &= \left(\left((1-\alpha)\sum_x x^r P_{reg}(X=x)\right) - (1-\alpha)k^r P_{reg}(X=k)\right) \\ &+ k^r\left(\alpha + (1-\alpha)P_{reg}(X=k)\right) \\ &= \alpha k^r + (1-\alpha)E_{reg}[X^r]. \end{aligned}$$

The theorem is proved. ■

Corollary 1. The variance of a k-inflated intershock time X distributed by (3) is given by

$$\begin{aligned} Var_+(X) = Var_{reg}(X) &+ \alpha^2\left(2kE_{reg}[X] - E^2_{reg}[X] - k^2\right) \\ &+ \alpha\left(2E^2_{reg}[X] - E_{reg}[X^2] - 2kE_{reg}[X] + k^2\right), \end{aligned}$$

where $Var_{reg}(X)$ is the variance of X in its regular mode.

Proof. Use Equations (4) and (5) and the definition of variance. This completes the proof. ■

In following, we calculate the probability generating function (pgf) of the inflated distribution in (3).

Theorem 5. If intershock time X follows the k-inflated distribution in (3), then its pgf is

$$G^+_X(z) = \alpha z^k + (1-\alpha)G^{reg}_X(z),$$

where $G^{reg}_X(z)$ is the pgf of X in its regular mode.

Proof. We have

$$\begin{aligned} G^+_X(z) = E_+[z^X] = \sum_x z^x P_+(X=x) &= \alpha z^k + (1-\alpha)\sum_x z^x P_{reg}(X=x) \\ &= \alpha z^k + (1-\alpha)E_{reg}(z^X) \\ &= \alpha z^k + (1-\alpha)G^{reg}_X(z). \end{aligned}$$

The theorem is proved. ■

Note that if $\alpha = 0$, then Equation (3) gives $P_+(X=x) = P_{reg}(X=x)$, $F_+(x) = F_{reg}(x)$, $\bar{F}_+(x) = \bar{F}_{reg}(x)$, $E_+[X^r] = E_{reg}[X^r]$, and $G^+_X(z) = G^{reg}_X(z)$. Therefore, when the inflation parameter α tends to zero, then the inflation property for the random variable X reduces the regular state.

4. Statistical Properties of the System's Lifetime

Following Section 2, when the intershock times are not equal to δ, the system continues to work safely. Indeed, if our observations of intershock times have overdispersion at point δ, then the system is in an critical situation and is expected to fail soon. Hence, it is important to study the lifetime of the system when the intershock times follow an δ-inflated distribution. This is done in the following.

The reliability function (or survival function) of the system's lifetime can be expressed by

$$P(T_\delta > t) = P\left(\sum_{i=0}^{N} X_i > t\right) = \sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n} X_i > t\right) P(N=n). \tag{6}$$

Since the distribution of intershock times is arbitrary, therefore, deriving a general explicit representation of the reliability function from Equation (6) is difficult, or very complex if obtained. Therefore, the probability generating function (pgf) can be useful for calculation of the probability mass function of the system's lifetime. Following theorem gives us the pgf of the system's lifetime.

Theorem 6. Let $X_1, X_2, \ldots$ be the intershock times in the model described in Section 2, and let also X denotes a generic random variable of X_i's. When the system is in the critical situation, the pgf of the system's lifetime becomes

$$G_{T_\delta}(z) = \frac{\left(\alpha z^\delta + (1-\alpha)G_X^{reg}(z)\right)\left(\alpha + (1-\alpha)P_{reg}(X=\delta)\right)}{1-(1-\alpha)P_{reg}(X \neq \delta)\left(\alpha z^\delta + (1-\alpha)G_X^{reg}(z)\right)}. \tag{7}$$

Proof. Since the system is in the critical situation, so X is δ-inflated distrebuted. The pgf of T_δ^+ can be written as (by using Equation (1))

$$\begin{aligned} G_{T_\delta^+}(z) = E[z^{T_\delta^+}] = E\left[z^{\sum_{i=1}^N X_i}\right] &= E\left[E_+\left[z^{\sum_{i=1}^N X_i}|N\right]\right] \\ &= E\left[\left(G_X^+(z)\right)^N\right] \\ &= G_N\left(G_X^+(z)\right), \end{aligned} \tag{8}$$

where $G_N(z)$ is the pgf of random variable N.

Since N has geometric distribution with parameter $P_+(X=\delta)$, therefore, the pgf of N is obtained as (use also (3))

$$G_N(z) = \frac{z\left(\alpha + (1-\alpha)P_{reg}(X=\delta)\right)}{1-\left(1-\left(\alpha+(1-\alpha)P_{reg}(X=\delta)\right)\right)z}. \tag{9}$$

Hence, by using Equation (9) in Equation (8), we obtain

$$G_{T_\delta^+}(z) = \frac{G_X^+(z)\left(\alpha + (1-\alpha)P_{reg}(X=\delta)\right)}{1-\left(1-\left(\alpha+(1-\alpha)P_{reg}(X=\delta)\right)\right)G_X^+(x)}. \tag{10}$$

Consequently, by applying Theorem 5 to Equation (10) we get the desired result. This completes the proof. ■

In the next theorem, we obtain an explicit formula for the mean lifetime of the system, which defines the system's mean time to failure. The second moment is also provided.

Theorem 7. Let $X_1, X_2, \ldots$ be intershock times in the model described in Section 2, and let also X denotes a generic random variable of X_i's. The mean lifetime of the system in the critical situation is

$$E[T_\delta^+] = \frac{\alpha\delta + (1-\alpha)E_{reg}[X]}{\alpha + (1-\alpha)P_{reg}(X=\delta)}, \tag{11}$$

and the second moment is

$$\begin{aligned} E[T_\delta^{+2}] &= \left(\frac{(1-\alpha)P_{reg}(X\neq\delta)}{\alpha+(1-\alpha)P_{reg}(X=\delta)}\right)(1-\alpha)\left(E_{reg}[X^2]-\delta^2 P_{reg}(X=\delta)\right) \\ &+ \delta^2\left(\alpha+(1-\alpha)P_{reg}(X=\delta)\right) \\ &+ \frac{2((1-\alpha)P_{reg}(X\neq\delta))^2}{(\alpha+(1-\alpha)P_{reg}(X=\delta))^2}\left((1-\alpha)\left(E_{reg}[X]-\delta P_{reg}(X=\delta)\right)\right)^2 \\ &+ 2\left(\frac{(1-\alpha)P_{reg}(X\neq\delta)}{\alpha+(1-\alpha)P_{reg}(X=\delta)}\right)\left((1-\alpha)\left(E_{reg}[X]-\delta P_{reg}(X=\delta)\right)\right) \\ &\times \delta\left(\alpha+(1-\alpha)P_{reg}(X=\delta)\right). \end{aligned} \tag{12}$$

Proof. Since N is a stopping time for X_i's ($i = 1, 2\ldots$), so the mean lifetime of the system can be computed as

$$E[T_\delta^+] = E\left[E\left[\sum_{i=1}^N X_i\right]|N\right] = \sum_{n=1}^\infty E_+\left[\sum_{i=1}^n X_i\right]P(N=n) = E_+[X]E[N]. \tag{13}$$

Since N follows the geometric distribution with parameter $P_+(X = \delta)$, therefore (use also (3))

$$E[N] = \frac{1}{P_+(X=\delta)} = \frac{1}{\alpha + (1-\alpha)P_{reg}(X=\delta)}, \tag{14}$$

and by using Equation (14) in Equation (13), we obtain

$$E[T_\delta^+] = \frac{E_+[X]}{\alpha + (1-\alpha)P_{reg}(X=\delta)}. \tag{15}$$

Using Theorem 4 we have $E_+[X] = \alpha\delta + (1-\alpha)E_{reg}[X]$, and by putting this in Equation (15), we get to Equation (11).

For the second moment, by conditioning on N, we can write

$$E[T_\delta^{+2}] = \sum_{n=1}^{\infty} E_+ \left(\left(\sum_{i=1}^{n} X_i \right)^2 |N = n \right) P_+(N = n). \tag{16}$$

A simple calculation show that

$$E_+ \left(\left(\sum_{i=1}^{n} X_i \right)^2 |N = n \right) = \sum_{i=1}^{n} E_+[X_i^2|N = n] + 2 \sum\sum_{1\le i<j\le n} E_+[X_iX_j|N = n]. \tag{17}$$

From the definition of random variable N (see (2)), we have

$$E_+[X_i^2|N = n] = \begin{cases} E_+[X^2|X \neq \delta], & i = 1, 2, \ldots, n-1, \\ E_+[X^2|X = \delta], & i = n, \end{cases}$$

thus,

$$\sum_{n=1}^{\infty} E_+[X_i^2|N = n] = (n-1)E_+[X^2|X \neq \delta] + E_+[X^2|X = \delta]. \tag{18}$$

Similarly,

$$\sum\sum_{1\le i<j\le n} E_+[X_iX_j|N = n] = \binom{n-1}{2} (E_+[X|X \neq \delta])^2 + (n-1)E_+[X|X \neq \delta]E_+[X|X = \delta]. \tag{19}$$

By using Equations (19) and (18) in (17) and then via Equation (16), we obtain

$$\begin{aligned} E[T_\delta^{+2}] &= \big(E[N]-1\big)E_+[X^2|X \neq \delta] + E_+[X^2|X = \delta] \\ &+ E[(N-1)(N-2)]\big(E_+[X|X \neq \delta]\big)^2 \\ &+ 2(E[N]-1)\Big(E_+[X|X \neq \delta]E_+[X|X = \delta]\Big). \end{aligned}$$

Since $E[N] = \frac{1}{P_+(X=\delta)}$ and $E[(N-1)(N-2)] = \frac{2(P_+(X=\delta)-1)^2}{(P_+(X=\delta))^2}$, therefore,

$$\begin{aligned} E[T_\delta^{+2}] &= \left(\frac{P_+(X \neq \delta)}{P_+(X = \delta)}\right)E_+[X^2|X \neq \delta] + E_+[X^2|X = \delta] \\ &+ \frac{2(P_+(X \neq \delta))^2}{(P_+(X = \delta))^2}\big(E_+[X|X \neq \delta]\big)^2 \\ &+ 2\left(\frac{P_+(X \neq \delta)}{P_+(X = \delta)}\right)E_+[X|X \neq \delta]E_+[X|X = \delta]. \end{aligned} \tag{20}$$

On the other hand, using Equation (3) and Theorem 4, we have

$$\begin{aligned} P_+(X=\delta) &= \alpha+(1-\alpha)P_{reg}(X=\delta), && (21)\\ P_+(X\neq\delta) &= 1-P_+(X=\delta)=(1-\alpha)P_{reg}(X\neq\delta), && (22)\\ E_+[X|X=\delta] &= \delta\left(\alpha+(1-\alpha)P_{reg}(X=\delta)\right), && (23)\\ E_+[X|X\neq\delta] &= E_+[X]-E_+[X|X=\delta]=(1-\alpha)\left(E_{reg}[X]-\delta P_{reg}(X=\delta)\right), && (24)\\ E_+[X^2|X=\delta] &= \delta^2\left(\alpha+(1-\alpha)P_{reg}(X=\delta)\right), && (25)\\ E_+[X^2|X\neq\delta] &= E_+[X^2]-E_+[X^2|X=\delta]=(1-\alpha)\left(E_{reg}[X^2]-\delta^2P_{reg}(X=\delta)\right). && (26) \end{aligned}$$

The desired result (12) is obtained by putting Equations (21)–(26) in Equation (20). The proof is complete. ■

Next, an explicit expression for the variance of lifetime of the system is given for the defined shock model.

Theorem 8. Let $X_1, X_2, \ldots$ be the intershock times in the model described in Section 2, and let also X denotes a generic random variable of X_i's. When the system is in the critical situation, the variance of the system's lifetime is

$$\begin{aligned} Var(T_\delta^+) &= \frac{1}{\alpha+(1-\alpha)P_{reg}(X=\delta)}\Bigg(Var_{reg}(X)+\alpha^2\left(2\delta E_{reg}[X]-E^2_{reg}[X]-\delta^2\right)\\ &+ \alpha\left(2E^2_{reg}[X]-E_{reg}[X^2]-2\delta E_{reg}[X]+\delta^2\right)\Bigg)\\ &+ \frac{\left(\alpha\delta+(1-\alpha)E_{reg}[X]\right)\left((1-\alpha)P_{reg}(X\neq\delta)\right)}{\left(\alpha+(1-\alpha)P_{reg}(X=\delta)\right)^2}. \end{aligned}$$

Proof. By using the second Wald's identity (see page 30 from [11]), the variance of the system's lifetime can be written as

$$Var(T_\delta^+)=Var\left(\sum_{i=1}^{N}X_i\right)=Var_+(X)E[N]+\left(E_+(X)\right)^2Var(N). \tag{27}$$

Since N has geometric distribution with parameter $P_+(X=\delta)$, therefore

$$Var(N)=\frac{1-P_+(X=\delta)}{\left(P_+(X=\delta)\right)^2}=\frac{P_+(X\neq\delta)}{\left(P_+(X=\delta)\right)^2}. \tag{28}$$

By using Equations (28) and (14) (for $E[N]$) in Equation (27), we obtain

$$Var(T_\delta^+)=\frac{Var_+(X)}{P_+(X=\delta)}+\frac{\left(E_+[X]\right)^2P_+(X\neq\delta)}{\left(P_+(X=\delta)\right)^2}. \tag{29}$$

From Equation (3) we have $P_+(X=\delta)=\alpha+(1-\alpha)P_{reg}(X=\delta)$, and thus, $P_+(X\neq\delta)=(1-\alpha)P_{reg}(X\neq\delta)$. Besides, by Theorem 4 and Corollary 8 we have, respectively,

$$\begin{aligned} E_+[X] &= \alpha\delta+(1-\alpha)E_{reg}[X],\\ Var_+(X) &= Var_{reg}(X)+\alpha^2\left(2\delta E_{reg}[X]-E^2_{reg}[X]-\delta^2\right)+\alpha\left(2E^2_{reg}[X]-E_{reg}[X^2]-2\delta E_{reg}[X]+\delta^2\right). \end{aligned}$$

Finally, by putting these identities in Equation (29) we obtain the desired result. The theorem is proved. ■

The Laplace transform (or the Laplace–Stieltjes transform) of T_δ^+ is derived in the following theorem.

Theorem 9. Let $X_1, X_2, \ldots$ be the intershock times in the model described in Section 2, and let also X denotes a generic random variable of X_i's. Assuming that the system is in the critical situation, the Laplace transform of the system's lifetime is

$$\mathcal{L}_{T^+_\delta}(s) = \frac{e^{-s\delta}\big(\alpha + (1-\alpha)P_{reg}(X=\delta)\big)}{1 - \sum_{x\neq\delta}\left[e^{-s\delta}\big(\alpha + (1-\alpha)P_{reg}(X=\delta)\big)\right]}.$$

Proof. By properties of conditional expectation and using the fact that the random variable N is independent of X_i's ($i = 1, 2, \ldots$), we have

$$\begin{aligned}\mathcal{L}_{T^+_\delta}(s) = E\left[e^{-sT}\right] = E\left[E\left[e^{-s\sum_{i=1}^{N} X_i}\middle|N\right]\right] &= \sum_{n=1}^{\infty} E\left[e^{-s\sum_{i=1}^{N} X_i}\middle|N=n\right]P(N=n)\\ &= \sum_{n=1}^{\infty} E_+\left[e^{-s\sum_{i=1}^{n} X_i} I_{(N=n)}\right].\end{aligned}$$

Using the definition of N (see Equation (2)) and the fact that X is δ-inflated distributed, we have

$$\mathcal{L}_X(s) = \sum_{n=1}^{\infty} E_+\left[e^{-s\sum_{i=1}^{n} X_i} I_{(X_1\neq\delta, X_2\neq\delta,\ldots,X_{n-1}\neq\delta, X_n=\delta)}\right] = \sum_{n=1}^{\infty}\left(E_+\left[e^{-sX} I_{(X\neq\delta)}\right]\right)^{n-1} E_+\left[e^{-sX} I_{(X=\delta)}\right].$$

Hence (by geometric series),

$$\mathcal{L}_{T^+_\delta}(s) = \frac{E_+\left[e^{-sX} I_{(X=\delta)}\right]}{1 - E_+\left[e^{-sX} I_{(X\neq\delta)}\right]}, \tag{30}$$

that is

$$\mathcal{L}_{T^+_\delta}(s) = \frac{e^{-s\delta}P_+(X=\delta)}{1 - \left(E_+[e^{-sX}] - e^{-s\delta}P_+(X=\delta)\right)} = \frac{e^{-s\delta}\left(\alpha + (1-\alpha)P_{reg}(X=\delta)\right)}{1 - \sum_{x\neq\delta}\left[e^{-sx}\big(\alpha + (1-\alpha)P_{reg}(X=x)\big)\right]}.$$

This completes the proof. ■

Remark 1. Note that if we consider $\alpha = 0$ in the above theorems, the distribution of intershock times reduces to a regular distribution, consequently, the above results investigate the system's lifetime in its regular situation.

5. Illustrative Example

Transistors are one of the crucial components that are almost always present in any electronic device. It is common knowledge that transistors tend to heat up during operation and that this temperature increase can significantly affect their performance and dependability. Understanding the thermal challenges that transistors confront is crucial for engineers and designers. One of the reasons for the heat increase of transistors is high voltage, and high voltages usually damage the transistor as shocks. Transistors are usually cooled by using cooling fans (often with an passive heat exchanger such as a heat sink) so that they don't get damaged. For more information, we refer the reader to [22] and [2].

Here, we consider a specific transistor as a system which is exposed to a sequence of external shocks in the form of high voltages. The transistor must operate within its specified voltage and current limits to prevent overheating and shocks occur randomly in any period of time $x = 1, 2, \ldots$, the values of x's are as minutes. If the interarrival time between two successive high voltages is more than 5 minutes, the transistor has enough time to cool itself. If the interarrival time between two successive high voltages is less than 5 minutes, the cooling fan will automatically turn on to cool the transistor. In the case when the interarrival time between two successive high voltages is equal to 5 minutes, the cooling fan will not turn on due to incorrect detection at the time length 5 minutes, so the transistor will be damaged. Observations show that the number of high voltages with interarrival times greater than 5 minutes and less than 5 minutes are approximately equal.

Indeed, the above example follows the shock model described in Section 2. Now, we present some illustrative computational results. Let us consider the case when the intershock times $X_1, X_2, \ldots$ are i.i.d. distributed by the geometric distribution with mean $\frac{1}{p}$, that is, the pmf of X_i $(i = 1, 2, \ldots)$ is $P(X_i = x_i) = p(1-p)^{x_i - 1}$ for $x_i = 1, 2, \ldots$. Assuming that the critical intershock time $x = \delta = 5$ is the median of X_i's (based on observations), some different values can be chosen for the parameter p. The median of a geometric distribution with parameter p is given by

$$\text{Median} = \left\lceil \frac{-1}{\log_2(1-p)} \right\rceil.$$

By solving the equation $\lceil \frac{-1}{\log_2(1-p)} \rceil = 5$, we find that $p \in [\frac{2-2^{4/5}}{2}, \frac{2-2^{3/4}}{2})$. Thus, some different choices for the parameter p can be $p = 0.135$, $p = 0.145$, and $p = 0.155$. Assuming that $T_\delta = \sum_{i=1}^{N} X_i$ is the lifetime of the above system, in Table 1 we present the pmf $P(T_\delta = t)$ and the reliability function $P(T_\delta > t)$ for $p = 0.135, 0.145, 0.155$, $\delta = 5$, and some values of t. Also, we have considered the lifetime of the system in its critical situation, that is, $T_\delta^+ = \sum_{i=1}^{N} X_i$, in which the distribution of X_i's is inflated at the critical time point $x = \delta = 5$.

From Table 1 we can see that the probability of system's lifetime in critical situation ($P(T_\delta^+ = t)$) at the critical time point $t = \delta = 5$ is greater than the probability of system's lifetime at other time points, and also the reliability function ($P(T_\delta^+ > t)$) at the critical point $t = \delta = 5$ is smaller than the reliability function at other time points. In general, the probability $P(T_\delta^+ = \delta)$ increases when the inflation parameter α increases. Also, as expected, an increase in inflation parameter α leads to a decrease in reliability function. By comparing $P(T_\delta > t)$ and $P(T_\delta^+ > t)$, it is clear that the system is more stable in the regular situation. This is consistent with what was expected from the theory, results, and purpose of the paper.

Table 1: The pmf and reliability function of the system's lifetime

p	δ	t	$P(T_\delta = t)$	$P(T_\delta > t)$	α	$P(T_\delta^+ = t)$	$P(T_\delta^+ > t)$
0.135	5	1	0.00571	0.99429	0.3	0.0333	0.9667
		2	0.00567	0.98869		0.0308	0.9359
		3	0.00564	0.98289		0.0286	0.9073
		4	0.00561	0.97737		0.0264	0.8809
		5	0.00558	0.97179		0.1304	0.7505
		6	0.00555	0.96624		0.0356	0.7149
		7	0.00551	0.96073		0.0334	0.6815
0.145	5	1	0.00600	0.99400	0.6	0.0365	0.9635
		2	0.00596	0.98804		0.0320	0.9315
		3	0.00593	0.98211		0.0281	0.9034
		4	0.00589	0.97622		0.0246	0.8788
		5	0.00586	0.97036		0.4001	0.4787
		6	0.00582	0.96454		0.0351	0.4436
		7	0.00579	0.95875		0.0309	0.4127
0.155	5	1	0.00624	0.99376	0.9	0.0140	0.9860
		2	0.00620	0.98756		0.0119	0.9741
		3	0.00616	0.98140		0.0100	0.9641
		4	0.00612	0.97528		0.0085	0.9556
		5	0.00609	0.96919		0.8243	0.1313
		6	0.00605	0.96314		0.0084	0.1229
		7	0.00601	0.95713		0.0071	0.1158

The following figure shows the mean lifetime of the above system versus p for critical time point $t = \delta = 5$ and some different values of the inflation parameter α. For $\alpha = 0$, we have the mean lifetime of the system in regular situation ($E[T_\delta]$). Figure 2 shows that the mean lifetime of the system in the regular state is higher than the mean lifetime of the system in the critical situation ($E[T_\delta^+]$). Also, we see that as the inflation parameter increases, the mean lifetime of the system decreases.

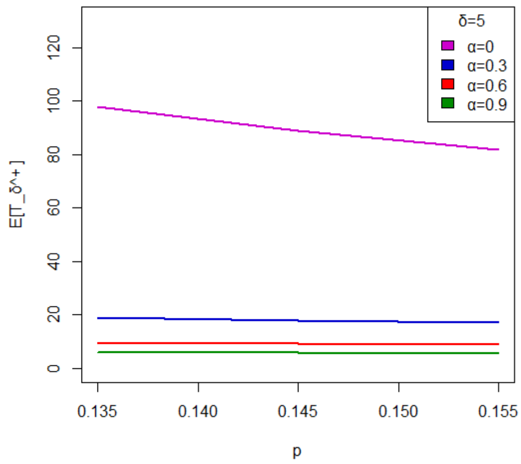

Figure 2: Plot of $E[T_\delta^+]$ versus p for $\delta = 5$ and some different values of inflation parameter α.

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