TOPP-LEONE EXPONENTIATED GOMPERTZ INVERSE RAYLEIGH DISTRIBUTION: PROPERTIES AND APPLICATIONS

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Abstract

This paper focused on deriving a new lifetime distribution having five parameters by compounding the Gompertz inverse Rayleigh model and the Topp-Leone exponentiated-G family of distributions. The new model is called Topp-Leone exponentiated Gompertz inverse Rayleigh (TLEGoIRa) distribution. The new model is very flexible and the shape of its pdf can be positively or negatively skewed and symmetric. Some statistical characteristics of the new model, such as the moments, incomplete moments, quantile function, rényi entropy and order statistics are derived and investigated. The pdf of the minimum and maximum order statistics of the new model were derived and studied. The model's parameters are estimated using the maximum likelihood approach. A simulation study was conducted to investigate the consistency of the newly proposed model, using the average bias and root mean square error (RMSE) as metrics. The outcome of the simulation suggested that as sample sizes increase, both the average bias and root mean square error (RMSE) decrease, indicating that the distribution is consistent. Finally, two real-life datasets were used to explore the new model's importance and adaptability in comparison to other competing models The results of the application revealed that the new distribution outperforms its competitors.

Keywords: Toppleone Exponentiated G., Gompertz Inverse Rayleigh, Quantile Function, Order Statistics, MLE.

I. Introduction

In the realm of distribution theory, the pursuit of developing models that accurately reflect the prevailing trends across various disciplines has proven challenging. Classical distributions, which form the foundation, often lack the required flexibility and robustness. This inherent limitation has spurred researchers within the distribution theory field to undertake the task of extending or generalizing existing distributions. The overarching goal is to imbue these distributions with greater flexibility and resilience, enabling them to effectively capture the evolving patterns present in datasets originating from diverse fields like engineering, environmental sciences, biological sciences, medical sciences, and beyond.

This process of extension or generalization entails the introduction of one or more additional parameters to the existing distributions. Contemporary approaches to distribution generalization frequently involve the utilization of distribution families. Examples of these families include the Topp-Leone exponentiated-G distribution by [1], Topp-Leone Kumaraswamy-G distribution by [2], Topp-Leone-G distribution by [3], type II half logistic-G distribution by [4], type I half logistic exponentiated-G distribution by [5], type II half logistic exponentiated-G distribution by [6], transmuted exponentiated generalized G distribution by [7], Topp-Leone odd Lindley G distribution by [8], and Topp-Leone Gompertz-G distribution by [9], among others. These families are constructed by introducing supplementary shape parameter(s) to the foundational distribution, thereby augmenting the efficacy and practicality of data modeling.

One significant continuous probability distribution, referred to as the inverse Rayleigh (IRa) distribution, was initially introduced by [10] and has since found extensive application in modeling system failure times. Notably, the IRa distribution is a specialized form of the broader inverse Weibull (IW) distribution. The statistical literature offers various adaptations and extensions of the IRa distribution, which can be explored further through references such as [11-13]. An extension of particular interest is the Gompertz inverse Rayleigh (GoIRa) distribution, developed by [14], which is considered as the baseline distribution in this study. Through the extension of the GoIRa distribution, we aim to develop a more adaptable compound distribution.

Reference [1] introduced the TLE-G, a distinct family of continuous distributions with the cumulative distribution function (cdf) and probability density function (pdf) given as:

$$F(x,\theta,\alpha)=\left(1-\left(1-G\left(x;\xi\right)^{\alpha}\right)^{2}\right)^{\theta} \tag{1}$$

$$f(x,\theta,\alpha)=2\theta\alpha g\left(x;\xi\right)G\left(x;\xi\right)^{\alpha-1}\left(1-G\left(x;\xi\right)^{\alpha}\right)\left(1-\left(1-G\left(x;\xi\right)^{\alpha}\right)^{2}\right)^{\theta-1} \tag{2}$$

The cdf and pdf corresponding to the baseline Gompertz inverse Rayleigh (GoIRa) distribution are given as:

$$G\left(x,\gamma,\beta,\lambda\right)=1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right)^{-\beta}\right)} \tag{3}$$

and

$$g\left(x,\gamma,\beta,\lambda\right)=2\gamma\lambda^{2}x^{-3}e^{-\left(\frac{\lambda}{x}\right)^{2}}\left(1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right)^{-\beta-1}e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right)^{-\beta}\right)} \tag{4}$$

Where $\lambda>0$ is the scale parameter and $\gamma,\beta>0$ are the shape parameters respectively.

The primary objective of this research is to utilize the GoIRa distribution as a foundational model within the TLE-G framework, aiming to develop a novel extension known as the TLEGoIRa distribution. This extension seeks to enhance the flexibility and applicability of the GoIRa distribution in capturing complex data patterns across various fields.

The remaining content of this article is organized as follows: Section 2 presents the development of the TLEGoIRa distribution, including the derivation of its properties and the method for estimating its parameters. Section 3 discusses simulation studies conducted to investigate the consistency of the estimates and the application of the model to two real datasets to demonstrate the practical potential of the new distribution. Finally, Section 4 provides concluding remarks.

II. Methods

2.1 Development of Topp-Leone Exponentiated Gompertz Inverse Rayleigh (TLEGoIRa) Distribution

To derive the cdf of the new model, equation (3) is inserted into equation (1) as:

$$F\left(x,\theta,\alpha,\gamma,\beta,\lambda\right)=\left(1-\left(1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right)^{-\beta}\right)}\right)^{\alpha}\right)^{2}\right)^{\theta} \tag{5}$$

To derive the PDF of the new model, equations (3) and (4) are inserted into equation (2) as follows:

$$f\left(x,\theta,\alpha,\gamma,\beta,\lambda\right)=4\gamma\theta\alpha\lambda^{2}x^{-3}e^{-\left(\frac{\lambda}{x}\right)^{2}}\left(1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right)^{-\beta-1}e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right)^{-\beta}\right)}\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right)^{-\beta}\right)}\right)^{\alpha-1}$$
$$\left(1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right)^{-\beta}\right)}\right)^{\alpha}\right)\left(1-\left(1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right)^{-\beta}\right)}\right)^{\alpha}\right)^{2}\right)^{\theta-1} \tag{6}$$

where $x \geq 0$, and $\theta, \alpha, \lambda, \beta, \gamma > 0$.

The hazard function for the TLEGoIRa distribution can be obtained using this expression:

$$h\left(x,\theta,\alpha,\gamma,\beta,\lambda\right)=\frac{f\left(x,\theta,\alpha,\gamma,\beta,\lambda\right)}{1-F\left(x,\theta,\alpha,\gamma,\beta,\lambda\right)}$$

The pdf and hazard function plots of the TLEGoIRa distribution are given figures 1 and 2 below:

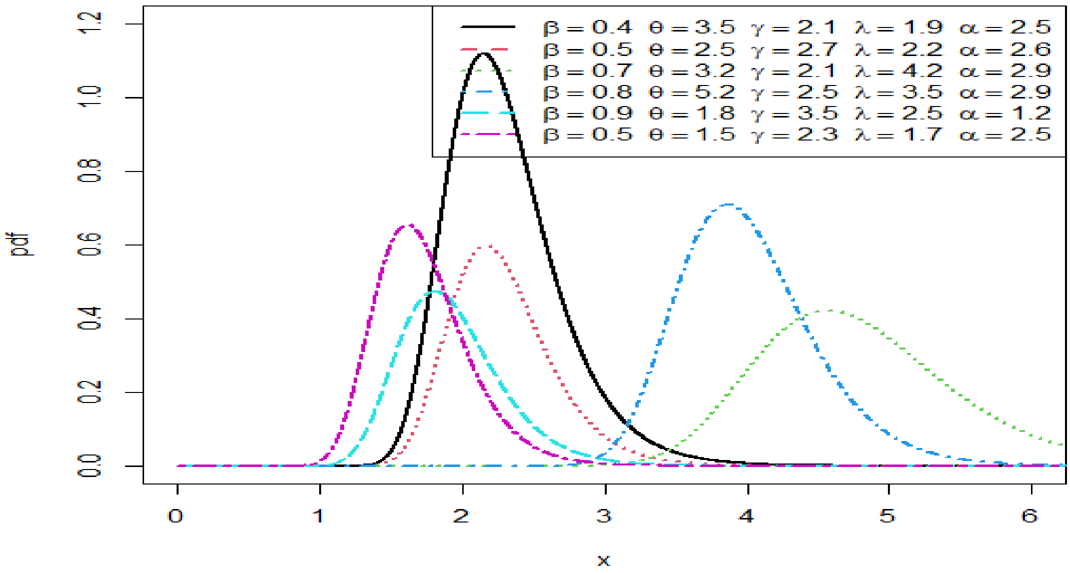

Figure 1: *pdf plots of the TLEGoIRa distribution with different parameter values*

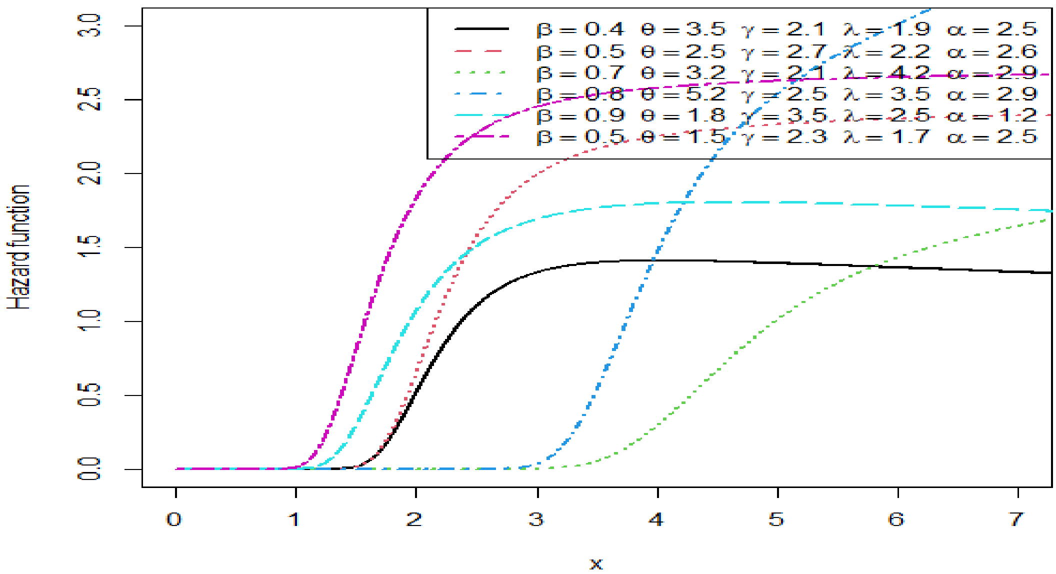

Figure 2: *hazard function plots of the TLEGoIRa distribution with different parameter values*

2.2 Statistical Properties of the TLEGoIRa Distribution

This section derives some statistical properties of the TLEGoIRa distribution including moments, survival function, hazard function, quantile functions, and order statistics.

2.2.1 Moment

The r^{th} moment of the TLEGoIRa model is computed using the following expression:

$$\dot{\mu}_r = E\left(X^r\right) = \int_0^\infty x^r f\left(x, \theta, \alpha, \lambda, \beta, \gamma\right) dx$$

First expanding equation (6) by using generalized binomial expansion $\left[1+u\right]^{-q} = \sum_{z=0}^{\infty}\left(-1\right)^z \binom{q}{z} u^z$,

$\left[1+u\right]^{-q} = \sum_{z=0}^{\infty}\left(-1\right)^z \binom{q}{z} u^z$, $\left[1-u\right]^{-q} = \sum_{w=0}^{\infty}\frac{\Gamma\left(q+w\right)}{w!\Gamma(q)} u^w$: $\left|u\right|<1, q>0$. [15-16]

$$\left(1-\left(1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha}\right)^2\right)^{\theta-1} = \sum_{i=0}^{\infty}\left(-1\right)^i \binom{\theta-1}{i}\left(1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha}\right)^{2i}$$

And

$$\left(1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha}\right)^{2i+1} = \sum_{j=0}^{\infty}\left(-1\right)^j \binom{2i+1}{j}\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha j}$$

Again, using the generalized binomial expansion and the exponential expansion formula $e^{-c} = \sum_{c=0}^{\infty}\frac{\left(-1\right)^c}{c!}u^c$ [17]

Then we get expansion pdf of TLEGoIRa distribution:

$$f\left(x, \theta, \alpha, \gamma, \beta, \lambda\right) = \psi_p x^{-3} e^{-(q+1)\left(\frac{\lambda}{x}\right)^2} \tag{7}$$

Where

$$\psi_p = 4\theta\alpha\gamma\lambda^2 \sum_{i=j=k=m=w==q=0}^{\infty}\frac{\left(-1\right)^{i+j+k+w+q}}{m!}\binom{\theta}{i}\binom{2i+1}{j}\binom{\alpha\left(j+1\right)-1}{k}\binom{m}{w}\binom{-\beta\left(1+w\right)-1}{q}$$

Hence,

$$\dot{\mu}_r = E\left(X^r\right) = \psi_p \int_0^\infty x^{r-3} e^{-(q+1)\left(\frac{\lambda}{x}\right)^2} dx$$

On solving the integral part in the equation above, then the $\dot{\mu}_r$ is:

$$\acute{\mu}_r = E\left(X^r\right) = \psi_p \frac{\lambda^2 \Gamma(1-\frac{r}{2})}{\left(q+1\right)^{1-\frac{r}{2}}} \tag{8}$$

2.2.2 Incomplete Moments

Equation (7) yields the incomplete moments for the TLEGoIRa distribution with r[th] (r > 0). [18-19]

$$\acute{\mu}_r\left(u\right) = \psi_p \int_0^u x^{r-3} e^{-(q+1)\left(\frac{\lambda}{x}\right)^2} dx$$

Let $t = (q+1)\left(\frac{\lambda}{x}\right)^2 \Rightarrow x = \left(\frac{(q+1)\lambda^2}{t}\right)^{\frac{1}{2}}$

When $x = 0 \Rightarrow t = 0$, *and if* $x = u \Rightarrow t = (q+1)\left(\frac{\lambda}{u}\right)^2$, then

$$\acute{\mu}_r\left(u\right) = \frac{\psi}{2\left((q+1)\lambda^2\right)^{1-\frac{r}{2}}} \gamma\left(1-\frac{r}{2}, (q+1)\lambda^2\right) \tag{9}$$

2.2.3 Quantile Function

The quantile function of TLEGoIRa distribution is given as

$$x = Q\left(u\right) = \lambda\left\{-\log\left(1-\left(1-\left(\frac{\beta}{\gamma} log\left(1-\left(1-u^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right)^{\frac{1}{\alpha}}\right)\right)^{-\frac{1}{\beta}}\right)\right\}^{-\frac{1}{2}} \tag{10}$$

The median of TLEGoIRa distribution is obtained by setting $u = 0.5$ in equation (10)

$$x_{median} = Q\left(0.5\right) = \lambda\left\{-\log\left(1-\left(1-\left(\frac{\beta}{\gamma} log\left(1-\left(1-\left(0.5\right)^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right)^{\frac{1}{\alpha}}\right)\right)^{-\frac{1}{\beta}}\right)\right\}^{-\frac{1}{2}}$$

Table 1: *Quantiles for given parameter values of the TLEGoIRa distribution.*

	(θ, λ, α, β, γ)		
U	(0.4, 0.8, 1.2, 1.3, 2.2)	(3, 1.3, 0.9, 0.7, 3.3)	(2.3, 3, 1.7, 1.4, 1.2)
0.1	0.45276	0.59975	1.33661
0.2	0.52743	0.63546	1.41855
0.3	0.59314	0.66110	1.48066
0.4	0.65832	0.68289	1.53586
0.5	0.72712	0.70315	1.58933
0.6	0.80314	0.72329	1.64475
0.7	0.89111	0.74472	1.70628
0.8	0.99973	0.76965	1.78139
0.9	1.15324	0.80387	1.89139

2.2.4 Renyi Entropy

Define the Rényi entropy of the TLEGoIRa distribution using the following formula. [20]

$$T_R(\tau)=\frac{1}{1-\tau}log\int_0^{\infty} f^{\tau}(x)dx \quad , \ \tau>0, \tau\neq 1$$

By substitution equation (7) into the equation above:

$$T_R(\tau)=\frac{1}{1-\tau}\log\left(\psi_p\int_0^{\infty} x^{-3\tau}e^{-(q+1)\tau\left(\frac{\lambda}{x}\right)^2}dx\right)$$

The last integral, we get

$$T_R(\tau)=\frac{1}{1-\tau}\log\left(\frac{\psi_p\Gamma\left(\frac{3}{2}(\tau-1)+1\right)}{2\left(\left(q+1\right)\tau\lambda^2\right)^{\frac{1}{2}(3\tau+1)}}\right) \tag{11}$$

2.2.5 Order Statistic

The pdf of the order statistics for the TLEGoIRa distribution is obtained as follows: [21-24]

$$g_{t:n}(x)=\frac{n!}{(t-1)!(n-t)!}\left[F(X))\right]^{t-1}\left[1-F(X)\right]^{n-t}f(x) \tag{12}$$

Substituting equations (5) and (6) into equation (12), we have:

$$g_{t:n}(x)=\frac{n!}{(t-1)!(n-t)!}\left[\left(1-\left(1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right)^{-\beta}\right)}\right)^{\alpha}\right)^{2}\right)^{\theta}\right]^{t-1}\left[1-\left(1-\left(1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right)^{-\beta}\right)}\right)^{\alpha}\right)^{2}\right)^{\theta}\right]^{n-t}$$
$$\left(4\gamma\theta\alpha\lambda^{2}x^{-3}e^{-\left(\frac{\lambda}{x}\right)^{2}}\left(1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right)^{-\beta-1}e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right)^{-\beta}\right)}\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right)^{-\beta}\right)}\right)^{\alpha-1}\left(1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right)^{-\beta}\right)}\right)^{\alpha}\right)\left(1-\left(1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right)^{-\beta}\right)}\right)^{\alpha}\right)^{2}\right)^{\theta-1}\right) \quad (13)$$

2.3 Maximum Likelihood Estimation (MLE)

This section provides the method of estimation of the unknown parameters of the TLEGoIRa distribution. Suppose that $x_1, x_2, \dots, x_n$ be n^{th} independent random sample from the TLEGoIRa distribution. Then, the log-likelihood function of the TLEGoIRa distribution is given as:

$$\log l = nlog(4)+nlog(\theta)+nlog(\alpha)+nlog(\gamma)+2nlog(\lambda)-3\sum_{i=1}^{n}\log(x_i)-\lambda^{2}\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^{2}$$
$$-(\beta+1)\sum_{i=1}^{n}\log\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^{2}}\right)+\frac{\gamma}{\beta}\sum_{i=1}^{n}\log\left(\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^{2}}\right)^{-\beta}\right)\right)+(\alpha-1)\sum_{i=1}^{n}\log\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^{2}}\right)^{-\beta}\right)}\right) \quad (14)$$
$$+\sum_{i=1}^{n}\log\left(1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^{2}}\right)^{-\beta}\right)}\right)^{\alpha}\right)+(\theta-1)\sum_{i=1}^{n}\log\left(1-\left(1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^{2}}\right)^{-\beta}\right)}\right)^{\alpha}\right)^{2}\right)$$

Differentiating equation (14) with respect to each unknown parameter and equating them zero, we have:

$$\frac{\partial(l)}{\partial\theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\log\left(1-\left(1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^{2}}\right)^{-\beta}\right)}\right)^{\alpha}\right)^{2}\right) \quad (15)$$

$$\frac{\partial(l)}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}\right)-\sum_{i=1}^{n}\frac{\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha}\log\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha}}{1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha}}$$

$$+2(\theta-1)\sum_{i=1}^{n}\frac{\left(1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha}\right)\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha}\log\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}\right)}{1-\left(1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha}\right)^{2}} \tag{16}$$

$$\frac{\partial(l)}{\partial\gamma}=\frac{n}{\gamma}+\frac{1}{\beta}\sum_{i=1}^{n}\log\left(\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)\right)-(\alpha-1)\sum_{i=1}^{n}\frac{e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}e^{\frac{1}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}}{1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}}+\alpha\sum_{i=1}^{n}\frac{\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha-1}e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}e^{\frac{1}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}}{1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha}}$$

$$+2\alpha(\theta-1)\sum_{i=1}^{n}\frac{\left(1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha}\right)\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha}e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}e^{\frac{1}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}}{1-\left(1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha}\right)^{2}} \tag{17}$$

$$
\begin{aligned}
\frac{\partial(l)}{\partial\lambda} = {} & \frac{2n}{\lambda} - 2\lambda\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^2 - 2\lambda(\beta+1)\sum_{i=1}^{n}\frac{e^{-\left(\frac{\lambda}{x_i}\right)^2}}{x_i^2\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)} - 2\lambda\gamma\sum_{i=1}^{n}\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-(\beta+1)} e^{-\left(\frac{\lambda}{x_i}\right)^2}\left(\frac{1}{x_i}\right)^2 \\
& + 2\lambda\beta(\alpha-1)\sum_{i=1}^{n}\frac{e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)} e^{\frac{\gamma}{\beta}\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-(\beta+1)}} e^{-\left(\frac{\lambda}{x_i}\right)^2}}{1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}} \\
& + 2\lambda\beta\alpha\sum_{i=1}^{n}\frac{\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha-1} e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)} e^{\frac{\gamma}{\beta}\left(\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-(\beta+1)}\right)} e^{-\left(\frac{\lambda}{x_i}\right)^2}}{x^2\left(1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha}\right)} \\
& + 4\lambda\beta\alpha(\theta-1)\sum_{i=1}^{n}\frac{\left(1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha}\right)\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha-1} e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)} e^{\frac{\gamma}{\beta}\left(\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-(\beta+1)}\right)} e^{-\left(\frac{\lambda}{x_i}\right)^2}}{1-\left(1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha}\right)^2}
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
\frac{\partial(l)}{\partial\beta} = {} & -\sum_{i=1}^{n}\log\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right) + \frac{\gamma}{\beta^2}\sum_{i=1}^{n}\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\log\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right) \\
& - (\alpha-1)\sum_{i=1}^{n}\frac{e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)} e^{\frac{\gamma}{\beta^2}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)} e^{\frac{\gamma}{\beta}\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}}\log\left(e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)}{1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}} \\
& + \alpha\sum_{i=1}^{n}\frac{\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha-1} e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)} e^{\frac{\gamma}{\beta^2}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)} e^{\frac{\gamma}{\beta}\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}}\log\left(e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)}{1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha}} \\
& - 2\alpha(\theta-1)\sum_{i=1}^{n}\frac{\left(1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha}\right)\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha-1} e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)} e^{\frac{\gamma}{\beta^2}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)} e^{\frac{\gamma}{\beta}\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}}\log\left(e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)}{1-\left(1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-\beta}\right)}\right)^{\alpha}\right)^2}
\end{aligned} \tag{19}
$$

Since equations (15), (16), (17), (18), and (19) are non-linear in parameters, an iterative technique is resorted to using Newton-Raphson iterative algorithm to obtain the estimate of the parameters.

III. Results

3.1 Simulation

This section describes the conclusions of a simulation research of the TLEGoIRa distribution. The study investigates five distinct sets of parameter values: (θ =0.9, α =1.2, γ =0.3, β =0.3, λ=0.5), (θ =1.2, α =0.9, γ =0.7, β =0.3, λ=0.5), (θ =1.3, α =1.3, γ =0.8, β =0.6, λ=0.4), and (θ =2, α =2, γ =0.6, β =0.75, λ=0.2). Each parameter set yields 1000 samples, with n = 50, 100, 150, and 300. We utilize these samples to compute the mean, average bias, and root mean square error (RMSE). To calculate bias and RMSE for the calculated parameters, use the formulas below:

$$Abias\left(\hat{\alpha}\right)=\frac{\sum_{i=1}^{N}\hat{\alpha}_i}{N}-\alpha,\ and\ RMSE\left(\hat{\alpha}\right)=\sqrt{\frac{\sum_{i=1}^{N}\left(\hat{\alpha}_i-\alpha\right)^2}{N}}.$$

Tables 2 and **3** illustrate the results, which demonstrate a clear pattern: as the sample size increases, the mean parameter estimates get more precise and closer to the true values. Simultaneously, the corresponding RMSEs and Abias approach zero, proving MLEs' reliability and consistency.

Table 2: *Results from Monte Carlo simulations of the TLEGoIRa distribution*

θ =0.9, α =1.2, γ =0.3, β =0.3, λ=0.5					θ =1.2, α =0.9, γ =0.7, β =0.3, λ=0.5		
Parameter	N	Mean	RMSE	Abias	Mean	RMSE	Abias
θ	50	3.3963	4.9674	2.4963	3.7293	3.6574	2.5293
	100	2.2823	2.9373	1.3823	3.7169	3.3882	2.5169
	150	1.9920	2.1471	1.0920	3.1464	2.3203	1.9464
	300	1.5072	1.4519	0.6072	2.2718	2.2284	1.0718
α	50	1.1618	1.3257	0.0981	1.1671	1.6299	0.2671
	100	1.2951	1.1776	0.0951	1.1055	1.3425	0.2055
	150	1.1993	1.1490	0.0644	1.1023	1.3182	0.2060
	300	1.3244	1.0653	0.0455	1.0609	1.0041	0.1609
γ	50	0.2144	0.1879	0.0855	0.5295	0.6465	0.1704
	100	0.2459	0.1461	0.0540	0.5168	0.4134	0.1531
	150	0.2632	0.1382	0.0367	0.5023	0.3071	0.1276
	300	0.2599	0.1155	0.0200	0.4994	0.2535	0.1005
β	50	0.3311	0.1891	0.0396	0.3246	0.2079	0.0246
	100	0.3274	0.1864	0.0374	0.3136	0.1747	0.0136
	150	0.3233	0.1581	0.0240	0.2956	0.1504	0.0043
	300	0.3209	0.1324	0.0233	0.3020	0.1368	0.0020
λ	50	0.9800	0.9588	0.4800	1.9086	1.6432	1.4086
	100	0.8343	0.6752	0.3343	1.2474	1.4809	0.7474
	150	0.7453	0.5092	0.2453	0.9105	1.0670	0.4105
	300	0.6237	0.3144	0.1237	0.7456	0.5470	0.2456

Table 3: *Results from Monte Carlo simulations of the TLEGoIRa distribution*

		θ =1.3, α =1.3, γ =0.8, β =0.6, λ=0.4			θ =2, α =2, γ =0.6, β =0.75, λ=0.2		
Parameter	N	Mean	RMSE	Abias	Mean	RMSE	Abias
θ	50	3.9020	3.2589	2.6020	4.6867	4.6171	2.6867
	100	3.1574	2.2916	1.8574	3.9532	3.5984	1.9532
	150	2.7310	2.6574	1.4310	3.4859	2.7741	1.4859
	300	2.9938	1.1012	0.6938	3.0303	2.0658	1.0303
α	50	1.5031	2.1812	0.2031	2.2077	3.9702	0.2077
	100	1.4930	2.1254	0.3930	2.6847	3.9583	0.0847
	150	1.4756	2.0739	0.2756	2.0827	3.9374	0.0827
	300	1.3273	1.3373	0.2273	2.4920	3.0661	0.0720
γ	50	0.7726	1.0620	0.9709	0.4879	0.3450	0.1920
	100	0.7423	0.5469	0.1073	0.4565	0.2704	0.1434
	150	0.7259	0.4181	0.0976	0.4367	0.2280	0.1132
	300	0.7016	0.3897	0.0594	0.5271	0.1969	0.0728
β	50	0.7850	0.5824	0.1850	1.3345	1.2143	0.5845
	100	0.7450	0.5133	0.1450	1.2569	1.1121	0.5069
	150	0.7342	0.4793	0.1342	1.3577	1.1337	0.4877
	300	0.6869	0.4226	0.0869	1.1989	0.9787	0.4489
λ	50	2.0712	2.7215	1.6719	1.0376	1.9009	0.8376
	100	1.3579	2.1418	0.9579	0.6940	1.0649	0.4940
	150	0.9976	1.3302	0.5976	0.6385	0.9570	0.4385
	300	0.7083	0.7030	0.3083	0.4223	0.4678	0.2223

3.2 Applications

In this section, the practical use of the TLEGoIRa distribution is explored via two real-life data sets. Table 4 displays the cdf of the models, which will be compared to the TLEGoIRa distribution.

Table 4: *CDF for the Comparative distributions*

Distribution	**CDF**
Truncated Exponentiated Exponential Gompertz inverse Rayleigh (TEEGoIRa) [15]	$\frac{\left(1-exp\left(\left(-\theta\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right)^{-\beta}\right)}\right)\right)\right)\right)^{\alpha}}{\left(1-exp\left(-\theta\right)\right)^{\alpha}}$
Beta Gompertz inverse Rayleigh (BeGoIRa) (New)	$pbeta\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right)^{-\beta}\right)},\theta,\alpha\right)$

Kumaraswamy Gompertz inverse Rayleigh (KuGoIRa) (New)	$1-\left(1-\left(1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right)^{-\beta}\right)}\right)^{\theta}\right)^{\alpha}$
Exponential Generalized Gompertz inverse Rayleigh (EGGoIRa) (New)	$\left(1-\left(1-1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right)^{-\beta}\right)}\right)^{\theta}\right)^{\alpha}$
Weibull Gompertz inverse Rayleigh (WeGoIRa) (New)	$1-exp\left(-\alpha^{-\theta}\left(-log\left(1-1-e^{\frac{\gamma}{\beta}\left(1-\left(1-e^{-\left(\frac{\lambda}{x}\right)^{2}}\right)^{-\beta}\right)}\right)\right)^{\theta}\right)$

The first dataset (I), shows the tensile strength in GPa of 69 carbon fibers evaluated at 20mm gauge lengths. It was utilized by Bader and Priest [25]
(1.312, 1.314, 1.479 ,1.552,1.700 ,1.803, 1.861 ,1.865 ,1.944, 1.958 ,1.966, 1.997 ,2.006, 2.021 ,2.027, 2.055, 2.063 ,2.098, 2.140, 2.179 ,2.224 ,2.240, 2.253 ,2.270, 2.272, 2.274, 2.301, 2.301 ,2.359 ,2.382 ,2.382 ,2.426 ,2.434, 2.435, 2.478 ,2.490, 2.511, 2.514, 2.535 ,2.554, 2.566, 2.570, 2.586, 2.629 ,2.633, 2.642, 2.648, 2.684 ,2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848 ,2.880, 2.954, 3.012, 3.067 ,3.084, 3.090, 3.096, 3.128, 3.233, 3.433 ,3.585, 3.585).

The second dataset (II), shown here represents COVID-19 mortality rate data for Mexico over a 108-day period from March 4th to July 20, 2020. It was utilized by Almongy et al. [26]

(8.826, 6.105 ,10.383, 7.267 ,13.220, 6.015 ,10.855, 6.122 ,10.685, 10.035, 5.242 ,7.630 ,14.604 ,7.903 ,6.327 ,9.391 ,14.962 ,4.730 , 3.215 ,16.498, 11.665 ,9.284, 12.878, 6.656,3.440 ,5.854, 8.813 , 10.043, 7.260, 5.985, 4.424 ,4.344 ,5.143 ,9.935 ,7.840 ,9.550 , 6.968 ,6.370 ,3.537 ,3.286 ,10.158, 8.108 ,6.697 ,7.151 ,6.560 , 2.988 ,3.336 ,6.814 ,8.325 ,7.854 ,8.551 ,3.228, 3.499 ,3.751, 7.486 ,6.625 ,6.140 ,4.909 ,4.661 ,1.867 ,2.838 ,5.392, 12.042, 8.696 ,6.412 ,3.395 ,1.815 ,3.327 ,5.406 ,6.182 ,4.949 ,4.089 , 3.359 ,2.070, 3.298 ,5.317 ,5.442 ,4.557 ,4.292 ,2.500 ,6.535 , 4.648 ,4.697 ,5.459 ,4.120, 3.922 ,3.219, 1.402 ,2.438, 3.257 , 3.632, 3.233 ,3.027, 2.352 ,1.205 ,2.077, 3.778, 3.218, 2.926, 2.601, 2.065, 1.041, 1.800, 3.029, 2.058, 2.326, 2.506, 1.923).

Tables 5, and 6 for data (I), and (II) show that the TLEGoIRa distribution beats Comparative distributions in several key criteria, including Akaike information criterion (AIC) , Consistent AIC (CAIC),Bayesian information criterion (BIC), Hannan-Quinn information (HQIC),Kolmogorov-Smimov (KS) ststistic, Anderson-Darling (A), and Cramer-von Mises (W) values. The lower values of these measures for the TLEGoIRa distribution are preferable for comparative distributions.

Figures 3, and **5** show the Fitted densities for Data I, and II, respectively, and **Figures 4**, and **5** show the empirical cdf plots for Data I and II. These visualizations enable us to evaluate the goodness of fit and see how well the model fits the data.

Table 5: *Goodness-of-Fit Statistics for Data* I

Dist.	MLEs	-2L	AIC	CAIC	BIC	HQIC	W	A	K-S	p-value
TLEGoIRa	$\hat{\theta}$:1.4173 $\hat{\alpha}$:1.4214 $\hat{\lambda}$:0.1896 $\hat{\beta}$:1.6041 $\hat{\gamma}$:2.0224	48.81	107.62	108.58	118.79	112.05	0.0178	0.1600	0.0414	0.9997
TEEGoIRa	$\hat{\theta}$:0.0443 $\hat{\alpha}$:0.8522 $\hat{\lambda}$:3.3049 $\hat{\beta}$:3.5479 $\hat{\gamma}$:4.1427	49.36	108.74	109.69	119.91	113.17	0.0356	0.2930	0.0530	0.9900
BeGoIRa	$\hat{\theta}$:1.7430 $\hat{\alpha}$:1.0087 $\hat{\lambda}$:0.0292 $\hat{\beta}$:0.7674 $\hat{\gamma}$:1.8205	50.50	111.03	111.99	122.21	115.47	0.0165	0.1487	0.0982	0.5177
KuGoIRa	$\hat{\theta}$:1.7872 $\hat{\alpha}$:1.0734 $\hat{\lambda}$:0.0238 $\hat{\beta}$:0.7657 $\hat{\gamma}$:1.8912	49.63	109.29	110.24	120.46	113.72	0.0164	0.1481	0.0750	0.8313
EGGoIRa	$\hat{\theta}$:0.9743 $\hat{\alpha}$:1.7609 $\hat{\lambda}$:0.0223 $\hat{\beta}$:0.7207 $\hat{\gamma}$:1.8454	50.12	110.28	111.23	121.45	114.71	0.0165	0.1483	0.0826	0.7330
WeGoIRa	$\hat{\theta}$:1.8695 $\hat{\alpha}$:1.0161 $\hat{\lambda}$:0.0322 $\hat{\beta}$:0.7167 $\hat{\gamma}$:1.4519	49.64	109.29	110.24	120.46	113.72	0.0308	0.2515	0.0598	0.9659

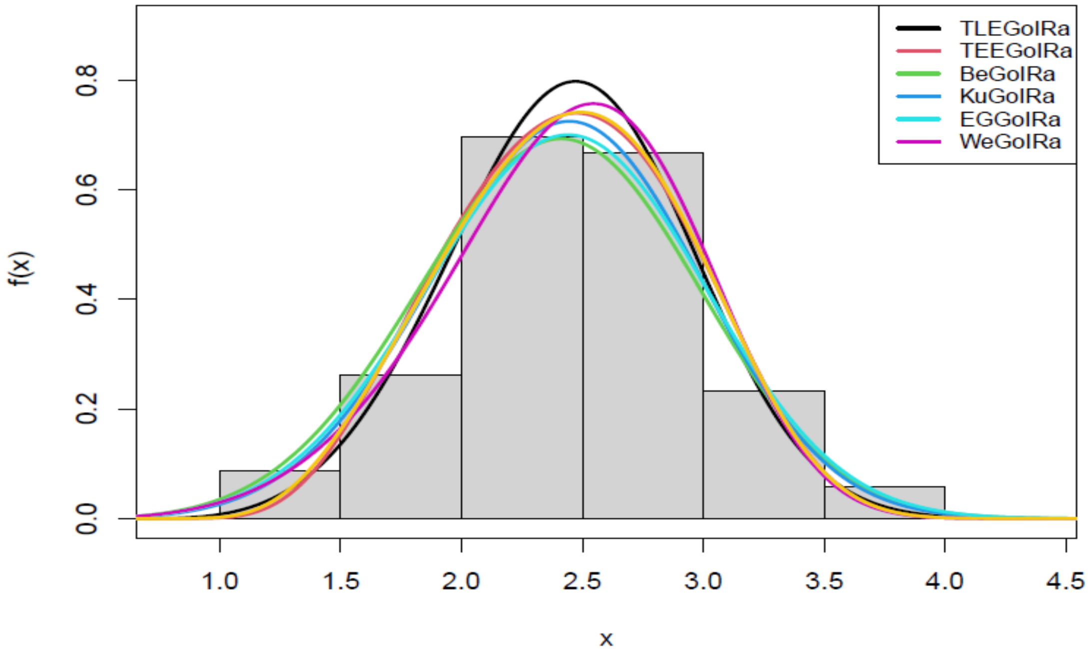

Figure 3: *Fitted densities for Data I*

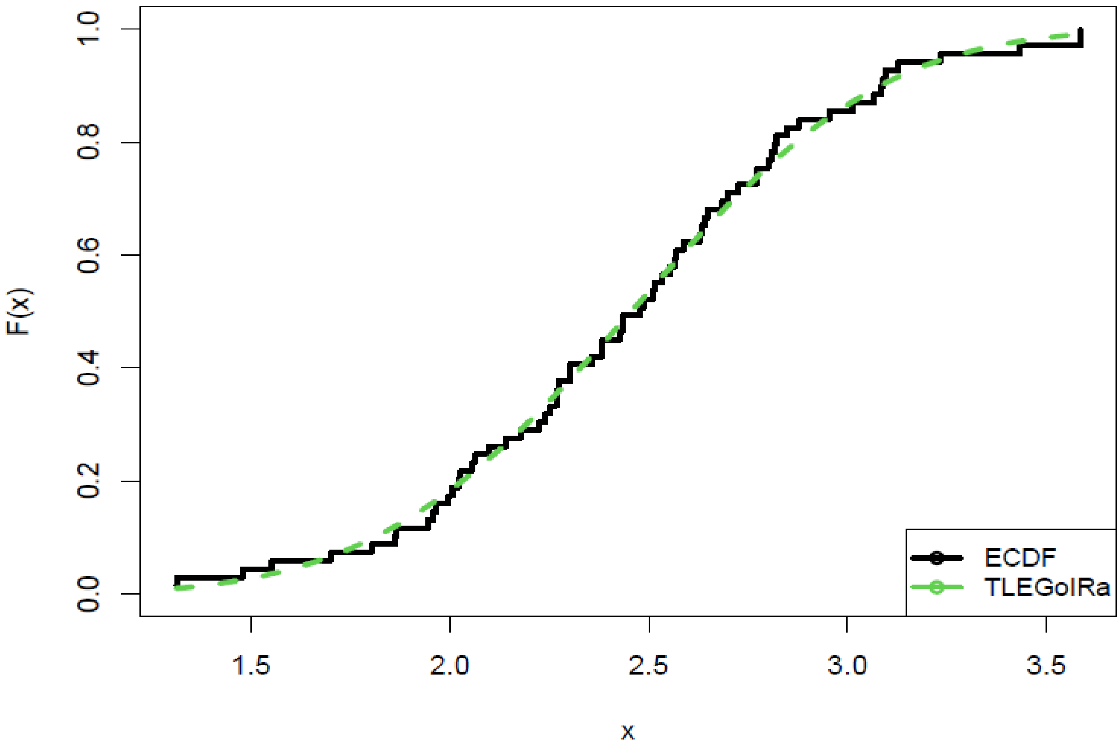

Figure 4: *Empirical cdf plot for Data I*

Table 6: *Goodness-of-Fit Statistics for Data II*

Dist.	MLEs	-2L	AIC	CAIC	BIC	HQIC	W	A	K-S	p-value
TLEGoIRa	$\hat{\theta}$:0.6769 $\hat{\alpha}$:1.4676 $\hat{\lambda}$:0.1046 $\hat{\beta}$:2.1082 $\hat{\gamma}$:0.7180	265.12	540.24	540.83	553.65	545.68	0.0357	0.2127	0.0565	0.8805
TEEGoIRa	$\hat{\theta}$:0.9677 $\hat{\alpha}$:1.1598 $\hat{\lambda}$:0.0450 $\hat{\beta}$:1.2991 $\hat{\gamma}$:0.8986	265.62	541.25	541.84	554.66	546.69	0.0517	0.3001	0.0646	0.7577
BeGoIRa	$\hat{\theta}$:1.2413 $\hat{\alpha}$:0.9903 $\hat{\lambda}$:0.0079 $\hat{\beta}$:0.3841 $\hat{\gamma}$:0.8618	267.77	545.54	546.13	558.95	550.98	0.0909	0.5702	0.0833	0.4415
KuGoIRa	$\hat{\theta}$:1.1977 $\hat{\alpha}$:0.9940 $\hat{\lambda}$:0.0108 $\hat{\beta}$:0.3686 $\hat{\gamma}$:0.7788	268.42	546.85	547.44	560.26	552.29	0.0802	0.4945	0.0683	0.6936
EGGoIRa	$\hat{\theta}$:0.9884 $\hat{\alpha}$:1.2186 $\hat{\lambda}$:0.0095 $\hat{\beta}$:0.3768 $\hat{\gamma}$:0.8146	267.83	545.66	546.25	559.07	551.10	0.0846	0.5260	0.0605	0.8236
WeGoIRa	$\hat{\theta}$:1.4253 $\hat{\alpha}$:1.0311 $\hat{\lambda}$:0.1373 $\hat{\beta}$:1.3555 $\hat{\gamma}$:0.51167	265.26	540.56	541.15	553.97	546.00	0.0433	0.2484	0.0607	0.8208

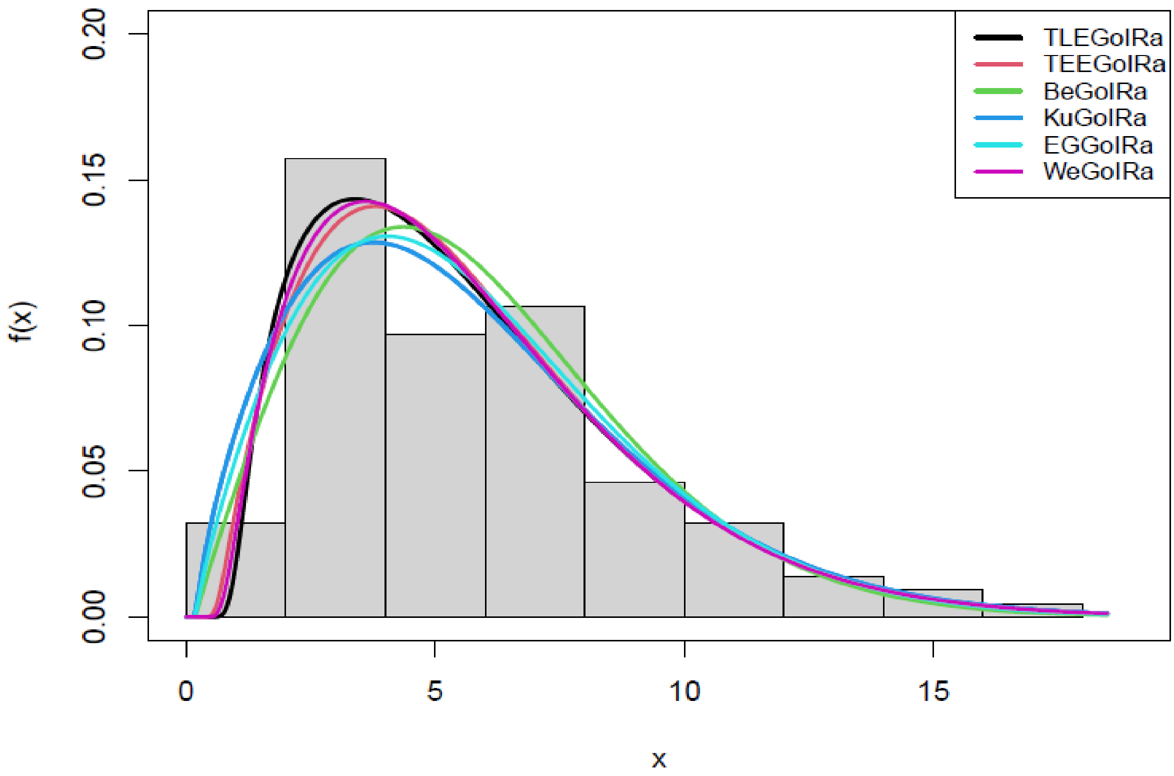

Figure 5: *Fitted densities for Data II*

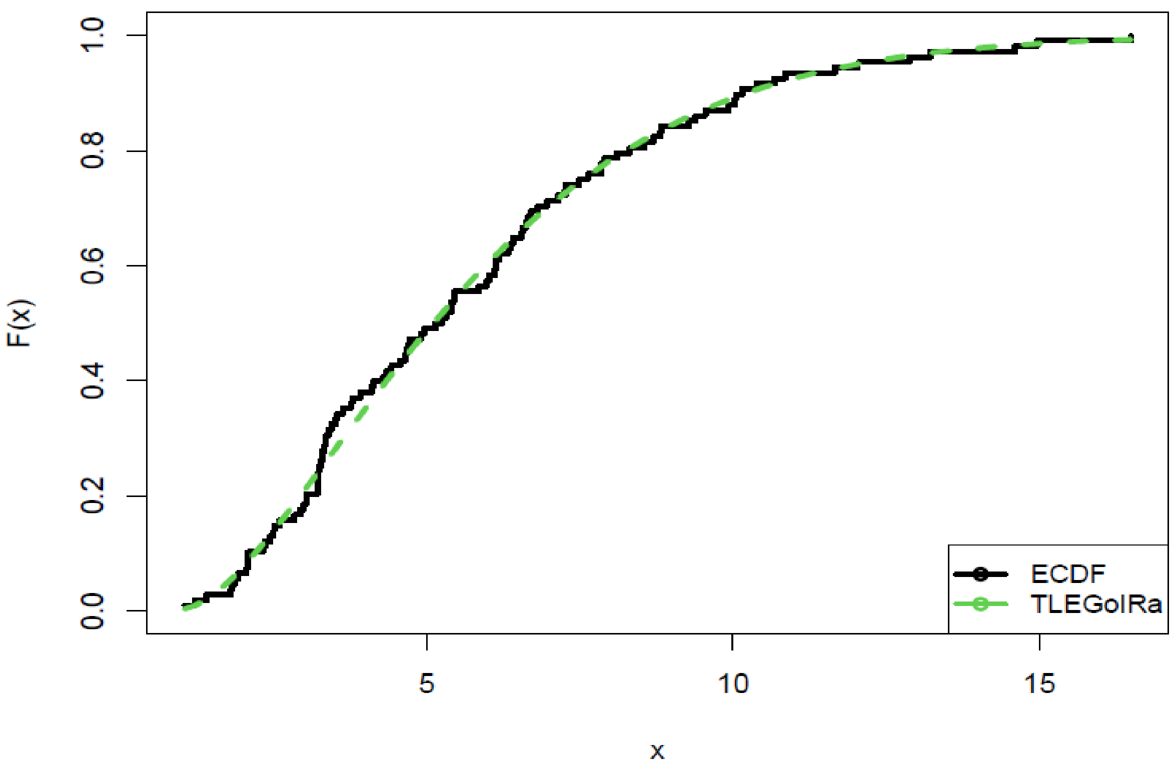

Figure 6: *Empirical cdf plot for Data II*

IV. Discussion

In this paper, we introduce a novel extension of the GoIRa model, referred to as the TLEGoIRa distribution. We provide explicit expressions for various statistical properties, including ordinary moments, incomplete moments, quantile function, rényi entropy and order statistics of the TLEGoIRa distribution. To estimate the unknown parameters, we employ the method of maximum likelihood estimation and undertake a simulation research to investigate the average bias and root mean square error (RMSE) as sample sizes rise. The results show a consistent model performance, with diminishing Abias and Rmse as the sample size grows. Furthermore, we validate the effectiveness of the proposed model through two real-life applications. Through these applications, we demonstrate that the proposed model outperforms several other competitive models in terms of goodness of fit. This empirical evidence underscores the enhanced flexibility and robustness of our novel model in accurately representing and modeling the characteristics of the given datasets when compared to the other competitive models under consideration.

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