RT&A, No 3 (79)
Volume 19, September 2024

ALGORITHMS FOR APPROXIMATING A FUNCTION BASED ON INACCURATE OBSERVATIONS

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Abstract

This paper is devoted to the approximation of a function by a trigonometric polynomial based on its inaccurate values at selected points. Two methods of observation are considered. The first method is to make observations at points evenly distributed on the segment where the function is specified. The second method is to take observations at the points of division into a finite number of equal parts of the neighbourhoods of the selected points. Upper estimates of the standard deviation of the function from trigonometric polynomials are constructed and the rate of their convergence is estimated. Differences were found in the computational complexity of these approximations and in the number of observations of the function values at the selected points. Thus, the problem of approximating a function from inaccurate observations of their values at selected points is a multi-criteria one and its solution depends on the choice of observation points.

Keywords: trigonometric functions, inaccurate observations, error in function estimation, experimental plan

1. Introduction

This paper is devoted to the approximation of the periodic function from inaccurate observations. To solve this problem, Chebyshev, Hermite, Jacobi, Laguerre polynomials, trigonometric polynomials are used for the exact values of the function at the selected points (see, for example, [1], [2]). However, the task becomes significantly more complicated if it is necessary to evaluate the function based on inaccurate observations at selected points. In this case, there are many different solutions that need to be compared by various indicators (solution error, computational complexity, number of observation points). In this paper, two solutions to this problem are proposed and compared.

Trigonometric polynomials were used to approximate the function from inaccurate observations. The first method of approximation consists in observing the function at points evenly distributed over the segment of its assignment. In the second method, observations are considered at the points of division into a finite number of equal parts of the neighbourhoods of the selected points. In both cases, upper estimates of the standard deviation of the approximation of the function from its exact value are constructed. Despite the proximity of the upper estimates obtained, differences in the computational complexity of these approximations are found in the number of observations of the function values at the selected points.

The proposed algorithm for estimating the value of a function by a trigonometric polynomial using inaccurate deterministic or stochastic observations, unlike classical algorithms, allows us to estimate the rate of convergence of the estimates obtained to the estimated parameters.

And considering a small interval of time observation makes it possible to build an experiment planning procedure. The authors previously used this idea to solve the problem of estimating the parameters of a number of ordinary differential equations and their systems, partial differential equations [3], [4].

The paper considers a function $f(x)$ that is continuously differentiable on the segment $[0, 2\pi]$. This function decomposes into a Fourier series in the space $L_2[0, 2\pi]$. Denote

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx)), \tag{1}$$

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(kx) dx, \ b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(kx) dx, \ k = 0, 1, ..., n.$$

It is known (see, for example, [5]) that under given conditions for the function $f(x)$ there exists a number C such that $|a_k| \le C/k, \ |b_k| \le C/k, \ k = 1, \dots,$ and, therefore,

$$\pi \sum_{k=n+1}^{\infty} (a_k^2 + b_k^2) = \int_0^{2\pi} (f(x) - f_n(x))^2 dx = D(n) = O(n^{-1}). \tag{2}$$

In this paper, we will consider two different estimates of the function $f(x)$ based on inaccurate observations. The first estimate of $\widehat{f}_n(x)$ of the function $f(x)$ is constructed as follows. Let $t^a_{p,k}, \ t^b_{p,k}, \ \varepsilon^a_{p,k}, \ \varepsilon^b_{p,k}, p = 1, \dots, \ k = 0, ...,$ independent random variables. Moreover, the random variables $t^a_{p,k}, \ t^b_{p,k}$ have a uniform distribution on the segment $[0, 2\pi]$, and random variables $\varepsilon^a_{p,k}, \ \varepsilon^b_{p,k}$ characterizing measurement errors have zero mean and variance σ^2. Then we define random variables

$$\widehat{a}_k = \frac{2}{m} \sum_{p=0}^{m-1} (f(t^a_{p,k}) + \varepsilon^a_{p,k}) \cos(kt^a_{p,k}), \ \widehat{b}_k = \frac{2}{m} \sum_{p=0}^{m-1} (f(t^b_{p,k}) + \varepsilon^b_{p,k}) \sin(kt^b_{p,k}) \tag{3}$$

and we will make the first assessment

$$\widehat{f}_n(x) = \frac{\widehat{a}_0}{2} + \sum_{k=1}^{n} (\widehat{a}_k \cos(kx) + \widehat{b}_k \sin(kx)). \tag{4}$$

The second estimate $\bar{f}(x)$ of the function $f(x)$ is based on inaccurate observations in the following way. Let $x_p = 2\pi p/m, \ p = 0, \dots, m-1,$and random variables $\varepsilon_{p,j}, \ p = 0, \dots, m-1, \ j = -(2N+1), \dots, 2N+1,$ characterizing measurement errors are independent, and have zero mean and variance σ^2.Let's assume random measurements of quantities $f(x_p)$ equal

$$\bar{f}(x_p) = \frac{1}{2N+1} \sum_{j=-N}^{N} (f(x_p + jh) + \varepsilon_{p,j}), \ h = N^{-\alpha}.$$

Let's construct a second estimate of the function $f(x)$

$$\bar{f}_n(x) = \frac{\bar{a}_0}{2} + \sum_{k=1}^{n} (\bar{a}_k \cos(kx) + \bar{b}_k \sin(kx)). \tag{5}$$

$$\bar{a}_k = \frac{2}{m} \sum_{p=0}^{m-1} \bar{f}_p \cos(kx_p), \ \bar{b}_k = \frac{2}{m} \sum_{p=0}^{m-1} \bar{f}_p \sin(kx_p), \ k = 0, 1, \dots, n.$$

Our task is to build upper bounds

$$M \int_0^{2\pi} (f(x) - \widehat{f}_n(x))^2 dx, \ M \int_0^{2\pi} (f(x) - \bar{f}_n(x))^2 dx$$

and compare them.

2. The error of the first estimate

Obviously, the inequality is fair

$$M\int_0^{2\pi}(f(x)-\widehat{f}_n(x))^2dx=\int_0^{2\pi}M[(f(x)-f_n(x))+(f_n(x)-\widehat{f}_n(x))]^2dx\le$$

$$\le 2\left[\int_0^{2\pi}(f(x)-f_n(x))^2dx+M\int_0^{2\pi}(f_n(x)-\widehat{f}_n(x))^2dx\right].$$

In turn,

$$M\int_0^{2\pi}(f_n(x)-\widehat{f}_n(x))^2dx=\pi M\left[\left(\frac{a_0-\widehat{a}_0}{2}\right)^2+\sum_{k=1}^n(a_k-\widehat{a}_k)^2+\sum_{k=1}^n(b_k-\widehat{b}_k)^2\right].$$

Then it is not difficult to get from the formulas (4)

$$M\widehat{a}_0=a_0\ \Rightarrow\ M\left(\frac{a_0-\widehat{a}_0}{2}\right)^2=\frac{1}{m}\left(\sigma^2+\frac{1}{2\pi}\int_0^{2\pi}f^2(x)dx-a_0^2\right),$$

$$M\widehat{a}_k=a_k\ \Rightarrow\ M(a_k-\widehat{a}_k)^2\le\frac{4}{m}\left(\sigma^2+\frac{1}{2\pi}\int_0^{2\pi}f^2(x)dx-a_k^2\right),\ k=1,2,\dots,$$

$$M\widehat{b}_k=b_k\ \Rightarrow\ M(b_k-\widehat{b}_k)^2\le\frac{4}{m}\left(\sigma^2+\frac{1}{2\pi}\int_0^{2\pi}f^2(x)dx-b_k^2\right),\ k=1,2,\dots.$$

It follows from this and from the formula (2) that

$$M\int_0^{2\pi}(f(x)-\widehat{f}_n(x))^2dx\le 2\left[D(n)+\frac{\pi(8n+1)}{m}\left(\sigma^2+\frac{1}{2\pi}\int_0^{2\pi}f^2(x)dx\right)\right] \tag{6}$$

and, therefore, due to the formulas (2), (6) we have

$$M\int_0^{2\pi}(f(x)-\widehat{f}_n(x))^2dx=O(n^{-1})+O(nm^{-1}). \tag{7}$$

In particular, for $n=[m^{1/2}]$ we have (here $[a]$ is the integer part of the real number a)

$$M\int_0^{2\pi}(f(x)-\widehat{f}_n(x))^2dx=O(m^{-1/2}). \tag{8}$$

3. The error of the second estimate

Let $\Delta f_p=\bar{f}(x_p)-f(x_p)$, first evaluate $M(\Delta f_p)^2$. From the continuous differentiability of the function $f(x)$ by $[0,2\pi]$ we have

$$\max(\sup_{0\le x\le 2\pi}|f(x)|,\ \sup_{0\le x\le 2\pi}|f'(x)|)=C<\infty.$$

Then

$$M(\Delta f_p)^2=M\left[\frac{1}{2N+1}\sum_{j=-N}^N(f(x_p+jh)+\varepsilon_{p,j})-f(x_p)\right]^2=$$

$$=M\left[\frac{1}{2N+1}\sum_{j=-N}^N(f(x_p+jh)+\varepsilon_{p,j}-f(x_p))\right]^2=$$

$$= M\left[\frac{1}{2N+1}\sum_{j=-N}^{N}(f(x_p+jh)-f(x_p))+\frac{1}{2N+1}\sum_{j=-N}^{N}\varepsilon_{p,j}\right]^2=$$

$$=\left[\frac{1}{2N+1}\sum_{j=-N}^{N}(f(x_p+jh)-f(x_p))\right]^2+\frac{\sigma^2}{2N+1}\leq$$

$$\leq\left[\frac{1}{2N+1}\sum_{j=-N}^{N}C|j|h\right]^2+\frac{\sigma^2}{2N+1}=O(h^2N^2)+O(N^{-1}).$$

Therefore, the relation is fulfilled

$$\sup_{0\leq p\leq m-1} M(\Delta f_p)^2=O(h^2N^2)+O(N^{-1}).$$

In particular, for $h=N^{-3/2}$ we get

$$\sup_{0\leq p\leq m-1} M(\Delta f_p)^2=O(N^{-1}). \tag{9}$$

Using this definition of the estimate $\bar{f}_p$, we construct an estimate of the error of the function $\bar{f}_n(x)$.

Denote

$$f_n^*(x)=\frac{a_0^*}{2}+\sum_{k=1}^{n}(a_k^*\cos(kx)+b_k^*\sin(kx)), \tag{10}$$

$$a_k^*=\frac{2}{m}\sum_{p=0}^{m-1}f_p\cos(kx_p),\ b_k^*=\frac{2}{m}\sum_{p=0}^{m-1}f_p\sin(kx_p),\ k=0,1,\dots,n.$$

Consider

$$M\int_0^{2\pi}(f(x)-\widehat{f}_n(x))^2dx=\int_0^{2\pi}M[(f(x)-f_n(x))+(f_n(x)-f_n^*(x))+(f_n^*(x)-\widehat{f}_n(x))]^2dx\leq$$

$$\leq 3\left[\int_0^{2\pi}(f(x)-f_n(x))^2dx+\int_0^{2\pi}(f_n(x)-f_n^*(x))^2dx+\int_0^{2\pi}M(f_n^*(x)-\widehat{f}_n(x))^2dx\right]. \tag{11}$$

Let's focus first on the assessment $\int_0^{2\pi}(f(x)-f_n^*(x))^2dx$.

$$\int_0^{2\pi}(f_n(x)-f_n^*(x))^2dx=\int_0^{2\pi}dx\left[\frac{a_0-a_0^*}{2}+\sum_{k=1}^{n}(a_k-a_k^*)\cos(kx)+\sum_{k=1}^{n}(b_k-b_k^*)\sin(kx)\right]^2=$$

$$=\frac{\pi}{2}\left[(a_0-a_0^*)^2+2\sum_{k=1}^{n}\left[(a_k-a_k^*)^2+(b_k-b_k^*)^2\right]\right].$$

We have

$$(a_0-a_0^*)^2=\left(\frac{1}{\pi}\int_0^{2\pi}f(x)dx-\frac{2}{m}\sum_{p=0}^{m-1}f(x_p)\right)^2=\left(\frac{1}{\pi}\sum_{p=0}^{m-1}\int_{x_p}^{x_{p+1}}f(x)dx-\frac{2}{m}\sum_{p=0}^{m-1}f(x_p)\right)^2=$$

$$=\left(\frac{1}{\pi}\sum_{p=0}^{m-1}\int_{x_p}^{x_{p+1}}(f(x)-f(x_p))dx\right)^2\leq\left(\frac{1}{\pi}\sum_{p=0}^{m-1}\int_{x_p}^{x_{p+1}}\frac{2\pi C}{m}dx\right)^2\leq\frac{16\pi^2C^2}{m^2}.$$

It is not difficult to get when $x_p\leq x\leq x_{p+1},\ k=0,1,\dots,n-1,$

$$|f(x)\cos(kx)-f(x_p)\cos(kx_p)|\leq|f(x)|\cdot|\cos(kx)-\cos(kx_p)|+$$

$$+|\cos(kx_p)|\cdot\left|f(x)-f(x_p)\right|\leq\frac{2\pi C(k+1)}{m}.$$

It follows that

$$|a_k - a_k^*| = \left| \frac{1}{\pi} \int_0^{2\pi} f(x)\cos(kx)dx - \frac{2}{m} \sum_{p=0}^{m-1} f(x_p)\cos(kx_p) \right| =$$

$$= \left| \frac{1}{\pi} \sum_{p=0}^{m-1} \int_{x_p}^{x_{p+1}} (f(x)\cos(kx) - f(x_p)\cos(kx_p)) \right| dx \le \frac{4\pi C(k+1)}{m}.$$

Then

$$\sum_{k=0}^{n} (a_k - a_k^*)^2 \le \frac{16C^2\pi^2(n+1)(n+2)(2n+3)}{6m^2} = O(n^3 m^{-2}),$$

By analogy, we have

$$\sum_{k=0}^{n} (b_k - b_k^*)^2 \le \frac{16C^2\pi^2(n+1)(n+1)(2n+3)}{6m^2} = O(n^3 m^{-2}).$$

Therefore,

$$\int_0^{2\pi} (f_n(x) - f_n^*(x))^2 dx = O(n^3 m^{-2}). \tag{12}$$

Let's now move on to the evaluation of $M \int_0^{2\pi} (f_n^*(x) - \bar{f}(x))^2 dx$, by calculating first

$$M \int_0^{2\pi} (f_n^*(x) - \bar{f}_n(x))^2 dx = \pi M \left(\frac{(a_0^* - \bar{a}_0)^2}{4} + \sum_{k=1}^{n} M(a_k^* - \bar{a}_k)^2 + \sum_{k=1}^{n} M(b_k^* - \bar{b}_k)^2 \right). \tag{13}$$

And then let's use the equalities

$$A = \sum_{k=1}^{n} M(a_k^* - \bar{a}_k)^2 = \sum_{k=1}^{n} M \left(\frac{2}{m} \sum_{p=0}^{m-1} \Delta f_p \cos(kx_p) \right)^2 =$$

$$= \frac{4}{m^2} \sum_{k=1}^{n} M \left(\sum_{p=0}^{m-1} \Delta f_p \cos(kx_p) \sum_{q=0}^{m-1} \Delta f_q \cos(kx_q) \right) =$$

$$= \frac{4}{m^2} \sum_{p=0}^{m-1} \sum_{q=0}^{m-1} M\Delta f_p \Delta f_q \sum_{k=1}^{n} \cos(kx_p)\cos(kx_q).$$

Therefore, we have

$$|A| \le \frac{4n}{m^2} \sum_{p=0}^{m-1} \sum_{q=0}^{m-1} |M\Delta f_p \Delta f_q| \le \frac{2n}{m^2} \sum_{p=0}^{m-1} \sum_{q=0}^{m-1} M((\Delta f_p)^2 + (\Delta f_q)^2) \le 4n \sup_{0 \le p \le m-1} M(\Delta f_p)^2.$$

From these relations and the formula (9) we obtain $|A| = O(nN^{-1})$.

In turn,

$$M(a_0^* - \bar{a}_0)^2 = \frac{4}{m^2} M \left(\sum_{p=0}^{m-1} \Delta f_p \right)^2 = O(N^{-1}).$$

Then

$$\sum_{k=0}^{n} M(a_k^* - \bar{a}_k)^2 = O(nN^{-1}). \tag{14}$$

Similarly, it is not difficult to obtain equality

$$\sum_{k=0}^{n} M(b_k^* - \bar{b}_k)^2 = O(nN^{-1}) \tag{15}$$

Combining the formulas (13) - (15), we get

$$M\int_0^{2\pi}(f_n^*(x)-\bar{f}_n(x))^2dx = O(nN^{-1}). \tag{16}$$

Finally from the formulas (2), (11), (12), (16) we come to the ratio

$$M\int_0^{2\pi}(f(x)-\bar{f}_n(x))^2dx = O(n^{-1})+O(n^3m^{-2})+O(nN^{-1}). \tag{17}$$

In particular, for $n = m^{1/2}$ and $N = m$ we get

$$M\int_0^{2\pi}(f(x)-\bar{f}_n(x))^2dx = O(m^{-1/2}).$$

4. Conslusion

From formulas (8), (17) it follows that the estimation error $\bar{f}_n(x)$, as well as the estimation error $\widehat{f}_n(x)$ are equal to $O(m^{-1/2})$. In turn, the number of observations in the first case is equal to $O(nm) = O(m^{3/2})$, and in the second case is equal to $O(Nm) = O(m^2)$. However, it should be noted that the formula for calculating the Fourier coefficients (5) can be made more economical using the fast Fourier transform method (see, for example, [6]).

This work was supported by the Ministry of Science and Higher Education of Russian Federation (Agreement No 075-02-2024-1440).

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