APPLICATION OF EXTENDED LOMAX DISTRIBUTION ON THE RELIABILITY ANALYSIS OF SOLAR PHOTOVOLTAIC SYSTEM

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Abstract

In this study, a novel distribution called the Extended-Lomax distribution which generalizes the existing Lomax distribution and has increasing and decreasing shapes for the hazard rate function was proposed. Various structural properties of the new proposed distribution are derived including the survival function, hazard function, and rth moment. The probability density function (PDF) plots indicated that the distribution is skewed to the right. To estimate the parameters of the newly proposed distribution, two estimation methods which include the Maximum likelihood approach and Method of Moments was employed. The main objective of the proposed distribution's construction was to increase the adaptability of the current Lomax distributions so that they could better suit reliability data sets than alternative candidate distributions with an equivalent number of parameters. This distribution should be able to eliminate the Heavy-tail of the current distribution and model both monotonic and non-monotonic patterns of failure rates. Solar photovoltaic system reliability data was used to evaluate the performance of the proposed Extended Lomax distribution as well as the estimation methods.

Keywords: Lomax distribution, Moments, Heavy-tail, monotonic, failure rates, reliability

1. Introduction

Reliability analysis plays a crucial role in ensuring the dependable performance of industrial systems. The Lomax distribution, a popular probability distribution for modeling lifetime data and reliability, has been employed in various applications. However, to improve the accuracy of reliability assessments for a repairable industrial system, it is essential to extend the Lomax distribution and adapt it to the characteristics of the data collected from that system. The moments and inference for the order statistics and generalized order statistics are given in [1], [2,] and [3], respectively. Assuming all distributional assumptions are satisfied, the goodness of fit between the probability distribution and the provided data sets plays a critical role in the precision of parametric statistical inference and data set modeling. A lot of studies have gone into creating distributions with more flexible and desirable features so that real-world data sets with different densities and failure rates can be adequately modeled. Currently, researchers are focused on creating new hybrid distributions that generalize existing ones, aiming to achieve better data modeling capabilities. These hybrid distributions are formed by combining a baseline distribution with a family distribution. Several authors have extensively reviewed different families of distributions Hamedani et al. [4]. The distributional properties, estimation and inference of the Lomax distribution are described in the literature as follows. In record value theory, some properties and moments for the Lomax distribution have been discussed in [5], [6], [7], [8]. Reliability analyses of solar photovoltaic system using Gumbel-Hougaard family copula distribution as studied by Maihulla et al. [16]. RAMD analyses was used by Anas and Yusuf [17] for analyzing the photovoltaic system. Performance prediction of small solar system for house used was studied by Anas and Yusuf [18].

This distribution was constructed with the primary goal of improving the flexibility of classical distributions so that they can fit survival data sets better than other candidate distributions with an equal number of parameters. This distribution should be capable of modeling various types of failure rates, including monotonic and non-monotonic patterns. The Lomax distribution, also known as the Pareto distribution of the second kind with two parameters (α, λ), has attracted significant attention from theoretical and statistical researchers due to its applications in reliability and lifetime testing studies. Lomax first introduced and studied this distribution in 1954, and it has since been utilized for analyzing business failures, as well as in economic, behavioral, scientific, and traffic modeling. The solar energy conversion into electricity is a very promising technique, knowing that the source is free, clean and abundant in several countries. However, the effect of the solar cell's temperature on the photovoltaic panel performance and lifespan remains one of the major disadvantages of this technology.

The present research has been categorized into seven main sections. Introduction is the first section. Proposed extended Lomax distribution is presented in section 2. Section 3 comprised up of the estimation methods used in the research. Mathematical properties of the proposed distribution were presented in section 4. Application of the proposed distribution to the Solar photovoltaic system were presented in section 5 of the paper. Summary and conclusion were presented in section 6 and 7 respectively.

1.1 Reliability theory

Reliability theory is a branch of applied mathematics that focuses on the study of the reliability and failure of systems, components, and processes. It is concerned with understanding and quantifying the probability that a system or component will operate without failure over a specified period or under given conditions. Reliability theory is widely used in engineering, quality control, and risk analysis to design, evaluate, and improve the dependability of various systems and products. Reliability theory encompasses issues such as:

1.1.1 Reliability engineering

Reliability engineering deals with the interdisciplinary use of probability, statistics and stochastic modeling, combined with engineering insights into the design and the scientific understanding of the failure mechanism, to study the various aspects of reliability. Frequently, it is desirable to understand and be able to predict the overall system failure characteristics for any given configuration.

1.1.2 Reliability modeling

Reliability modeling deals with model building to obtain solutions to problems in predicting, estimating and optimizing the survival or performance of an unreliable system, the impact of the unreliability, and actions to mitigate this impact [19][20].

1.1.3 Reliability management

Reliability management deals with the various management issues in the context of managing the design, manufacture and/or operation of reliable products and systems. The emphasis is on the business viewpoint, as unreliability has consequences in cost; time wasted, and in certain cases the welfare or an individual or the security of a nation.

1.2 Standard Lomax distribution

The Lomax distribution, also known as the Pareto Type II distribution or the shifted Pareto distribution, is a probability distribution used in statistics and probability theory. It is often used to model heavy-tailed or long-tailed data and is closely related to the Pareto distribution. The Lomax distribution is defined by the probability density function (PDF)[9][21]:

$$f(x) = \frac{\lambda\kappa}{(1+\lambda\kappa)^{(\kappa+1)}} \quad (1)$$

The cumulative distribution function associated with the (1) above is;

$$F(x) = 1-(1+\lambda\kappa)^{-\kappa} \quad (2)$$

Survival function (Reliability) as:

$$R(x) = (1+\lambda\kappa)^{-\kappa} \quad (3)$$

Hazard rate function corresponding to (3) will be;

$$h(x) = \frac{f(x)}{R(x)} = \frac{\lambda\kappa}{(1+\lambda\kappa)} \quad (4)$$

One of the key characteristics of the Lomax distribution is its heavy tail, which means that it has a higher probability of extreme values compared to many other probability distributions. The tail index λ controls the heaviness of the tail. When λ is small, the tail is heavier, indicating that extreme values are more likely. An important aspect of the Lomax distribution is how the values of the shape and the scale parameter affect such distribution characteristics as the shape of the PDF curve, the reliability, and the failure rate.

2. The Proposed Extended Lomax Distribution

Consider a continuous distribution G with density g and the Weibull cdf

$$F(x) = 1-e^{-\alpha x^{\beta}} \quad (5)$$

With positive parameters α and $\boldsymbol{\beta}$. Based on this density, by replacing x with $\frac{G(x)}{\bar{G}(x)}$ for $\bar{G}(x) = 1-G(x)$ Silver et al. [15] define the cdf family by:

$$F(x;\alpha,\beta,\zeta) = \int_0^{\frac{G(x)}{\bar{G}(x)}} \alpha\beta t^{\beta-1}e^{-\alpha t^{\beta}}dt = 1-\exp[-\alpha[\frac{G(x;\zeta)}{\bar{G}(x;\zeta)}]^{\beta}] \quad (6)$$

Where $G(x;\zeta)$ is a baseline cdf, which depends on a parameter vector $\boldsymbol{\zeta}$. The family pdf is reduced to:

$$(x;\alpha,\beta,\zeta) = \alpha\beta g(x;\zeta)\frac{G(x;\zeta)^{\beta-1}}{\bar{G}(x;\zeta)^{\beta+1}}\exp[-\alpha[\frac{G(x;\zeta)}{\bar{G}(x;\zeta)}]^{\beta}] \quad (7)$$

The hazard rate function is given by:

$h(x;\alpha,\beta,\zeta)=\frac{\alpha\beta g(x;\zeta)G(x;\zeta)^{\beta-1}}{\bar{G}(x;\zeta)^{\beta+1}}$ (8)

Inserting (4) into (6) we have;

$F(x)=1-\exp[-\alpha\left[\frac{1-(1+\lambda\kappa)^{-\kappa}}{(1+\lambda\kappa)^{-\kappa}}\right]^{\beta}]$ (9)

From the above equation, we've

$F(x)=1-\exp[-a[1-(1+\lambda\kappa)^{\kappa}]^{b}$ (10)

From the above CDF we've

$R(x)=\exp[-a[1-(1+\lambda\kappa)^{\kappa}]^{b}$ (11)

To evaluate for $f(x)$ i.e probability density function

$f(x)=\frac{-dR(x)}{dt}$ (12)

We need to find the derivative of $R(t)$ with respect to t. Given expression for $R(t)$ and $R(x)=\exp[-a[1-(1+\lambda\kappa)^{\kappa}]^{b}$ using chain rule:

$\frac{dR(x)}{dt}=\exp[-a[1-(1+\lambda\kappa)^{\kappa}]^{b}\frac{d}{dx}(1-(1+\lambda\kappa)^{\kappa})$ (13)

We next find $\frac{d}{dx}(1-(1+\lambda\kappa)^{\kappa})$ using chain rule again:

$$\frac{d}{dx}(1-(1+\lambda\kappa)^{\kappa})=-ab(1-(1+\lambda\kappa)^{\kappa})^{b-1}\frac{d}{dx}(1+\lambda\kappa)^{\kappa}$$

Simplifying the equation we've

$=ab\kappa\lambda((1+\lambda\kappa)^{\kappa})^{b-1}(1+\lambda\kappa)^{\kappa-1}$ (14)

Therefore, the probability density function for the newly generated extended Lomax distribution is:

$f(x)=\exp[-a[1-(1+\lambda\kappa)^{\kappa}]^{b}[ab\kappa\lambda((1+\lambda\kappa)^{\kappa})^{b-1}(1+\lambda\kappa)^{\kappa-1}]$ (15)

The Hazard rate function h(x) represents the instantaneous failure rate at time x. It is defined as the conditional probability that a system or component fails at time x given that it has survived up to that time. For continuous distribution, the hazard rate can be calculated as the ratio of the PDF to the survival function.

$h(x)=\frac{f(x)}{R(x)}$ (16)

To find the hazard rate function, we need to differentiate the negative logarithm of the reliability function with respect to time (x)

$-lnR(x)=-\ln[\exp[-a[1-(1+\lambda\kappa)^{\kappa}]^{b}]=[a[(1+\lambda\kappa)^{\kappa}]^{b}]$ (17)

Differentiating the above equation with respect to x using chain rule;

$\frac{d}{dx}[a[(1+\lambda\kappa)^{\kappa}]^{b}]=[ab\kappa\lambda((1+\lambda\kappa)^{\kappa})^{b-1}(1+\lambda\kappa)^{\kappa-1}]$ (18)

Therefore, the hazard rate function for the newly generated extended Lomax distribution corresponding to (10) and (15);

$h(x)=ab\kappa\lambda((1+\lambda\kappa)^{\kappa})^{b-1}(1+\lambda\kappa)^{\kappa-1}$ (19)

3. Parameter estimation

The choice of parameter estimation method depends on the nature of the data, the statistical model, the sample size, and the specific goals of the analysis. Each method has its own strengths

and weaknesses, we consider these factors when selecting an appropriate technique.

3.1.1 Maximum Likelihood Estimation (MLE)

MLE is a widely used method for parameter estimation. It seeks to find the values of parameters that maximize the likelihood function, which measures how well the observed data fit the model. MLE is often used for a wide range of statistical models and has desirable properties, such as asymptotic efficiency.

$$L(a,b,\kappa,\lambda;t) \propto [ab\kappa\lambda] + n_2 lna(b+1)\sum_{j=1}^{n}\ln\left(1+\lambda x_j\right) + \sum_{j=1}^{n} \begin{matrix} \ln(1-(1+\lambda x)^{k}) + n_2(\kappa-1)\ln(ab\kappa\lambda)\left((1+\lambda x)^{\kappa}\right)^{b-1}(1+\lambda x)^{\kappa-1} + \\ \sum_{j=n_1+1}^{n}\ln(1-(1+\lambda x)^{k}) + n_2(\kappa-1)\ln(ab\kappa\lambda)\left((1+\lambda x)^{\kappa}\right)^{b-1}(1+\lambda x)^{\kappa-1} \end{matrix} \tag{20}$$

Based on the above equation, by solving the likelihood equations with respect to a,b,κ,λ after equating them to zero, the MLEs $(\bar{a},\bar{b},\bar{\kappa},\bar{\lambda})$ of a,b,κ,λ can be obtained. This procedure can be done as follows:

$$\frac{\partial L}{\partial a} = \frac{n_2}{a} - n_2\ln(ab\kappa\lambda)\left(\left((1+\lambda x)^{\kappa}\right)^{b-1}(1+\lambda x)^{\kappa-1}\right) + \sum_{j=n_1+1}^{n}\ln(ab\kappa\lambda)\left((1+\lambda x)^{\kappa}\right)^{b-1}+(1+\lambda x)^{\kappa-1} \tag{21}$$

$$\frac{\partial L}{\partial b} = \frac{n}{b} + (b-1)\frac{(1+\lambda x)^{-\kappa-1}-(1+\lambda x)^{-1}}{1-(1+\lambda x)^{-1}}\sum_{j=1}^{n}\frac{x_j}{1+\lambda x_j}\left(\kappa+1-\frac{\kappa}{\left(1+\lambda x_j\right)^{\kappa}-1}\right) + \kappa(a-1)\sum_{j=n_1}^{n}x_j\frac{(1+\lambda x)^{-\kappa-1}-(1+\lambda x)^{-1}}{1-(1+\lambda x)^{-1}} \tag{22}$$

$$\frac{\partial L}{\partial \kappa} = \frac{n}{k} + (1-b)\ln(1+\lambda t)\frac{(1+\lambda t)^{-\kappa}-1}{1-(1+\lambda x)^{-\kappa}}\sum_{j=1}^{n}\ln\left(1+\lambda t_j\right)\left(1-\frac{1}{\left(1+\lambda t_j\right)^{\kappa}-1}\right) + (b-1)\sum_{j=n_1+1}^{n}\ln\left(1+\lambda t_j\right)\frac{\left(1+\lambda t_j\right)^{-\kappa}-1}{1-\left(1+\lambda t_j\right)^{-\kappa}} \tag{23}$$

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} - n(b-1)\ln(ab\kappa)\left((1+\lambda x)^{\kappa}\right)^{b-1} + (\kappa-1)\sum_{j=n_1}^{n}\ln\left(ab\kappa(1+\lambda t)^{b}\right) + \left(1-\frac{1}{\left(1+\lambda t_j\right)^{\kappa-1}-1}\right) \tag{24}$$

From equation (21), the following equation can be used to calculate $\bar{a}$ as a function of b,κ,λ.

$$\frac{d}{da}\left(\frac{n_2}{a} - n_2\ln(ab\kappa\lambda)\left((1+\lambda x)^{\kappa}\right)^{b-1}(1+\lambda x)^{\kappa-1}\right) + \sum_{j=n_1+1}^{n}\ln(ab\kappa\lambda)\left((1+\lambda x)^{\kappa}\right)^{b-1} = 0 \tag{25}$$

To simplify this, we'll first find the derivatives of the individual terms with respect to a, and then set the expression equal to zero.

$$\frac{d}{da}\left(-n_2\ln(ab\kappa\lambda)\left((1+\lambda x)^{\kappa}\right)^{b-1}(1+\lambda x)^{\kappa-1}\right) = -n_2\frac{d}{da}\left(\ln(ab\kappa\lambda)\left((1+\lambda x)^{\kappa}\right)^{b-1}(1+\lambda x)^{\kappa-1}\right) = 0 \tag{26}$$

Now, differentiating the natural logarithm term. Differentiating the expression inside the logarithm with respect to

$$\frac{d}{da}\left(\ln(ab\kappa\lambda)(1+\lambda x)^{b-1}\right) + (1+\lambda x)^{\kappa-1}\Big) = \frac{1}{\left((ab\kappa\lambda(1+\lambda x)^{b-1}+(1+\lambda x)^{\kappa-1}\right)}\frac{d}{da}\left[\left((ab\kappa\lambda(1+\lambda x)^{b-1}+(1+\lambda_t)^{\kappa-1}\right)\right] \tag{27}$$

Differentiating the expression inside the logarithm. Summing up the individual derivatives:

$$\bar{a}(b\kappa\lambda) = \frac{n_2}{\kappa^2} - n_2\frac{\left[(b\kappa\lambda(1+\lambda x)^{b-1}+(1+\lambda_t)^{\kappa-1}\right]}{\left((ab\kappa\lambda(1+\lambda x)^{b-1}+(1+\lambda_t)^{\kappa-1}\right)} + \sum_{j=n_1+1}^{n}\frac{\left[(ab\kappa\lambda(1+\lambda x)^{b-1}\right]}{\left((ab\kappa\lambda(1+\lambda x)^{b-1}+(1+\lambda_t)^{\kappa-1}\right)} = 0 \tag{28}$$

From equation (28), the above equation can be used to calculate $\bar{a}$ as a function of $b,\kappa\ and\ \lambda$.By substituting $\bar{a}(b\kappa\lambda)$ in (22), (23), and (24) the MLE of $b\kappa\lambda$ can be produced by solving the likelihood equations$\frac{\partial L}{\partial b}=0$, $\frac{\partial L}{\partial \kappa}=0$, and $\frac{\partial L}{\partial \lambda}=0$ with regard to $b,\kappa\ and\ \lambda$ by utilizing Newton-rampson numerical iteration method. The MLEs of the reliability function, and Hazard rate function at t_0, denoted by $\bar{R}(t_0)$ and $\bar{h}(t_0)$ are obtained by substituting $(\bar{a},\bar{b},\bar{\kappa},\bar{\lambda})$ in equations (11) and (19) respectively.

4. Mathematical Properties

The major mathematical properties of the proposed extended Lomax distribution are derived and presented in this section.

4.1 Moment

In the method of moments, we set the sample moments equal to the corresponding population moments. This approach provides estimates for the parameters based on simple algebraic equations. Some of the most important features and characteristics of a distribution can be studied through moments. (e.g tendency, dispersion, and skewness) [10].

$$\mu_r = E(X^r) = \int_0^{\infty} x^r f(x)dx \tag{29}$$

$$\mu_r = E(X^r) = \int_0^\infty x^r \exp[-a[1-(1+\lambda\kappa)^\kappa]^b [ab\kappa\lambda((1+\lambda\kappa)^\kappa)^{b-1}(1+\lambda\kappa)^{\kappa-1}]dx \quad (30)$$

In any statistical analysis, especially in the field of applied statistics, it is crucial to keep in mind the importance of moments. Moments can be used to examine key distributional properties like kurtosis, skewness, dispersion, and tendency, among others. Assume that M is a random variable in a Lomax distribution with parameters $\boldsymbol{\kappa}$ and λ, in that case, the r^{th} moment of M is given as: (σ B') B $(r+1, \sigma - r)$

$$E(M') = (\frac{\lambda}{\kappa})\kappa(r+1, \lambda - r) \quad (31)$$

Also, suppose X is random variable that assumes the proposed extended Lomax distribution. From equation (30), the r^{th} moment of X is therefore obtained as:

$$E(X^r) = ab\kappa\lambda((1+\lambda\kappa)^\kappa)^{b-1}\frac{ab\kappa\lambda(i+1)}{\kappa^r}\kappa(r+1, ab\kappa\lambda(i+1) - r) \quad (32)$$

The mean is obtained my equating $r = 1$ in equation (32)

5. Application to Solar Photovoltaic System

A statistical model called the Extended Lomax distribution is used in many different domains, such as survival analysis, reliability analysis, and modeling of extreme occurrences. It is an effective instrument for comprehending and forecasting the performance and dependability of solar photovoltaic (PV) systems over time.

The Extended Lomax distribution was used to simulate the possibility of extreme events in the context of solar PV systems over 360 days, such as abrupt declines in efficiency, component failures, or unforeseen changes in energy output. Engineers and academics can estimate the chance of future occurrences and obtain insights into the probability distribution of solar PV system performance by fitting historical data to the Extended Lomax distribution. The efficiency of the proposed life distribution is demonstrated in this section using real-life data sets. The data set contains information about the reliability of Solar photovoltaic system by times (in days) [11]. The data set consists of twenty (12) observations as presented in Table 1 below:

Table 1: *Computation of Reliability for different values of time*

Time (In days)	Reliability
30	0.99912
60	0.94321
90	0.89824
120	0.83345
150	0.74561
180	0.71984
210	0.70125
240	0.66257
270	0.62152
300	0.59452
330	0.54529
360	0.41984

Table 2: *The MLEs and Information Criteria of the models based on the solar data set*

Models	$\bar{a}$	$\bar{b}$	$\bar{\kappa}$	$\bar{\lambda}$
Proposed model	13.237	8.648	9.183	5.984
Mohamed and Essam [12]	3.978	2.967	4.826	5.034
Abdelaziz Alsubie 2021 [13]	6.284	6.936	6.920	3.961
F. Hashem et al. [14]	0.284	-	4.783	0.947

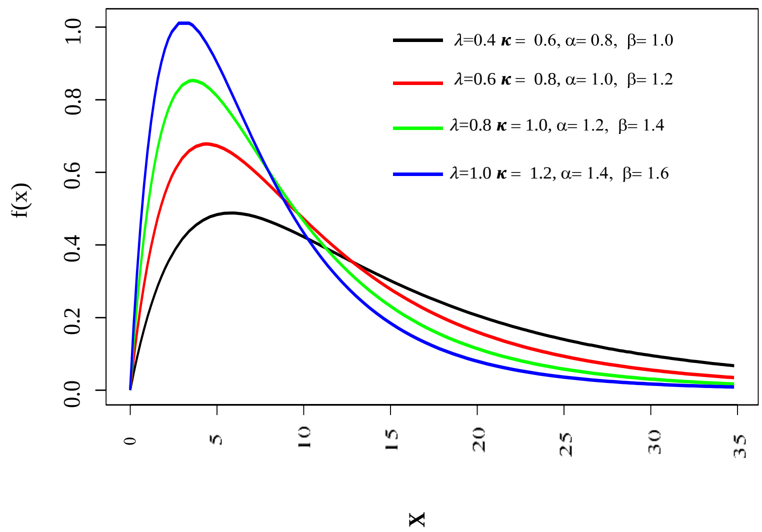

Figure 1: *PDF of the Proposed Extended Distribution*

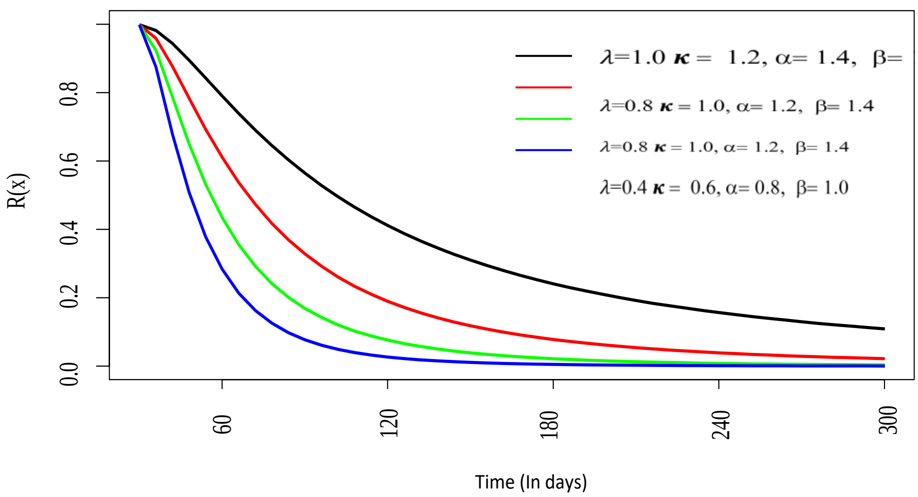

Figure 2: *Reliability Function of the Proposed Extended Distribution*

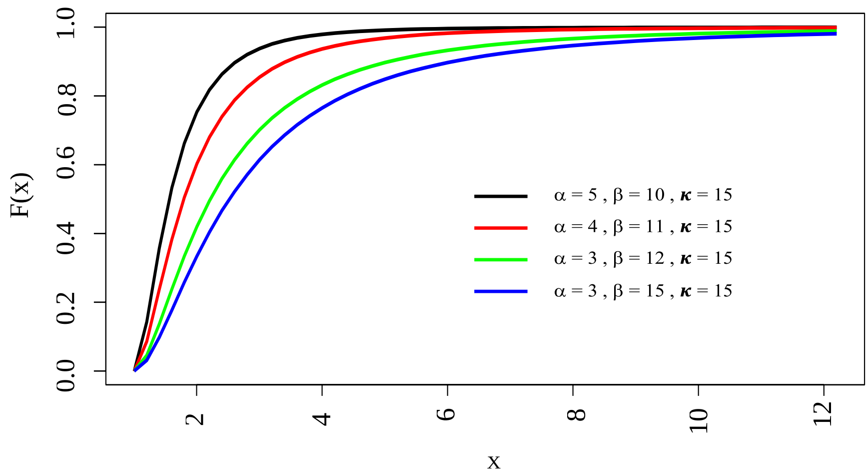

Figure 3: *CDF of the Extended Lomax Distribution*

6. Summary

In this study, we introduce a three-parameter Extended Lomax distribution by compounding the Weibull-G Family of Distributions and the Lomax distributions. The shape of the hazard function of the new compounding distribution can be monotonically decreasing or upside-down bathtub. Some mathematical and statistical properties of the new model are studied. We estimate the model parameters by the Maximum likelihood (MLE) approach, and Method of Moments. We present a simulation study to illustrate the performance of estimators. The flexibility and potentiality of the proposed model are illustrated by means of a real data set (from the reliability analysis of a repairable solar photovoltaic system). We hope that the Extended Lomax distribution may attract a wider range of applications in areas such as engineering, survival and lifetime data, economics, meteorology, hydrology, and others. PDF, Reliability plot and cumulative distribution function plots for the proposed extended Lomax distribution were displayed in Figure 1, 2, and 3 respectively.

7. Conclusion

By addressing the limitations of the standard Lomax distribution and extending it to fit the specific data from an industrial system, this research aims to enhance the reliability analysis and support better decision-making in the context of the industrial system's operation and maintenance. The proposed distribution was compared to some existing distributions, it was discovered that the proposed life distribution is right-skewed and that changing the parameters' values results in different shapes. The proposed Extended Lomax distribution is more efficient than the competing distribution's pdfs and reliability for different values of shape parameters from a real-life data set, as shown in Figures 1.

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10. Conflict of interest

The authors declare that they have no conflicts of interest.

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References

[1] Saran, J. and Pushkarna, N. Moments of order statistics from doubly truncated Lomax distribution, Journal of Statistical Research 33, 57–66, 1999.

[2] Childs, A., Balakrishnan, N. and Moshref, M. Order statistics from non-identical right-truncated Lomax random variables with applications, Statistical Papers 42, 187–206, 2001.

[3] Moghadam, M.S., Yahmaei, F. and Babanezhad, M. Inference for Lomax distribution under generalized order statistics, Applied Mathematical Sciences 6, 5241–5251, 2012.

[4] Hamedani G. G., Yousof H. M., Rasekhi M., Alizadeh M. and Najibi S. M. (2018). Type I general exponential class of distributions. Pakistan Journal of Statistics and Operation Research, 14(1): 39-55.

[5] Ahsanullah, M. Recoed values of the Lomax distribution, Statistica Neerlandica 45, 21–29, 1991.

[6] Balakrishnan, N. and Ahsanullah, M. Relations for single and product moments of record values from Lomax distribution, Sankhy˜a Series B 17, 140–146, 1994.

[7] Lee, M-Y. and Lim, E-K. Characterization of the Lomax, exponential and Pareto distributions by conditional expectations of recoed values, Journal of the Chungcheong Mathematical Society 22, 149–153, 2009.

[8] Amin, E.A. Kth upper record values and their moments, International Mathematical Forum 6, 3013–3021, 2011.

[9] Bourguignon, M., Silva, R.B. and Cordeiro, G.M. The Weibull–G family of probability distributions, Journal of Data Science 12, 53–68, 2014.

[10] Halid O.Y., and Sule O.B. (2022). A Classical and Bayesian Estimation Techniques for Gompertz Inverse Rayleigh Distribution: Properties and Application. Pakistan Journal of Statistics, 38(1): 49-76.

[11] Maihulla, A.S., Yusuf, I. & Khoo, M.B.C. Reliability and performance analysis on stages to control manufacturing system using Copula technique. *Life Cycle Reliab Saf Eng* 12, 197–205 (2023). https://doi.org/10.1007/s41872-023-00224-8

[12] Mohamed S. Eliwa and Essam A. Ahmed "Reliability analysis of constant partially accelerated life tests under progressive first failure type-II censored data from Lomax model: EM and MCMC algorithms, AIMS Mathematics, 8(1): 29–60. DOI: 10.3934/math.2023002

[13] Abdelaziz Alsubie, "Properties and Applications of the Modified Kies–Lomax Distribution with Estimation Methods," Hindawi Journal of Mathematics Volume 2021, Article ID 1944864, 18pages https://doi.org/10.1155/2021/1944864.

[14] Laila A. Al-Essa1, Alaa H. Abdel-Hamid, Tmader Alballa1, and Atef F. Hashem,

"Reliability analysis of the triple modular redundancy system under step-partially accelerated life tests using Lomax distribution," scientific reports https://doi.org/10.1038/s41598-023-41363-3 (2023) 13:14719

[15] Bourguignon, M., Silva, R.B. and Cordeiro, G.M. The Weibull–G family of prob- ability distributions, Journal of Data Science 12, 53–68, 2014.

[16] Maihulla, A.S., Yusuf, I. and Salihu Isa, M. (2022), "Reliability modeling and performance evaluation of solar photovoltaic system using Gumbel–Hougaard family copula", *International Journal of Quality & Reliability Management*, Vol. 39 No. 8, pp. 2041-2057. https://doi.org/10.1108/IJQRM-03-2021-0071

[17] Maihulla, A.S., Yusuf, I. Reliability, availability, maintainability, and dependability analysis of photovoltaic systems. *Life Cycle Reliab Saf Eng* 11, 19–26 (2022). https://doi.org/10.1007/s41872-021-00180-1

[18] Maihulla, A.S., Yusuf, I. Reliability and performance prediction of a small serial solar photovoltaic system for rural consumption using the Gumbel-Hougaard family copula. *Life Cycle Reliab Saf Eng* 10, 347–354 (2021). https://doi.org/10.1007/s41872-021-00176-x

[19] Maihulla, Anas Sani; Yusuf, Ibrahim; Bala, Saminu I. "Performance Evaluation of a Complex Reverse Osmosis Machine System in Water Purification using Reliability, Availability, Maintainability and Dependability Analysis." Reliability: Theory & Applications. Sep2021, Vol. 16 Issue 3, p115-131. 17p.

[20] Maihulla, A. S. and Yusuf, I. (2022). RELIABILITY ANALYSIS OF REVERSE OSMOSIS FILTRATION SYSTEM USING COPULA. *Reliability: Theory & Applications*, *17*(2): 163-177. Retrieved from http://www.gnedenko.net/RTA/index.php/rta/article/view/890.

[21] Anas Sani Maihulla, Ibrahim Yusuf, Saminu I. Bala WEIBULL COMPARISON BASED ON RELIABILITY, AVAILABILITY, MAINTAINABILITY, AND DEPENDABILITY (RAMD) ANALYSIS. *Reliability: Theory & Applications.* **2023**, *March 1(72):* 120-132. https://doi.org/10.24412/1932-2321-2023-172-120-132