MAXWELL-GOMPERTZ DISTRIBUTION: PROPERTIES AND APPLICATIONS

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Abstract

This paper proposed a three parameter Maxwell-Gompertz distribution as an extension of Gompertz distribution. Some statistical properties of the distribution such as moments, survival and hazard functions, quantile function, Rényi entropy and order statistics were derived. Maximum likelihood method was used to estimate the model parameters. A simulation study was carried out in order to gain an insight into the performance on small, moderate and large samples. The flexibility of the new distribution was empirically demonstrated in comparison to four other extensions of Gompertz distributions using two real life datasets.

Keywords: Maxwell-Gompertz, generator, skewness, Rényi entropy, maximum likelihood

I. Introduction

Gompertz distribution is a popular distribution commonly used in many applied problems, particularly in modelling lifetime data [1]. The distribution is often characterized by an increasing hazard function and it is commonly used to describe the distribution of adult life spans by actuaries and demographers [2]. It is also considered for modelling survival data in some sciences such as gerontology [3], computer science [4], biology [5], and marketing science [6]. For more details about the Gompertz distribution and its applications, see [7], [8]. The cumulative distribution function (cdf) and probability density function (pdf) of the Gompertz random variable X are respectively given as

$$T(x,\vartheta)=1-e^{-\frac{c}{b}(e^{bx}-1)}, \tag{1}$$

and

$$t(x,\vartheta)=ce^{bx}e^{-\frac{c}{b}(e^{bx}-1)}, \qquad \vartheta>0;x>0 \tag{2}$$

where $\vartheta=(b,c)$ with b denting the shape parameter and c the scale parameter.

The development of new families of distributions has become an important trend in the theory and application of distributions. Such new families of distributions are often compounded by adding one or more parameters to the well-known standard baseline distributions. This has become necessary because the resulting extended new distributions provide greater flexibility in modelling observed data. A few of such families of distributions which have been explored in the recent times include, among others the Beta-G of [9], a new generalized odd log-logistic family of distributions by [10], The generalized odd half-Cauchy family of distributions by [11], a New Kumaraswamy generalized family of distributions by [12].

Several other families of distributions can be mentioned such as Odd F family of distributions by [13], Odd Beta Prime family of distributions by [14], Generalized odd Maxwell family of distributions by [15], Generalized beta-generated distributions of [16], Garhy-generated family of distributions by [17], Gamma-G Type-3 of [18], The Logistic-X family of [19], a new Weibull-X family of [20], a-Zubair-G family of [21], and a new Alpha power transformed family distribution by [22].

Gompertz distribution has been extended by some authors in the literature through the addition of one or more other parameters. Some of such studies in the recent time include the modified beta Gompertz distribution by [23], the generalized Gompertz distribution by [24] which was based on an idea of [25], the cubic transmuted Gompertz distribution by [26], the odd generalized exponential-Gompertz distribution by [27], the transmuted Gompertz distribution by [28] and the odd lindley-Gompertz distribution by [29]. This article seeks to develop a distribution that has the characterization of the Gompertz and Maxwell distributions in a unified framework. The Maxwell distribution was introduced by [30], and it has the cdf given as

$$G(u,a)=\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{u^2}{2a^2}\right), \tag{3}$$

with $\gamma(u,b)=\int_0^u s^{b-1}e^{-s}ds$ denoting the lower incomplete gamma function. The associated pdf of (3) is

$$g(u,a)=\sqrt{\frac{2}{\pi}}\frac{u^2 e^{-\frac{u^2}{2a^2}}}{a^3}, \qquad u,a>0 \tag{4}$$

where a is the scale parameter.

Studies involving Maxwell generalized family of distributions have not been widely covered in the literature. However, [31] proposed Maxwell–Weibull distribution by applying the odd ratio link approach of [32]. Also, [33] developed Maxwell-Dagum distribution while [34] developed Maxwell-Lomax distribution.

II. Methods

2.1.The Maxwell-Gompertz (Mgom) Distribution

Consider a random variable X which follows the Gompertz distribution with the cumulative and probability density functions as defined in (1) and (2) respectively. Following [31] who proposed Maxwell family of distributions for continuous generator, we can present the cumulative density function of Maxwell-G family as

$$F(x;a,\vartheta)=\int_0^{\frac{T(x,\vartheta)}{1-T(x,\vartheta)}}\sqrt{\frac{2}{\pi}}\frac{u^2 e^{-\frac{u^2}{2a^2}}}{a^3} = \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2a^2}\left(\frac{T(x,\vartheta)}{1-T(x,\vartheta}\right)^2\right), \quad a>0;\mathrm{x}>0 \tag{5}$$

and the corresponding pdf is given as

$$f(;a,\vartheta)=\frac{2t(x,\vartheta)}{a^3\sqrt{2\pi}(1-T(x,\vartheta))^2}\left(\frac{T(x,\vartheta)}{1-T(x,\vartheta}\right)^2\exp\left(-\frac{1}{2a^2}\left(\frac{T(x,\vartheta)}{1-T(x,\vartheta}\right)^2\right), \tag{6}$$

where $\vartheta=(b,c)$ denotes the vector of parameters of the baseline Gompertz distribution.

Substituting (1) in (5) gives the proposed cdf of the Maxwell-Gompertz (MGom) distributions as

$$F(x;a,b,c)=\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2a^2}\left(\frac{1-e^{-\frac{c}{b}\left(e^{bx}-1\right)}}{e^{-\frac{c}{b}\left(e^{bx}-1\right)}}\right)^2\right),\quad a,b,c>;x>0 \tag{7}$$

and on substituting (1) and (2) in (6), the pdf can be obtained as

$$f(x;a,b,c)=\frac{2ce^{bx}\left(1-e^{-\frac{c}{b}\left(e^{bx}-1\right)}\right)^2}{a^3\sqrt{2\pi}\left(e^{-\frac{c}{b}\left(e^{bx}-1\right)}\right)^3}\exp\left(-\frac{1}{2a^2}\left(\frac{1-e^{-\frac{c}{b}\left(e^{bx}-1\right)}}{e^{-\frac{c}{b}\left(e^{bx}-1\right)}}\right)^2\right). \tag{8}$$

2.2 Linear Representation of MGom Density

Consider the power series expansion of the exponential function

$$e^{-x}=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}x^i \tag{9}$$

Putting (9) in (8) and dropping (*a*,*b*,*c*) in *f* (*a*,*b*,*c*) for simplicity, we have

$$f(x)=\frac{2ce^{bx}\,\mathrm{e}^{-\frac{c}{b}\left(e^{bx}-1\right)}}{a^3\sqrt{2\pi}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!\left(2a^2\right)^i}\frac{\left(1-e^{-\frac{c}{b}\left(e^{bx}-1\right)}\right)^{2+2i}}{\left(e^{-\frac{c}{b}\left(e^{bx}-1\right)}\right)^{4+2i}}. \tag{10}$$

Considering the generalized binomial expansion in power of positive real number μ, expressed as

$$(1-x)^{\mu}=\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(u+1)}{k!\Gamma(u+1-k)}x^k. \tag{11}$$

By applying (11) to (10) we obtain

$$f(x)=\sum_{i,j,k=0}^{\infty}\frac{2c(-1)^{i+k}\Gamma(4+2i+j)\Gamma(3+2i+j)}{i!\,j!\,k!\left(2a^2\right)^i a^3\sqrt{2\pi}\Gamma(4+2i)\Gamma(3+2i+j-k)}e^{bx}e^{-\frac{c}{b}(1+k)\left(e^{bx}-1\right)}. \tag{12}$$

Thus, the pdf of the MGom distribution expressed as a linear representation is obtained by applying (9) to (12) which gives

$$f(x)=\Omega_{i,j,k}e^{bx}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\left(\frac{c}{b}(1+k)\right)^l\left(e^{bx}-1\right)^l \tag{13}$$

$$=\sum_{l,m=0}^{\infty}\Phi_{l,m}e^{-bx(m-l-1)}, \tag{14}$$

where $\Phi_{l,m}=\frac{(-1)^{l+m}\Gamma(l+1)}{l!m!\Gamma(l+1-m)}\left(\frac{c}{b}(1+k)\right)^l\Omega_{i,j,k}$

and $\Omega_{i,j,k}=\sum_{i,j,k=0}^{\infty}\frac{2c(-1)^{i+k}\Gamma(4+2i+j)\Gamma(3+2i+j)}{i!j!k!\left(2a^2\right)^i a^3\sqrt{2\pi}\Gamma(4+2i)\Gamma(3+2i+j-k)}$.

The plots of the pdf and cdf of MGom distribution using different parameter values are displayed in Figures 1 and 2 respectively. From Figure 1, it is observed that the pdf of the MGom distribution is skewed to the right and therefore will be a good model for different kinds of positively skewed data sets.

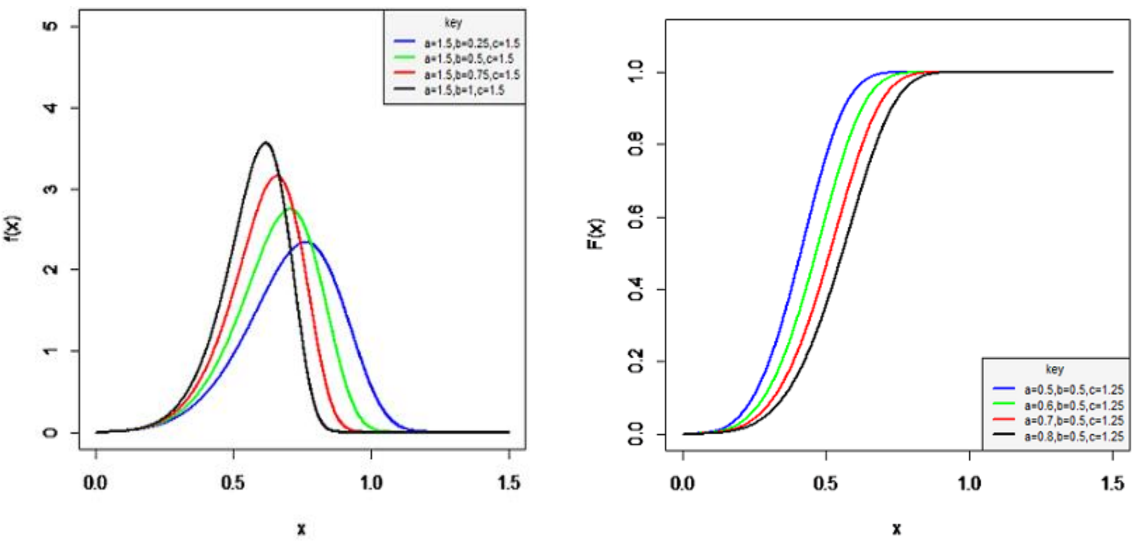

Figure 1: *Plots of pdf and cdf of EGILx distribution*

2.3 Statistical Properties

Some structural properties of the Maxwell-Gompertz distribution are discussed in this section.

2.3.1 Moments

Suppose that X denote a continuous random variable, the r^th non-central moment of X is given by

$$E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx . \tag{15}$$

Taking f(x) as the pdf of the MGom distribution given in (14), the r^th moments of X is given as

$$E(X^r) = \sum_{l,m=0}^{\infty} \Phi_{l,m} \int_0^{\infty} x^r e^{-bx(m-l-1)} dx . \tag{16}$$

Let,

$$y = bx(m-l-1), \Rightarrow x = \frac{y}{b(m-l-1)}, \text{ so that } dx = \frac{dy}{b(m-l-1)} . \tag{17}$$

By inserting (17) into (16), we obtain

$$E(X^r) = \sum_{l,m=0}^{\infty} \Phi_{l,m} \int_0^{\infty} \left(\frac{y}{b(m-l-1)} \right)^r e^{-y} \frac{dy}{b(m-l-1)} \tag{18}$$

$$= \frac{\sum_{l,m=0}^{\infty} \Phi_{l,m}}{\left(b(m-l-1)\right)^{r+1}} \Gamma(r+1) . \tag{19}$$

which is the moments of MGom distribution.

2.3.2. Quantile function

Quantile function of MGom can be derived by inverting the cdf given in (7).
If we let

$$F(x)=\frac{2}{\sqrt{2\pi}}\gamma\left(\frac{3}{2},\frac{1}{2a^2}\left(\frac{1-e^{-\frac{c}{b}\left(e^{bx}-1\right)}}{e^{-\frac{c}{b}\left(e^{bx}-1\right)}}\right)^2\right)=\text{u}\,, \tag{20}$$

then by solving (20) for x we obtain

$$x_q=Q(u)=b^{-1}\log\left(1-\frac{b}{c}\log\left(1-\frac{\left(2a^2\gamma^{-1}\left(\frac{3}{2},u\Gamma\left(\frac{3}{2}\right)\right)\right)^{0.5}}{1+\left(2a^2\gamma^{-1}\left(\frac{3}{2},u\Gamma\left(\frac{3}{2}\right)\right)\right)^{0.5}}\right)\right), \tag{21}$$

where u is a uniform random variable defined on interval $(0,1)$.

We can obtain the three quartiles Q1, Q2 and Q3 from (21) by using u = 0.25, 0.50 and 0.75 respectively.

2.3.3 Survival function

The survival function for the MGom random variable $X\sim\text{MGom}(a,b,c)$ from the cdf in (7) is obtained as

$$S(x,a,b,c)=1-F(x)=1-\frac{2}{\sqrt{2\pi}}\gamma\left(\frac{3}{2},\frac{1}{2a^2}\left(\frac{1-e^{-\frac{c}{b}\left(e^{bx}-1\right)}}{e^{-\frac{c}{b}\left(e^{bx}-1\right)}}\right)^2\right). \tag{22}$$

The plot of the survival function of MGom for different parameter values is displayed in Figure 3.

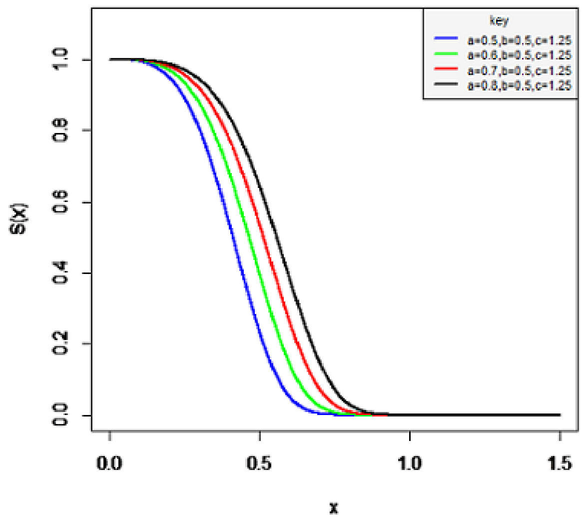

Figure 2: *Plots of the survival function of MGom distribution*

As observed from the plots in Figure 2, the value of the survival function equals one at initial value of zero, it decreases as x increases and degenerates to zero as x becomes larger, which is a major characteristic of survival function.

2.3.4 Hazard function

The hazard function can be obtained using the pdf in (8) and survival function in (22) as

$$h(x)=\frac{f(x)}{S(x)}=\frac{2ce^{bx}\left(1-e^{-\frac{c}{b}\left(e^{bx}-1\right)}\right)^{2}\exp\left(-\frac{1}{2a^{2}}\left(\frac{1-e^{-\frac{c}{b}\left(e^{bx}-1\right)}}{e^{-\frac{c}{b}\left(e^{bx}-1\right)}}\right)^{2}\right),}{a^{3}\sqrt{2\pi}\left(e^{-\frac{c}{b}\left(e^{bx}-1\right)}\right)^{3}\left[1-\frac{2}{\sqrt{2\pi}}\gamma\left(\frac{3}{2},\frac{1}{2a^{2}}\left(\frac{1-e^{-\frac{c}{b}\left(e^{bx}-1\right)}}{e^{-\frac{c}{b}\left(e^{bx}-1\right)}}\right)^{2}\right)\right]}. \tag{23}$$

The plots for the hazard function of MGom distribution for different parameter values are shown in Figure 4.

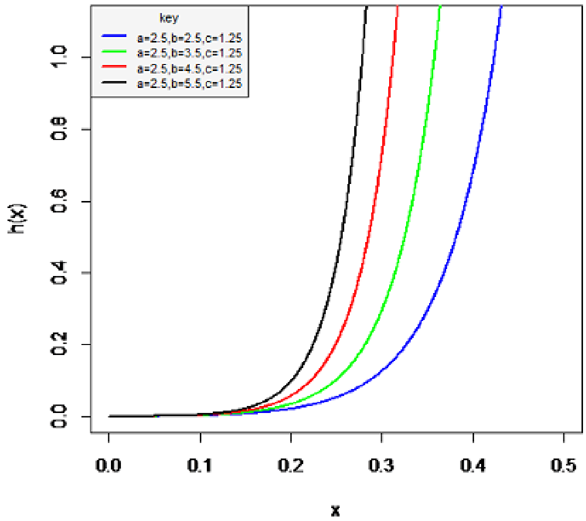

Figure 3: *Plot of the hazard function of MGom distribution*

From the plots in Figure 3, it is observed that the value of the hazard function increases as X increases, meaning that the conditional probability of failure within a given interval of time for a random variable following MGom distribution increases as life ages.

2.3.5 Rényi entropy

If X is a random variable with density function $f(x)$ as defined in (8), then the Rényi entropy of the MGom distribution is defined as

$$R_{\iota}=\frac{1}{1-\iota}\left[\int_{-\infty}^{\infty}f^{\iota}(x)dx\right]. \qquad t>0, t\neq 1$$

(24)

The term $f^{\iota}(x)$ in (24) can be simplified as

$$f^{\iota}(x)=\frac{(2c)^{\iota}e^{\iota bx}\,\mathrm{e}^{-\frac{c\iota}{b}\left(e^{bx}-1\right)}}{\left(a^{3}\sqrt{2\pi}\right)^{\iota}}\sum_{i=0}^{\infty}\frac{(-\iota)^{i}}{i!\left(2a^{2}\right)^{i}}\frac{\left(1-e^{-\frac{c}{b}\left(e^{bx}-1\right)}\right)^{2\iota+2i}}{\left(e^{-\frac{c}{b}\left(e^{bx}-1\right)}\right)^{4\iota+2i}}. \tag{25}$$

By applying (9) to (25), we have

$$f^{\iota}(x)=\frac{(2c)^{\iota}e^{\iota bx}e^{-\frac{c\iota}{b}(e^{bx}-1)}}{\left(a^3\sqrt{2\pi}\right)^{\iota}}\sum_{i=0}^{\infty}\frac{(-\iota)^i}{i!\left(2a^2\right)^i}\sum_{j=0}^{\infty}\frac{\Gamma(4\iota+2i+j)}{j!\Gamma(4\iota+2i)}\left(1-e^{-\frac{c}{b}(e^{bx}-1)}\right)^{2\iota+2i+j}. \quad (26)$$

Using binomial expansion defined in (11), equation (26) becomes

$$f^{\iota}(x)=\frac{(2c)^{\iota}}{\left(a^3\sqrt{2\pi}\right)^{\iota}}\sum_{i,j,k=0}^{\infty}\frac{(-\iota)^{i+k}\Gamma(4\iota+2i+j)\Gamma(2\iota+2i+j+1)}{i!j!k!\left(2a^2\right)^i\Gamma(4\iota+2i)\Gamma(2\iota+2i+j+1-k)}e^{\iota bx}e^{-\frac{c(\iota+k)}{b}(e^{bx}-1)}. \quad (27)$$

which on simplification becomes

$$f^{\iota}(x)=\sum_{l,m=0}^{\infty}\Xi_{i,j,k,l,m}e^{-bx(m-\iota-l)}, \quad (28)$$

where $\Xi_{i,j,k,l,m}=\frac{(2c)^{\iota}}{\left(a^3\sqrt{2\pi}\right)^{\iota}}\sum_{i,j,k=0}^{\infty}\frac{(-\iota)^{i+k+l+m}\Gamma(4\iota+2i+j)\Gamma(2\iota+2i+j+1)\Gamma(l+1)}{i!j!k!l!m!\left(2a^2\right)^i\Gamma(4\iota+2i)\Gamma(2\iota+2i+j+1-k)\Gamma(l+1-m)}\left(\frac{c}{b}(\iota+k)\right)^l$. By taking the integral of (28) and substituting (17) gives

$$\int_{-\infty}^{\infty}f^{\iota}(x)dx=\sum_{l,m=0}^{\infty}\Xi_{i,j,k,l,m}\int_0^{\infty}e^{-bx(m-\iota-l)}dx$$

$$=\frac{\sum_{l,m=0}^{\infty}\Xi_{i,j,k,l,m}}{b(m-l-1)}\int_0^{\infty}e^{-y}dy=\frac{\sum_{l,m=0}^{\infty}\Xi_{i,j,k,l,m}}{b(m-l-1)}. \quad (29)$$

Substituting (29) in (24), the Rényi entropy of the MGom distribution can be given as

$$R_{\iota}=\frac{1}{1-\iota}\left[\frac{\sum_{l,m=0}^{\infty}\Xi_{i,j,k,l,m}}{b(m-l-1)}\right]. \qquad \iota\neq 1 \quad (30)$$

2.3.6 Order statistics

Suppose that $X_1, X_2, ..., X_n$ is a random sample of size n from MGom distribution and $X_{(1)}, X_{(2)}, ..., X_{(n)}$ denote the corresponding order statistics of the sample, then the pdf of the i^{th} order statistics is given as

$$f(x)=\frac{n!}{(i-1)!(n-1)!}f(x)[F(x)]^{i-1}[1-F(x)]^{n-1}, \quad (31)$$

where $F(x)$ and $f(x)$ are defined in (7) and (8). Using the definition of binomial expansion for the term $[1-F(x)]^{n-1}$, (31) can be expressed as

$$f(x)=\frac{n!}{(i-1)!(n-1)!}\sum_{i=0}^{n-i}(-1)^k\binom{n-i}{k}f(x)F(x)^{i+k-1}. \quad (32)$$

Consequently, using (7) and (8), the pdf of i^{th} order statistics for the MGom distribution can be obtained as

$$f_{i,n}^{MGom}=\frac{n!}{(i-1)!(n-1)!}\sum_{i=0}^{n-i}(-1)^k\binom{n-i}{k}f_{MGom}(x)F_{MGom}(x)^{i+k-1}. \quad (33)$$

From (33), The pdf of the smallest and largest order statistics can be obtained by setting i = 1 and i = n respectively.

2.3.7 Parameter estimation

This section derives the maximum likelihood estimator of MGom distribution. Let $X_1, X_2, ..., X_n$ be a random sample of size n drawn from X~ MGom(Θ) with observed values $x_1, x_2 \cdots x_n$, where $\Theta = (a,b,c)^T$ is a $p \times 1$ vector of parameters to be estimated. The likelihood function is given as

$$L(\Theta) = \left(\frac{2c}{a^3\sqrt{2\pi}}\right)^n \prod_{i=1}^{n} e^{bx} e^{-\frac{c}{b}\left(e^{bx}-1\right)} \left(\frac{1-e^{-\frac{c}{b}\left(e^{bx}-1\right)}}{e^{-\frac{c}{b}\left(e^{bx}-1\right)}}\right)^2 \prod_{i=1}^{n} \exp\left(-\frac{1}{2a^2}\left(\frac{1-e^{-\frac{c}{b}\left(e^{bx}-1\right)}}{e^{-\frac{c}{b}\left(e^{bx}-1\right)}}\right)^2\right). \tag{34}$$

The log likelihood function $(\ell\ell)$ is obtained as

$$\ell\ell = n\log(2) + n\log(c) - \frac{n}{2}\log(2\pi) - 3n\log(a) + b\sum_{i=1}^{n}(x_i) + \frac{c}{b}\sum_{i=1}^{n}\log\left(e^{bx_i}-1\right)$$
$$+2\sum_{i=1}^{n}\log\left(\frac{1-e^{-\frac{c}{b}\left(e^{bx_i}-1\right)}}{e^{-\frac{c}{b}\left(e^{bx_i}-1\right)}}\right) - \frac{1}{2a^2}\sum_{i=1}^{n}\left(\frac{1-e^{-\frac{c}{b}\left(e^{bx_i}-1\right)}}{e^{-\frac{c}{b}\left(e^{bx_i}-1\right)}}\right)^2. \tag{35}$$

Taking the partial derivatives of (35) with respect to *a, b* and *c* to obtain

$$\frac{\partial \ell\ell}{\partial a} = \frac{-3n}{a} + \frac{1}{a^3}\sum_{i=1}^{n} w_i^2 , \tag{36}$$

$$\frac{\partial \ell\ell}{\partial b} = \sum_{i=1}^{n} x_i + \left[\frac{c}{b}\sum_{i=1}^{n}\left(e^{bx_i}\log\left(e^{x_i}\right)\right) - \frac{c}{b}\sum_{i=1}^{n}(\vartheta_i)\right] + 2\sum_{i=1}^{n}\left(\frac{\varpi_i}{w_i e^{-c\vartheta_i}}\right) - \frac{1}{a^2}\sum_{i=1}^{n}\left(\frac{w_i \varpi_i}{e^{-c\vartheta_i}}\right), \tag{37}$$

and

$$\frac{\partial \ell\ell}{\partial c} = \frac{n}{c} + \sum_{i=1}^{n}(\vartheta_i) + 2\sum_{i=1}^{n}\left(\frac{\vartheta_i}{e^{-c\vartheta_i}}\right) - \frac{1}{a^2}\sum_{i=1}^{n}\left(\frac{w_i \varpi_i}{e^{-c\vartheta_i}}\right), \tag{38}$$

where $\vartheta_i = \frac{e^{bx_i}-1}{b}$, $w_i = \frac{1-e^{-c\vartheta_i}}{e^{-c\vartheta_i}}$ and $\varpi = \frac{c}{b}\left(x_i e^{bx_i} - \frac{\vartheta_i}{b}\right)$.

Setting $\frac{\partial \ell\ell}{\partial a} = 0$, $\frac{\partial \ell\ell}{\partial b} = 0$ and $\frac{\partial \ell\ell}{\partial c} = 0$, and solving the resulting nonlinear system of equations, we can obtain the maximum likelihood estimates $\hat{a}, \hat{b}, \hat{c}$. However, these equations cannot be solved analytically, thus statistical software can be used to solve them numerically using iterative methods.

III. Results

3.1 Simulation study

A simulation study is carried out here to investigate the performance of the maximum likelihood estimates of MGom distribution. The simulation is based on the quantile function defined in (21) for four sets of parameter vector $\Theta = (a,b,c)$. We generate 1000 replications of random samples of sizes 50, 100, 200 and 500. The four sets of the parameter's values are assigned as follows:

Set 1: $a = 0.5$, $b = 0.5$, $c = 0.5$
Set 2: $a = 1.0$, $b = 1.0$, $c = 1.\ 0$
Set 3: $a = 2.0$, $b = 2.0$, $c = 2.0$
Set 4: $a = 0.5$, $b = 2.0$, $c = 1.0$.

The maximum likelihood estimates $\hat{\Theta} = (\hat{a}, \hat{b}, \hat{c})$ are determined based on each generated sample, by maximizing the log-likelihood function in (35). The average estimates, average bias, denoted Bias and Root mean square error (RMSE) are then determined where

$$\text{Bias}(\hat{\Theta}) = \frac{1}{1000}\sum_{j=1}^{1000}(\hat{\Theta}_j - \Theta) \quad \text{and} \quad \text{RMSE}(\hat{\Theta}) = \left[\frac{1}{1000}\sum_{j=1}^{1000}(\hat{\Theta}_j - \Theta)^2\right]^{1/2}.$$

The results of the simulation study are displayed in Tables 1 and 2.

Table 1: *The parameter estimates (Est), Bias and RMSE.*

		a = 0.5, b = 0. 5, c = 0.5			a = 1.0, b =1. 0, c = 1.0		
N	Parameter	Est	Bias	RMSE	Est	Bias	RMSE
50	a	0.5374	0.0374	1.9486	1.2385	0.2385	2.1963
	b	0.5187	0.0187	1.5791	1.2189	0.2189	1.7649
	c	0.5265	0.0265	1.8411	1.3019	0.3019	1.9253
100	a	0.5210	0.0210	1.2814	1.2273	0.2273	1.8189
	b	0.5146	0.0146	1.3166	1.1916	0.1916	1.4729
	c	0.5158	0.0158	1.5778	1.2342	0.2342	1.6071
200	a	0.5113	0.0113	1.1519	1.1218	0.1218	1.4658
	b	0.5138	0.0138	1.1608	1.1126	0.1126	1.2075
	c	0.5114	0.0114	1.2764	1.1480	0.1480	1.4526
500	a	0.5037	0.0037	0.7342	1.0490	0.0490	0.8903
	b	0.5069	0.0069	0.6969	1.0307	0.0307	0.5933
	c	0.5023	0.0023	0.7564	1.0657	0.0657	0.7505

Table 2: *The parameter estimates (Est), Bias and RMSE.*

		a = 2.0, b = 2. 0, c = 2.0			a = 0.5, b =1.0, c = 2.0		
N	Parameter	Est	Bias	RMSE	Est	Bias	RMSE
50	a	2.2882	0.2882	2.2462	0.5164	0.0164	1.8385
	b	2.1828	0.1828	1.8214	1.2062	0.2062	2.0963
	c	2.3944	0.3944	2.0236	2.1642	0.1642	2.1462
100	a	2.1901	0. 1901	2.1038	0.5099	0.0099	1.1713
	b	2.1643	0.1643	1.6454	1.1616	0.1616	1.7189
	c	2.2729	0.2729	1.9016	2.1001	0.1001	2.0038
200	a	2.1421	0.1421	1.9643	0.5056	0.0056	1.0418
	b	2.1226	0.1226	1.4325	1.0823	0.0823	1.3658
	c	2.1933	0.1933	1.7031	2.0992	0.0992	1.8643
500	a	2.0442	0.0442	0.8606	0.5021	0.0021	0.6241
	b	2.0320	0.0320	0.6542	1.0361	0.0361	0.7903
	c	2.0580	0.0580	0.7730	2.0542	0.0542	0.7606

3.2 Data Application

Application of the MGom distribution to two real life data sets are provided to show how it can be applied in practice in comparison to other distributions in the family. The proposed distribution is compared with four other Gompertz distribution extensions, namely: power Gompertz (powGom), exponentiated Gompertz (expGom), Marshall-Olkin Gompertz (M-OGom) and odd-logistic Gompertz (Odd-loGom). The goodness-of- fit criteria and tests used in the choice of the most appropriate distribution include Akaike's Information Criterion (AIC), Consistent Akaike's Information Criterion (CAIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC), as well as Anderson-Darling $\left(A^*\right)$ and Cramér-von Mises $\left(W^*\right)$ tests. These can be computed as follows

$\text{AIC} = -2\ell + 2p$, $\text{CAIC} = -2\ell + \dfrac{2np}{n-p-1}$, $BIC = -2\ell + p\log(n)$, and

$\text{HQIC} = -2\ell + 2p\log(\log(n))$, $A^* = \left(\dfrac{9}{4n^2} + \dfrac{3}{4n} + 1\right)\left\{n + \dfrac{1}{n}\sum_{j=1}^{n}(2j-1)\log[z_i(1-z_{n-j+1})]\right\}$,

$W^* = \left(\dfrac{1}{2n} + 1\right)\left\{\sum_{j=1}^{n}\left(z_i - \dfrac{2j-1}{2n}\right)^2 + \dfrac{1}{2n}\right\}$, where $z_i = F(x_i)$ and x_i's are the ordered observations, ℓ is the maximized log likelihood of the parameter vector $\Theta = (a,b,c)$, n is the number of observations, and p is the number of estimated parameters.
The model with the smallest value of these measures is preferred to other models.

Dataset 1: This dataset is taken from [35]. The data represent the time between failures of 30 repairable items.
1.43,0.11,0.71,0.77,2.63,1.49,3.46,2.46,0.59,0.74,1.23,0.94,4.36,0.40,1.74,4.73,2.23,0.45,0.70,1.06,1.46,0.30,1.82,2.37,0.63,1.23,1.24,1.97,1.86,1.17.
Dataset 2: The dataset consists of 100 observations of breaking stress of carbon fibers (in Gba) given by [36] as given below:
0.39, 0.81, 0.85, 0.98, 1.08, 1.12, 1.17, 1.18, 1.22, 1.25, 1.36, 1.41, 1.47, 1.57, 1.57, 1.59, 1.59, 1.61, 1.61, 1.69, 1.69, 1.71, 1.73, 1.80, 1.84, 1.84, 1.87, 1.89, 1.92, 2.00, 2.03, 2.03, 2.05, 2.12, 2.17, 2.17, 2.17, 2.35, 2.38, 2.41, 2.43, 2.48, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.76, 2.77, 2.79, 2.81, 2.81, 2.82, 2.83, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.51, 3.56, 3.60, 3.65, 3.68, 3.68, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90, 4.91, 5.08, 5.56.

Dataset 1

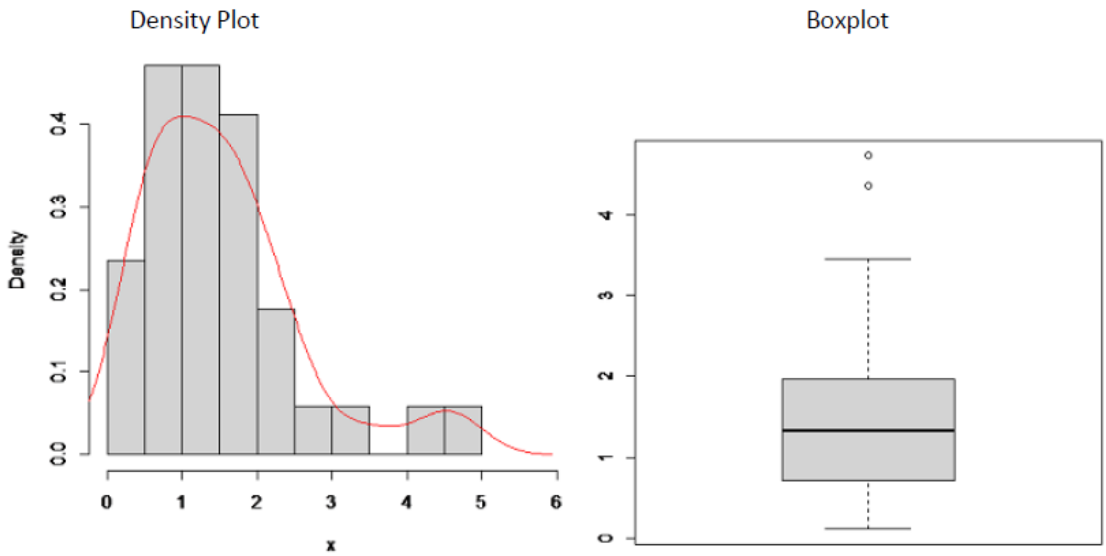

Figure 4: *Density and boxplots for dataset 1*

Dataset 2

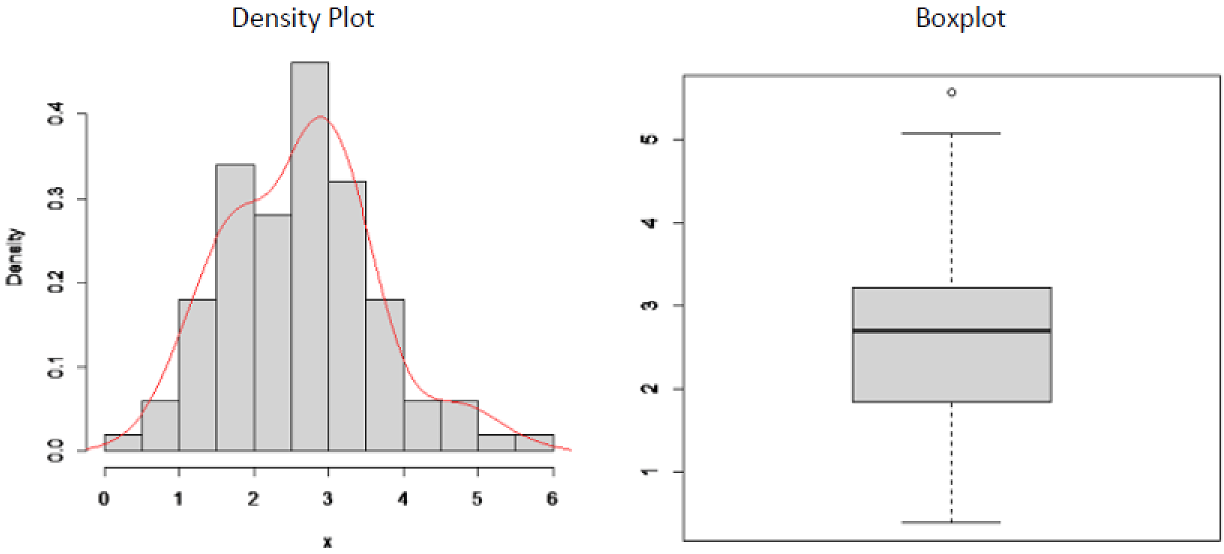

Figure 5: *Density and boxplots for dataset 2*

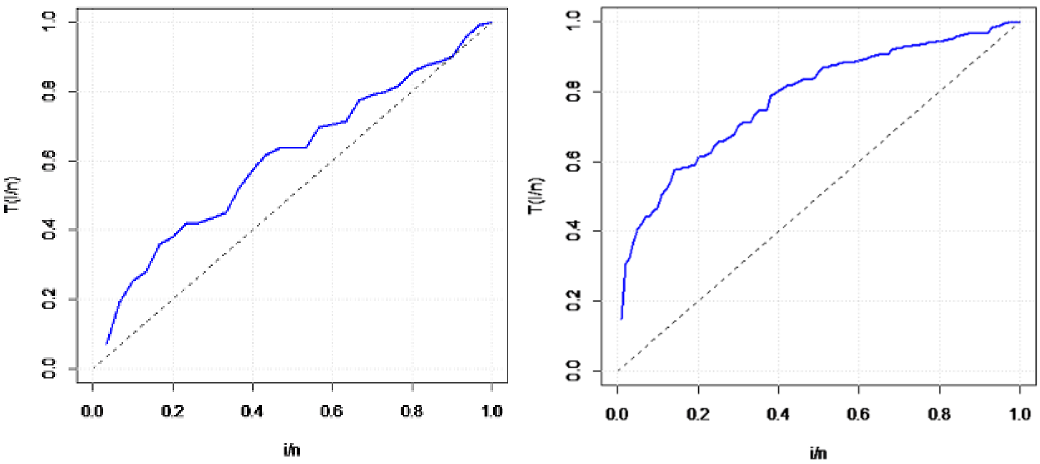

Figure 6: *TTT Plots for datasets 1 and 2*

Table 3: *MLEs and goodness-of-fit-statistics for dataset 1.*

Model	*a*	*B*	*C*	AIC	CAIC	BIC	HQIC	A*	W*
MGom	0.5463	0.1282	1.2171	85.3735	85.6965	89.5771	86.7183	0.1290	0.0173
PowGom	0.4806	0.0131	1.3732	86.9282	86.9513	91.6231	89.3729	0.3642	0.0532
ExpGom	1.8355	0.2814	1.3073	86.2447	86.8677	91.1483	88.0894	0.3466	0.0471
M-OGom	0.3422	0.3516	0.2785	88.6075	89.1306	92.3111	89.9522	0.4120	0.0868
Odd-lGom	0.5555	0.0223	1.5030	86.0282	86.2965	90.2318	87.2183	0.2114	0.0279

Table 4: *MLEs and goodness-of-fit-statistics for dataset 2*

Model	*a*	*B*	*C*	AIC	CAIC	BIC	HQIC	A*	W*
MGom	0.2628	0.1015	1.8936	278.528	278.779	286.344	281.691	0.3542	0.0563
PowGom	0.1010	0.0561	1.8671	298.771	299.022	306.587	301.934	0.5600	0.0791
ExpGom	0.3620	0.5327	0.7557	293.555	293.805	301.371	296.718	0.5109	0.0711
M-OGom	0.0835	0.7208	0.2346	307.515	307.765	315.331	310.678	0.8052	0.1309
Odd-lGom	0.1329	0.5124	1.6392	291.109	291.359	298.924	294.272	0.4564	0.0674

IV. Discussion

As observed from Tables 1 and 2, for all the different parameter settings, the average of the estimates for a, b, and c get closer to the true parameter values as the sample size increases. Also, the average Bias and the RMSE decrease as the sample size increases. These results validate the asymptotic properties of maximum likelihood estimators.

As observed from the density plot as well as box plot depicted in Figures 4 and 5, it is clear that dataset 1 is heavily skewed to the right and, dataset 2 is moderately skewed to the right, hence the two datasets are could be good for a flexible model like MGom distribution. The total time on test (TTT) curve of the datasets are also plotted in Figure 6 to obtain the empirical behaviour of the hazard function. As observed, the shapes of the hazard function of both datasets are concave showing increasing hazards, and this could also be a good candidate for Gompertz distribution and any of its compound distributions.

Tables 3 and 4 present the maximum likelihood estimates and the values of goodness-of-fit statistics for datasets 1 and 2 respectively. It was found that MGom distribution had the smallest values of all these measures (AIC, CAIC, BIC, HQIC, A^* and W^*) and therefore can be best used in comparison to other Gompertz extensions for modelling real life situations of positively skewed data with increasing hazard rates.

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