ON CERTAIN CLASSES OF CONFORMALLY FLAT LORENTZIAN PARA-KENMOTSU MANIFOLDS

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Abstract

In this present paper, we classify and explore the geometrical significance of a class of Lorentzian almost paracontact metric manifolds namely Lorentzian para-Kenmotsu (briefly LP-Kenmotsu) manifolds whenever the manifolds are either conformally flat or conformally symmetric. It was found that a conformally flat LP-Kenmotsu manifold is of constant curvature and a conformally symmetric LP-Kenmotsu manifold is locally isomorphic to a unit sphere. At the end, we obtain the scalar curvature of ϕ-conformally flat LP-Kenmotsu manifolds.

Keywords: Lorentzian para-Kenmotsu manifold, Weyl-conformal curvature tensor, Riemanninan curvature tensor, ϕ-conformal curvature tensor, η-Einstein manifold.

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I. Introduction

In 1995, Sinha and Sai Prasad [11] defined a class of almost paracontact metric manifolds namely para-Kenmotsu (briefly P-Kenmotsu) and special para-Kenmotsu (briefly SP-Kenmotsu) manifolds in similar to P-Sasakian and SP-Sasakian manifolds. In 1989, K. Matsumoto [3] introduced the notion of Lorentzian paracontact and in particular, Lorentzian para-Sasakian (LP-Sasakian) manifolds. Later, these manifolds have been widely studied by many geometers such as Matsumoto and Mihai [4], Mihai and Rosca [5], Mihai, Shaikh and De [6], Venkatesha and Bagewadi [13], Venkatesha, Pradeep Kumar and Bagewadi [14, 15].

In 2018, Abdul Haseeb and Rajendra Prasad defined a class of Lorentzian almost paracontact metric manifolds namely Lorentzian para-Kenmotsu (briefly LP-Kenmotsu) manifolds [1, 2] and they studied ϕ-semisymmetric LP-Kenmotsu manifolds with a quarter-symmetric non-metric connection admitting Ricci solitons [7, 8]. As an extension, Sai Prasad *et al.*, [9] have studied LP-Kenmotsu manifolds admitting the Weyl-projective curvature tensor of type (1, 3). Further, they also have studied and shown that the LP-Kenmotsu manifolds admitting both irrotational and conservative pseudo-projective curvature tensors are Einstein manifolds of constant scalar curvature [10].

In 2023, Sunitha and Sai Prasad [12] have defined a class of Lorentzian para-Kenmotsu manifolds admitting a quarter-symmetric metric connection and shown that it is either ϕ-symmetric or concircular ϕ-symmetric with respect to quarter-symmetric metric connection if and only if it is symmetric with respect to the Riemannian connection, provided the scalar curvature of Riemannian connection is constant. Recently, Rao, Sunitha and Sai Prasad [16] have studied ϕ-conharmonically flat and ϕ-projectively flat LP-Kenmotsu manifolds. They have shown that ϕ-conharmonically flat LP-Kenmotsu manifold is an η-Einstein manifold with zero-scalar curvature and ϕ-projectively flat LP-Kenmotsu manifold is an Einstein manifold with the scalar curvature $r = n(n-1)$.

In this work we explore a class of conformally flat Lorentzian para-Kenmotsu (LP-Kenmotsu) manifolds. The following is the layout of the current paper: Following the introduction, Section 2 includes some preliminaries on Lorentzian para-Kenmotsu manifolds. In section 3, we study conformally flat Lorentzian para-Kenmotsu manifolds and shown that they are of constant curvature. Further in section 4, we study and have shown that Lorentzian para-Kenmotsu manifold satisfying the condition $R(X,Y).C = 0$ is locally isomorphic to a unit sphere $S^n(1)$. Finally in section 5, it is shown that ϕ-conformally flat LP-Kenmotsu manifold is an η-Einstein manifold with the scalar curvature $r = n(n-1)$.

II. Preliminaries

An n-dimensional differentiable manifold M_n admitting a $(1,1)$ tensor field ϕ, contravariant vector field ξ, a 1-form η and the Lorentzian metric $g(X,Y)$ satisfying

$$\eta(\xi) = -1, \tag{1}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{3}$$

$$g(X,\xi) = \eta(X), \tag{4}$$

$$\phi\xi = 0, \;\; \eta(\phi X) = 0, \;\; rank\ \phi = n-1 \tag{5}$$

is called Lorentzian almost paracontact manifold [3].

In a Lorentzian almost paracontact manifold, we have

$$\Phi(X,Y) = \Phi(Y,X), \qquad where \qquad \Phi(X,Y) = g(\phi X, Y). \tag{6}$$

A Lorentzian almost paracontact manifold M_n is called Lorentzian para-Kenmotsu manifold if [1]

$$(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{7}$$

for any vector fields X and Y on M_n and ∇ is the operator of covariant differentiation with respect to the Lorentzian metric g.

It can be easily seen that in a LP-Kenmotsu manifold M_n, the following relations hold [1]:

$$\nabla_X \xi = -\phi^2 X = -X - \eta(X)\xi, \tag{8}$$

$$(\nabla_X \eta)Y = -g(X,Y)\xi - \eta(X)\eta(Y), \tag{9}$$

for any vector fields X and Y on M_n.

Also, in an LP-Kenmotsu manifold, the following relations hold [1]:

$$\begin{aligned} g(R(X,Y)Z,\xi) &= \eta(R(X,Y)Z) \\ &= g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \end{aligned} \tag{10}$$

$$R(\xi, X)Y = g(X,Y)\xi - \eta(Y)X, \tag{11}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{12}$$

$$S(X,\xi) = (n-1)\eta(X) \tag{13}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{14}$$

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{15}$$

for any vector fields X, Y and Z and S is the Ricci tensor of M_n.

The Riemannian Christoffel curvature tensor R of type $(1,3)$ is given by:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{16}$$

where ∇ be its Levi-Civita connection.

III. *LP*-Kenmotsu manifolds with $C(X,Y)Z = 0$

In this section, we consider conformally flat Lorentzian para-Kenmotsu manifolds.

The Weyl-conformal curvature tensor $C(X,Y)Z$ is given by

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z \\ &- \frac{1}{(n-2)}[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{17}$$

where

$$S(X,Y) = g(QX,Y).$$

Using (16), we get from (17)

$$\begin{aligned} R(X,Y)Z = &\frac{1}{(n-2)}[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{18}$$

By taking $Z = \xi$ in (18) and on using (4), (12) and (13), we get

$$\begin{aligned} \eta(Y)X - \eta(X)Y = &\frac{1}{(n-2)}[\eta(Y)QX - \eta(X)QY] + \frac{(n-1)}{(n-2)}[\eta(Y)X - \eta(X)Y] \\ &- \frac{r}{(n-1)(n-2)}[\eta(Y)X - \eta(X)Y]. \end{aligned} \tag{19}$$

Taking $Y = \xi$ and using (1), we get

$$QX = \Big(\frac{1}{n-1} - 1\Big)X + \Big(\frac{r}{n-1} - 1\Big)\eta(X)\xi. \tag{20}$$

It shows that the manifold is η-Einstein.

Further on contracting (20), we have

$$r = n(n-1). \tag{21}$$

Now, by using (21) in (20), we get

$$QX = (n-1)\,X. \tag{22}$$

Then by putting (22) in (19), we get

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{23}$$

Thus, a conformally flat *LP*-Kenmotsu manifold is of constant curvature. The value of this constant is +1. Hence, we can state

Theorem 1. A conformally flat *LP*-Kenmotsu manifold is locally isometric to a unit sphere $S^n(1)$.

IV. *LP*-Kenmotsu manifold satisfying $R(X,Y).C = 0$

Using (4), (11) and (13) we find from (17) that

$$\begin{aligned} \eta(C(X,Y)Z) = \frac{1}{n-2}[(\frac{r}{n-1}-1)(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)) \\ -(S(Y,Z)\eta(X) - S(X,Z)\eta(Y))]. \end{aligned} \tag{24}$$

Putting $Z = \xi$ in (24) and on using (4), (13) we get

$$\eta(C(X,Y)\xi) = 0. \tag{25}$$

Again, taking $X = \xi$ in (24), we get

$$\begin{aligned} \eta(C(\xi,Y)Z) = \frac{1}{n-2}[S(Y,Z) + (n-1)\eta(Y)\eta(Z)] \\ - \frac{1}{n-2}\Big(\frac{r}{n-1}-1\Big)[g(Y,Z) + \eta(Y)\eta(Z)]. \end{aligned} \tag{26}$$

Now,

$$\begin{aligned} (R(X,Y)C)(U,V)W = R(X,Y)C(U,V)W - C(R(X,Y)U,V)W \\ - C(U,R)(X,Y)V)W - C(U,V)R(X,Y)W. \end{aligned} \tag{27}$$

Using $R\,(X,Y)\,.C = 0$, we find from above that

$$\begin{aligned} g[R(\xi,Y)C(U,V)W,\xi] - g[C(R(\xi,Y)U,V)W,\xi] \\ - g[C(U,R(\xi,Y)V)W,\xi] - g[C(U,V)R(\xi,Y)W,\xi] = 0. \end{aligned}$$

Using (4) and (11) we get

$$\begin{aligned} &- C(U,V,W,Y) - \eta(Y)\eta(C(U,V)W) - g(Y,U)\eta(C(\xi,V)W) \\ &+ \eta(U)\eta(C(Y,V)W) - g(Y,V)\eta(C(U,\xi)W) + \eta(V)\eta(C(U,Y)W) \\ &- g(Y,W)\eta(C(U,V)\xi) + \eta(W)\eta(C(U,V)Y) = 0, \end{aligned} \tag{28}$$

where

$$C(U,V,W,Y) = g(C(U,V)W,Y).$$

Putting $U = Y$ in (28), we get

$$\begin{aligned} &- C(U,V,W,U) - \eta(U)\eta(C(U,V)W) + \eta(U)\eta(C(U,V)W) \\ &+ \eta(V)\eta(C(U,U)W) + \eta(W)\eta(C(U,V)U) - g(U,U)\eta(C(\xi,V)W) \\ &- g(U,V)(C(U,\xi)W) - g(U,W)\eta(C(U,V)\xi) = 0. \end{aligned} \tag{29}$$

Let $\{e_i$: i= $1,\ldots,n\}$ be an orthonormal basis of the tangent space at any point, then the sum for $1{\le}i{\le}n$ of the relations (29) for $U = e_i$ gives

$$(1-n)\eta(C(\xi,V)W) = 0,$$

which implies

$$\eta(C(\xi, V)W) = 0 \;\; as \;\; n > 3. \tag{30}$$

Using (25) and (30), (28) takes the form

$$\begin{aligned} &- C(U,V,W,Y) - \eta(Y)\eta(C(U,V)W) + \eta(U)\eta(C(Y,V)W) \\ &+ \eta(V)\eta(C(U,Y)W) + \eta()\eta(C(U,V)Y) = 0. \end{aligned} \tag{31}$$

Using (24) in (31) we get

$$\begin{aligned} -C(U,V,W,Y) + \eta(W)\frac{1}{n-2}\Big[\Big(\frac{r}{n-1} - 1\Big)\big(\eta(U)g(V,Y) - \eta(V)g(U,Y)\big) \\ - \big(\eta(U)S(V,Y) - \eta(V)S(U,Y)\big)\Big] = 0. \end{aligned} \tag{32}$$

In virtue of (30),(26) reduces to

$$S(Y,Z) = \Big(\frac{r}{n-1} - 1\Big)g(Y,Z) + \Big(\frac{r}{n-1} - n\Big)\eta(Y)\eta(Z). \tag{33}$$

Using (33), (31) reduces to

$$C(U,V,W,Y) = 0, \tag{34}$$

which proves that the manifold is conformally flat. Hence, by using the Theorem 1, we state

Theorem 2. If in an *LP*-Kenmotsu manifold $M_n(n> 3)$ the relation $R(X,Y).C= 0$ holds, then it is locally isometric with a unit sphere $S^n(1)$.

For a conformally symmetric Riemannian manifold, we have $\nabla C= 0$. Hence for such a manifold $R(X,Y).C= 0$ holds. Thus, we have the following corollary of the above theorem.

Corollary 1. A conformally symmetric *LP*-Kenmotsu manifold M_n $(n> 3)$ is locally isometric with a unit sphere $S^n(1)$.

V. ϕ-conformally flat *LP*-Kenmotsu manifold

Let C be the Weyl conformal curvature tensor of M_n. Since at each point $p\in M_n$ the tangent space $T\left(M_n\right)$ can be decomposed into the direct sum $T_p\left(M_n\right) =\phi\left(T_p\left(M_n\right)\right) \oplus L\left(\xi_p\right)$, where $L\left(\xi_p\right)$ is a 1-dimensional linear subspace of $T_p\left(M_n\right)$ generated by ξ_p, we have a map:

$$\mathrm{C} : T_p\left(M_n\right) \times T_p\left(M_n\right) \times T_p\left(M_n\right) \to \varphi\left(T_p\left(M_n\right)\right) \oplus L\left(\xi_p\right)$$

It may be natural to consider the following particular cases:

1. C:$T_p\left(M_n\right) \times T_p\left(M_n\right) \times T_p\left(M_n\right) \to L\left(\xi_p\right)$, that is, the projection of the image of C in $\phi\left(T_p(M_n)\right)$ is zero.
2. C:$T_p\left(M_n\right) \times T_p\left(M_n\right) \times T_p\left(M_n\right) \to \phi\left(T_p\left(M_n\right)\right)$, that is, the projection of the image of C in $L\left(\xi_p\right)$ is zero.
3. C:$\phi\left(T_p\left(M_n\right)\right) \times \phi\left(T_p\left(M_n\right)\right) \times \phi\left(T_p\left(M_n\right)\right) \to L\left(\xi_p\right)$, that is, when C is restricted to $\left(T_p\left(M_n\right)\right) \times \phi\left(T_p\left(M_n\right)\right) \times \varphi\left(T_p\left(M_n\right)\right)$, the projection of the image of C in $\phi\left(T_p\left(M_n\right)\right)$ is zero. This condition is equivalent to

$$\phi^2 C(\phi X, \varphi Y)\phi Z= 0. \tag{35}$$

Definition 1. A differentiable manifold (M_n, g) , $n > 3$, satisfying the condition (35) is called ϕ-conformally flat.

Now our aim is to find the characterization of *LP*-Kenmotsu manifolds satisfying the condition (35).

Theorem 3. Let M_n be an n-dimensional, $(n > 3)$, ϕ-conformally flat *LP*-Kenmotsu manifold. Then M_n is an η-Einstein manifold.

Proof. Suppose that (M_n, g) , $n > 3$, is a ϕ-conformally flat $LP-$ Kenmotsu manifold. It is easy to see that $\phi^2 C(\phi X, \phi Y)\phi Z = 0$ holds if and only if $g(C(\phi X, \phi Y)\phi Z, \varphi W) = 0$ for any $X, Y, Z, W \in \chi(M_n)$. So, by the use of (17), ϕ-conformally flat means

$$\begin{aligned} g(R(\phi X, \phi Y)\phi Z, \phi W) = \frac{1}{n-2}&[g(\phi Y, \phi Z)S(\phi X, \phi W) - g(\phi X, \phi Z)S(\phi Y, \phi W) \\ &+ g(\phi X, \phi W)S(\phi Y, \phi Z) - g(\phi Y, \phi W)S(\phi X, \phi Z)] \\ &- \frac{r}{(n-1)(n-2)}[g(\phi Y, \phi Z)g(\phi X, \phi W) \\ &- g(\phi X, \phi Z)g(\phi Y, \phi W)]. \end{aligned} \tag{36}$$

Let $\{e_1, \ldots, e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M_n. Using that $\{\phi e_1, \ldots, \phi e_{n-1}, \xi\}$ is also a local orthonormal basis, if we put $X = W = e_i$ in (36) and sum up with respect to i, then

$$\begin{aligned} \sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = \frac{1}{n-2}&\sum_{i=1}^{n-1}[g(\phi Y, \phi Z)S(\phi e_i, \phi e_i) \\ &- g(\phi e_i, \phi Z)S(\phi Y, \phi e_i) + g(\phi e_i, \phi e_i)S(\phi Y, \phi Z) \\ &- g(\phi Y, \phi e_i)S(\phi e_i, \phi Z)] \\ &- \frac{r}{(n-1)(n-2)}\sum_{i=1}^{n-1}[g(\phi Y, \phi Z)g(\phi e_i, \phi e_i) \\ &- g(\phi e_i, \phi Z)g(\phi Y, \phi e_i)]. \end{aligned} \tag{37}$$

It can be easily verified that

$$\sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = S(\phi Y, \phi Z) + g(\phi Y, \phi Z), \tag{38}$$

$$\sum_{i=1}^{n-1} S(\phi e_i, \phi e_i) = r + n - 1, \tag{39}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z) S(\phi Y, \phi e_i) = S(\phi Y, \phi Z), \tag{40}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n + 1, \tag{41}$$

and

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z) g(\phi Y, \varphi e_i) = g(\phi Y, \phi Z). \tag{42}$$

So, by virtue of (38)-(42) the equation (37) can be written as

$$S(\phi Y, \phi Z) = \left(\frac{r}{n-1} - 1\right) g(\phi Y, \phi Z). \tag{43}$$

Then by making use of (3) and (14), the equation (43) takes the form

$$S(Y,Z)=\left(\frac{r}{n-1}-1\right)g(Y,Z)+\left(\frac{r}{n-1}-n\right)\eta(Y)\eta(Z). \tag{44}$$

Therefore from (44), by contraction, we obtain

$$r=n(n-1). \tag{45}$$

Then by substituting (45) in (44), we get

$$S(Y,Z)=(n-1)g(Y,Z)$$

which implies M_n is an η-Einstein manifold with the scalar curvature $r=n(n-1)$.

This completes the proof of the theorem.

VI. Conclusion

The present work explores the geometrical significance of a new class of Lorentzian paracontact metric manifolds namely the Lorentzian para-Kenmotsu manifolds whenever these manifolds are either conformally symmetric or conformally flat. The concepts and various geometrical properties of these manifolds can be applied in various aspects of Applied Mathematics such as Computational Fluid Dynamics, in designing the Super Resolution Sensors in Communications Engineering, and also in the field of General Theory of Relativity.

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