# APPLICATION OF POLAR COORDINATES IN THE SUMMATION OF THE GAUSSIAN DISTRIBUTION

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**Abstract**

*This work applies the polar coordinates system of advanced calculus in the summation of the Gaussian distribution. In trying to achieve this aim, sub-concepts such as complex variables, gamma function of half, error function, and the relation between the error function and the standard normal distribution were defined and explained at various stages of the work. The embedded theorem which seems to be a new theorem also came up in the body of the work.*

**Keywords:** Normal distribution, Standard Normal Distribution, Gaussian Distribution, gamma Function of Half, Embedded Theorem, Polar Coordinates.

## 1. Introduction

When Mathematics is used to study observational phenomena, a mathematical model is constructed for the phenomena. This involves an idealization and simplification of the original phenomena to the extent that a mathematical problem is developed. The mathematical solution obtained, eventually has to be interpreted in terms of the original problem.There are essentially two types of mathematical models:the deterministic model and the non-deterministic or probabilistic model [12]. The deterministic model is a model which stipulates that the conditions under which an experiment is performed determine the outcome of the experiment. Example, a body is allowed to fall freely from a height above ground level, the distance(s) traveled is completely determined by the time $t$ (seconds) during which the body has been in motion and the initial velocity $u$ with acceleration $a$, is given as $S = a_t^2 + u_t$. Based on the given expression, it is possible to determine the value of $S$ for known values of $u$ and $t$. this shows that for deterministic models, the results of the experiment depend only on the physical conditions operating [4, 2].

However, non-deterministic or probabilistic models introduce uncertainty into the mathematical problem [7]. In the context of probabilistic models, the Gaussian distribution, also known as the normal distribution, plays a vital role in various fields such as statistics, physics, finance, and engineering [6]. In recent years, there has been a growing interest in developing efficient methods for the summation of the Gaussian distribution. One such method is the application of polar coordinates in the summation of the Gaussian distribution[8]. [10] proposed a Bayesian inferential method for directional data modelled by projected normal distributions. [18] Projected normal distributions, also referred to as angular Gaussian distributions, are created by imposing different constraints on the parameter space associated with a multivariate Gaussian distribution. This resolves the non-identifiability issue that arises when the support of a random variable changes from an Euclidean space to a spherical space [9]. In mathematics and statistics, the Gaussian distribution, also known as the normal distribution, is a crucial concept used to model various real-world phenomena that exhibit a bell-shaped curve [17] [13].

The Gaussian distribution is characterised by its mean and standard deviation, which determine the central tendency and spread of the distribution, respectively [11]. In the work by [10], they proposed a Bayesian inferential method for directional data modeled by projected normal distributions, which are also referred to as angular Gaussian distributions. These distributions are created by imposing different constraints on the parameter space associated with a multivariate Gaussian distribution, allowing for the resolution of the non-identifiability issue when the support of a random variable changes from a Euclidean space to a spherical space. The general projected normal distribution, a simple and intuitive model for directional data in any dimension, is discussed by [10]. They describe a new parameterisation of the general projected normal distribution that makes inference in any dimension tractable, including the important three-dimensional case. This new parameterisation allows for closed-form full conditionals of the unknown parameters and proposes a slice sampler to draw the latent lengths without rejection. The work by [10] demonstrates the applicability and effectiveness of the projected normal distribution in modeling directional data, particularly in higher dimensions.

## 1.1. Statement of the Problem

In an attempt to prove that

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp^{-\left(\frac{(x-\mu)^2}{2\sigma^2}\right)} \delta_k = 1 \tag{1.1}$$

one will meet the following problems:

1. One must understand the meaning of the gamma function of half which is defined by [15] as

$$\Gamma\left(1/2\right) = \int_{0}^{\infty} t^{-1/2} \exp^{-t} \delta_t \tag{1.2}$$

2. The proof of the integral function

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp^{-(1/2)t^2} \delta_t = 1 \tag{1.3}$$

   must be known.

This work will make these problems easy to see.

## 1.2. Aim and Objectives of the Study

The aim of this work is to show clearly that equation 1.1 is equal to 1 without making assumptions of any kind. The main objectives of this work is as follows:

1. The derivation of the Gaussian distribution.
2. To prove the Gaussian distribution using the direct integration method.

## 2. Materials and Methods

### 2.1. Binomial Distribution

**Proposition 1.** If $X$ is a binomial random variable, then the probability of obtaining $x$ successes in n trials of a binomial experiment with probability of success $P$ is given by

$$f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}; \quad x = 0, 1, 2, \cdots, n; \quad 0 < p < 1 \\ 0, \quad \text{otherwise} \end{cases} \tag{2.1}$$

We show that $f(x)$ is a probability distribution function with parameters $n$ and $P$. At this stage, n is a positive integer and $0 < P < 1$, it is clear that $f(x) \geq 0$

$$\begin{aligned} \sum_{i=0}^{n} f(x) &= \sum_{i=0}^{n} \binom{n}{x} p^x (1-p)^{(n-x)} \\ &= [(1-p)+p]^n \\ \therefore \sum_{i=0}^{n} f(x) &= 1 \end{aligned} \tag{2.2}$$

Proposition 1 is called the binomial distribution.

**Theorem 2.1.** If $X$ has binomial distribution, then the moment-generating function of the random variable $X$ is $M_X t = \left[(1-p) + Pe^t\right]^n$ **Proof.**

$$\begin{aligned} M_X t &= \mathbb{E}\left(e^{tX}\right) \\ &= \sum_{i=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} \\ &= \sum_{i=0}^{n} e^{te} \binom{n}{x} \left(pe^t\right)^x (1-p)^{n-x} \\ M_X t &= \left[(1-p) + pe^t\right]^n \end{aligned}$$

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**Corrolary 1.** If $X$ has a binomial distribution, then

$$\mathbb{E}(X) = np \tag{2.3}$$

$$\mathbb{V}\text{ar}(X) = np(1-p) \tag{2.4}$$

### 2.2. The Derivation of the Normal Distribution

[1] states the limit of the symmetrical binomial distribution using theorem 2.2 below.

**Theorem 2.2.** If $X$ has a symmetrical binomial distribution with mean $\mu$ and variance $\sigma^2$,then as $n$ tends to infinity,

$$Z = \tfrac{(x-\mu)}{\sigma^2} \tag{2.5}$$

Equation 2.5 approaches the standard normal distribution.

**Proof.**

$$\begin{aligned} &\mu = np = \tfrac{1}{2}n; \\ &\sigma = \sqrt{np(1-p)} = \tfrac{1}{2}\sqrt{n}; \\ &Z = \frac{x-\mu}{\sigma} = \frac{x - \frac{1}{2}n}{\frac{1}{2}\sqrt{n}}; \end{aligned}$$

Now the distance $\Delta Z$ between successive values of Z is given by

$$\Delta Z = \frac{(x+1) - \frac{1}{2}n}{\frac{1}{2}\sqrt{n}} - \frac{x - \frac{1}{2}n}{\frac{1}{2}\sqrt{n}} = \frac{1}{\frac{1}{2}\sqrt{n}} \tag{2.6}$$

$$\lim_{n\to\infty} \Delta Z = 0$$

Hence the symmetrical binomial histogram will appears to become more like a curve as n tends to infinity

.

We take the value of $f(x) = Y$ . Then the distance $\Delta Y$ is the value between two successive values of Y.

We take the values corresponding to x and $x+1$ and multiply them by $\sigma$ .

$$\begin{aligned}
Y &= \binom{n}{x} p^x \left(1-p\right)^{n-x} \sigma \\
&= \frac{n!}{(n-x)!\,x!} \left(\tfrac{1}{2}\sqrt{n}\right) \left(\tfrac{1}{2}n\right) \\
\Delta Y &= \frac{n!}{(n-x+1)!\,x+1!} \left(\tfrac{1}{2}n\right)\left(\tfrac{1}{2}\sqrt{n}\,\right) \\
&\quad - \frac{n!}{(n-x)!\,x!} \left(\tfrac{1}{2}n\right)\left(\tfrac{1}{2}\sqrt{n}\,\right) \\
&= \left(\tfrac{1}{2}n\right)\left(\tfrac{1}{2}\sqrt{n}\,\right) n! \\
&\quad \times \left[\frac{(n-x)!\,x! - (n-x+1)!\,(n-x)!}{(n-x+1)!\,(n-x)!\,(n-x)!\,x!}\right] \\
&= \left(\tfrac{1}{2}n\right)\left(\tfrac{1}{2}\sqrt{n}\,\right) n! \\
&\quad \times \left[\frac{\cancel{(n-x+1)!\,x!}[(n+x) - (x+1)]}{\cancel{(n-x+1)!}\,(x+1)!\,(n-x)!\,\cancel{x!}}\right] \\
&= \left(\tfrac{1}{2}n\right)\left(\tfrac{1}{2}\sqrt{n}\,\right)\sqrt{n}\big)\left[\frac{n!}{(x+1)!\,(n-x)!}\right] \\
&\quad \times \left[(n-x) - (x+1)\right] \\
&= \left(\tfrac{1}{2}n\right)\left(\tfrac{1}{2}\sqrt{n}\,\right)\left[\frac{n!}{(n-x)!\,x!}\right] \\
&\quad \times \frac{(n-x-x-1)}{(x+1)} \\
\Delta Y &= Y\left[\frac{(n-2x-1)}{(x+1)}\right]
\end{aligned}$$

From equation 2.6

$$\begin{aligned}
\Delta Z &= \frac{1}{\frac{1}{2}\sqrt{n}} \\
\therefore \frac{\Delta Y}{\Delta Z} &= Y\left[\frac{(n-2x-1)}{(x+1)}\right]\tfrac{1}{2}\sqrt{n}
\end{aligned} \tag{2.7}$$

From equation 2.5

$$\begin{aligned} Z &= \frac{(x-\mu)}{\sigma^2} = Z\sigma + \mu \\ x &= Z\tfrac{1}{2}\sqrt{n} + \tfrac{1}{2}n \end{aligned} \tag{2.8}$$

Substitute equation 2.8 into equation 2.7, we have

$$\begin{aligned} \frac{\Delta Y}{\Delta Z} &= Y\left[\frac{\left(n-\sqrt{n}Z-n-1\right)}{\left(\frac{1}{2}\sqrt{n}Z+\frac{1}{2}n+1\right)}\right]\tfrac{1}{2}\sqrt{n} \\ \frac{\Delta Y}{\Delta Z} &= Y\left[\frac{-\left(\frac{1}{2}n\right)Z-\left(\frac{1}{2}\sqrt{n}\right)}{\left(\frac{1}{2}\sqrt{n}\right)Z+\left(\frac{1}{2}n\right)+1}\right] \end{aligned}$$

$\lim_{n\to\infty}\dfrac{\Delta Y}{\Delta Z}$ tends to $\dfrac{\delta Y}{\delta Z} = -YZ$ separating the variables

$$\begin{aligned} \int \frac{\delta Y}{Y} &= \int -Z\delta Z \\ log_e Y &= \frac{-Z^2}{2} + log_e K \end{aligned}$$

where K is the constant of Integration

$$\begin{aligned} log_e \frac{Y}{K} &= \frac{-Z^2}{2} \\ \therefore Y &= Ke^{-\frac{1}{2}Z^2} \end{aligned} \tag{2.9}$$

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### 2.3. The Proof of the Standard Normal Distribution Using Substitution Method

[16] states the standard normal distribution as in theorem 2.3 below.

**Theorem 2.3.** The random variable Z is said to have a standard normal distribution if its pdf is $\varphi(Z) = f(Z;0,1) = \dfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}Z^2}$ We show below that $\varphi(z)$ is a valid pdf **Proof.**

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}Z^2}\delta Z = \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-\frac{1}{2}Z^2}\delta Z \tag{2.10}$$

let $x = \frac{1}{2}Z^2$, so that

$$\delta Z = \frac{\sqrt{2}}{2\sqrt{x}\delta x} \tag{2.11}$$

Substitute equation 2.11 into equation 2.10

$$\begin{aligned}
\frac{2}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}Z^2}\delta Z &= \frac{1}{\sqrt{\pi}}\int_0^\infty x^{-\frac{1}{2}}e^{-x}\delta x \\
&= \tfrac{1}{2}\sqrt{\pi}\Gamma\left(\tfrac{1}{2}\right) \\
&= \frac{\sqrt{\pi}}{\sqrt{\pi}} \\
\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\frac{1}{2}Z^2}\delta Z &= 1
\end{aligned} \tag{2.12}$$

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[16] is silent about the origin of the standard normal distribution. Also there is no attempt to show Let us call equation 2.13

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \tag{2.13}$$

In the proof, we shall called equation 2.13 assumption 1. [16] also stated theorem 2.4 below.

**Theorem 2.4.** Let Z have a standard normal distribution. Define x to be $x = \sigma Z + \mu$. Then it can be shown that x is a random normal variable with pdf given as

$$f\left(x;\mu,\sigma^2\right) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

**Proof.**

$$\begin{aligned}
x &= \sigma z + \mu \\
Z &= \frac{x-\mu}{\sigma} \\
\frac{\delta z}{\delta x} &= \frac{1}{\sigma}
\end{aligned}$$

The density function for x is

$$\begin{aligned}
f\left(x;\mu,\sigma^2\right) &= \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \\
&\begin{cases} -\infty < \mu < \infty \\ \sigma > 0 \end{cases}
\end{aligned} \tag{2.14}$$

Now we now show that $f\left(x;\mu,\sigma^2\right) = 1$

$$f\left(x;\mu,\sigma^2\right) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\delta x \tag{15}$$

Let $y = \dfrac{x-\mu}{\sigma}$, $x = y\sigma + \mu$

$$\delta_x = \sigma\delta_y \tag{16}$$

Substituting equation 16 into equation 15 we have

$$\int_{-\infty}^{\infty} f(y;0,1) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\delta_y$$

From Theorem 2.3

$$\int_{-\infty}^{\infty} f(y;0,1) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\delta_y = 1 \tag{2.17}$$

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His entire work rest on the assumption 1 of Theorem 2.3. Assumption 1 is the **gamma function of half**. Theorem 2.5 below is the proof of the gamma function of half as resented by [5].

**Theorem 2.5.** $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

**Proof.**

$$\Gamma(n) = \int_{0}^{\infty} x^{n-1}e^{-x}\delta_x$$

Let $n = \frac{1}{2}$

$$\Gamma(\tfrac{1}{2}) = \int_{0}^{\infty} x^{-\frac{1}{2}}e^{-x}\delta_x \tag{18}$$

Put $x^{\frac{1}{2}} = \frac{1}{\sqrt{2}}t$, $x = \frac{1}{2}t^2$, $\frac{\delta_x}{\delta_t} = t$

$$\delta_x = t\delta_t \tag{19}$$

Substituting equation 19 into equation 18 we have

$$
\begin{aligned}
\Gamma(1/2) &= \int_0^\infty \left(\tfrac{1}{2}t^2\right)^{-1/2} e^{-\frac{1}{2}t^2} t\delta_t \\
&= \int_0^\infty e^{-\frac{1}{2}t^2} \left(\tfrac{1}{2}t^2\right)^{-1/2} t\delta_t \\
&= \sqrt{2}\int_0^\infty e^{-\frac{1}{2}t^2} t\delta_t \\
&= \tfrac{1}{2}\sqrt{2}\int_0^\infty e^{-\frac{1}{2}t^2} t\delta_t \\
&= \frac{1}{2}\sqrt{2}\sqrt{2\pi}\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}\delta_t \qquad (20)\\
&= \frac{1}{2}\sqrt{2}\sqrt{2\pi}\times 1 \\
&= \frac{1}{2}\sqrt{2}\sqrt{2}\sqrt{\pi} \\
\Gamma(1/2) &= \sqrt{\pi}
\end{aligned}
$$

■

At equation 20 she made the assumption that $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}\delta_t = 1$. Let called this **assumption 2**. The success of the proof of Theorem 2.5 depends on assumption 2. Now for [16] to prove assumption 2 (2.3), he made assumption 1 [(2.5). Also for [5] to prove assumption 1 (2.5) she made assumption 2 (2.3). Let us see how [5] presents the proof of the normal distribution. She used theorem 2.6 below.

**Theorem 2.6.** A random variable X has a normal distribution and is referred to as a normal random variable if and only if its probability density is given by $f\left(x;\mu,\sigma^2\right) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

**Proof.** Since $e^x$ is always positive, it follows that f(x) $\geq 0$ as long as $\sigma > 0$.
We show that the total area under the curve is equal to 1. That is, to show that

$$\int_{-\infty}^{\infty} f(x)\delta_x = 1$$

Let $Z = \frac{(x-\mu)}{\sigma^2}$ and $\delta_x = \sigma\delta_z$

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\delta_x &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\delta_z \\
&= \int_0^\infty \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\delta_z \qquad (21)
\end{aligned}
$$

But $\int_0^\infty e^{-\frac{1}{2}z^2}\delta_z = \frac{\Gamma(1/2)}{\sqrt{2}}$

(22)

Substituting equation 22 into equation 21 we have

$$= \frac{2}{\sqrt{2\pi}} \times \frac{\Gamma(1/2)}{\sqrt{2}}$$
$$= \frac{2 \times \sqrt{\pi}}{\sqrt{2\pi} \times \sqrt{2}} \tag{23}$$
$$\therefore \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \delta_x = 1$$

(2.24)

■

Again we can see the assumption 1 at the point of equation 23. That is $\Gamma(1/2) = \sqrt{\pi}$. Just like the work of [16], [5] also made the same assumption 1 in order to prove the standard normal distribution, which in this case is known as assumption 2.

#### 2.3.1 The Complex Number System

[14, 3] stated that there is no real number x that satisfies the polynomial equation

$$x^2 + 1 = 0 \tag{2.25}$$

To permit solution of equation 2.25 and other similar equations, the set of complex number is introduced. A complex number takes the form.

$$z = a + bi \tag{2.26}$$

Where a and b are real numbers and i which is called the imaginary unit has the property that.

$$i = -1 \tag{2.27}$$

From equation 2.26, a is called the real part of z and b is called the imaginary part of z. z is called a complex variable.

#### 2.3.2 The Argand Diagram

The real number can be graphically represented as a point on the real line. By using the cartesian coordinate system, a pair of real numbers can be graphically represented by a point in the plane. The Argand diagram is a device which represents complex numbers in the plane of the Cartesian coordinate system. The pair of real numbers $a$ and $b$ of equation 2.26 are plotted as a point in the plane and then joined that point to the origin with a straight line. See figure 2.1 below.

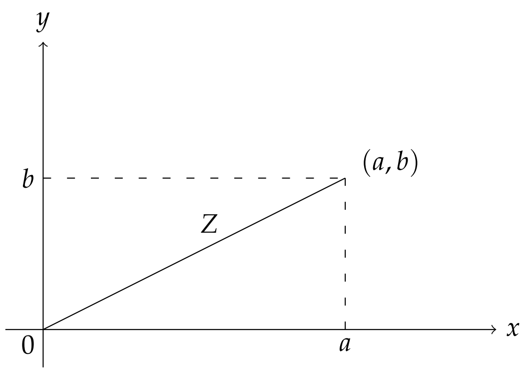


**Figure 2.1:** *Argand Diagram 1*

Figure 2.1 presents a visual representation of a complex number in polar coordinates on an Argand diagram. The diagram consists of a complex plane with a horizontal real axis ($X - axis$) and a vertical imaginary axis ($Y - axis$). A complex number $'z'$ is depicted as a point in this plane. The distance $'r'$ from the origin to the point represents the modulus of the complex number, which is the magnitude of the vector. The angle $'\theta'$ (theta) between the positive real axis and the line segment connecting the origin to the point 'z' represents the argument of the complex number, which indicates its direction. The coordinates $'a'$ and $'b'$ on the real and imaginary axes, respectively, correspond to the real and imaginary parts of the complex number. The polar form of the complex number is expressed as $'z = r(\cos\theta + \sin\theta)'$, which provides an alternative way to represent complex numbers using the magnitude and angle instead of the traditional rectangular form $'a + b_i'$.

According to [3], this straight line is the graphical representation of the complex variable z of equation refeq2.14. The plane it is plotted against is referred to as the complex plane. The entire diagram is called an Agrand Diagram.

## 2.4. Polar Form of a Complex Variable

We can express the complex variable of equation refeq2.14 in a different form on an Argand diagram. Let $Oz$ be a complex variable. Let $r$ be the length of complex variable and 0 the angle made with $OX$. See figure 2.2 below.

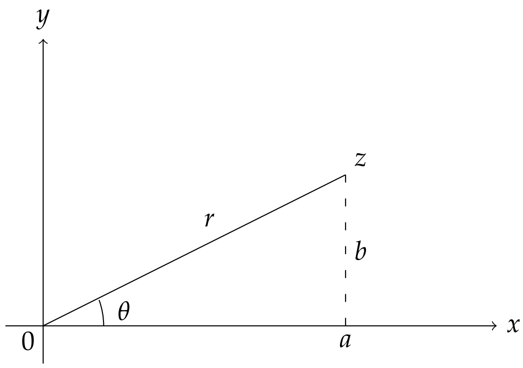


**Figure 2.2:** *Argand Diagram 2*

Figure 2.2 demonstrates the Cartesian coordinate system with the X-axis and Y-axis representing the real and imaginary parts of complex numbers, respectively. The figure is used to explain the concept of integrating the function $e^( - x?)$ over the entire range of x to obtain the value of the integral 'I'. The shaded area under the curve of the function $e^( - x?)$ in the first quadrant of the $(X?Y)$ plane represents the geometric interpretation of the integral. The integral 'I' is a key component in the derivation of the standard normal distribution and is related to the gamma function and the area under the normal curve.

From Figure 2.2

$$r = \sqrt{a^2 + b^2} \tag{2.28}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) \tag{2.29}$$

$$a = r\cos\theta \tag{2.30}$$

$$b = r\sin\theta \tag{2.31}$$

Substituting equation 2.30 and equation 2.30 into equation 2.26 we have

$$z = r\left(\cos\theta + \sin\theta\right) \tag{2.32}$$

Equation 2.32 is the polar form of equation 2.26, r is called the modulus of the complex variable z and is often abbreviated to 'Mod z' or indicated by $|z|$ . 0 is called the argument of the complex variable and can be abbreviated to 'arg z'.

## 2.5. Integral Functions

The gamma function $\Gamma(x)$ is defined by the integral

$$\Gamma(x) = \int_{0}^{\infty} t^{x-1}e^{-t}\delta_t \text{ for } x > 0 \tag{33}$$

Integrating equation 33 by part we have

$$\Gamma(x+1) = x\Gamma(x) \tag{34}$$

When $x = n$, a positive integer greater than 1, equation 34 becomes

$$\Gamma(n+1) = n!\Gamma(1) \tag{35}$$

From equation 33 we have that

$$\Gamma(1) = 1 \tag{36}$$

Substitute equation 36 into equation 35 we have

$$\Gamma(n+1) = n! \tag{37}$$

When $x = 1/2$

equation 33 becomes

$$\Gamma(1/2) = \int_{0}^{\infty} t^{(\frac{1}{2})}e^{-t}\delta_t \tag{38}$$

# 3. Analysis and Result

## 3.1. The Direct Intergration Method

### 3.1.1 The Gamma Function of Half $\Gamma(1/2)$

**Theorem 3.1.** The gamma function of half defined as follows:

$$\begin{aligned}\Gamma(1/2) &= \int_{0}^{\infty} t^{(1/2)}e^{-t}\delta_t \\ &= \Gamma(\pi)\end{aligned}$$

**Proof.**

$$\Gamma(1/2) = \int_0^\infty t^{(1/2)} e^{-t} \delta_t$$

Let $t = u^2$; $\delta_t = 2u\delta_u$, $\Gamma(1/2) = \int_0^\infty u^{-1} e^{-u^2} 2u\delta_u$

$$\Gamma(1/2) = 2\int_0^\infty e^{-u^2} \delta_u \tag{1}$$

Unfortunately, $\int_0^\infty e^{-u^2}\delta_u$ cannot easily be determined by normal means. It is however, important, so we have to find a way of getting round the difficulty. We now convert equation 1 into the polar coordinates form. See figure 3.1 below.

Let $I = \int_0^\infty e^{-x^2}\delta_x$ Then also $I = \int_0^\infty e^{-y^2}\delta_y$

$$\begin{aligned} I^2 &= \left(\int_0^\infty e^{-x^2}\delta_x\right)\left(\int_0^\infty e^{-y^2}\delta_y\right) \\ &= \left(\int_0^\infty e^{-x^2}\delta_x\right)\left(\int_0^\infty e^{-y^2}\delta_y\right) \\ &= \int_0^\infty \int_0^\infty e^{\left(x^2+y^2\right)}\delta_x\delta_y \end{aligned} \tag{2}$$

$\delta_a = \delta_x\delta_y$ represent an element of area in the $(X - Y)$ plane and the integration with the stated limit covers the whole of the first quadrant. See figure 3.2 below

Now converting to polar coordinates, the element of area becomes $\delta_a = r\delta\theta\delta_r$

$$r^2 = x^2 + y^2 \tag{3}$$
$$e^{-(x^2+y^2)} = e^{-r^2}$$

Form figure 3.2 below the limit of $r$ are $0 \leq r \leq \infty$. The limit of $\theta$ are $O \leq \theta \leq \pi/2$. Equation 2 becomes

$$I^2 = \int_0^{\left(\frac{\pi}{2}\right)} \int_0^\infty e^{-r^2} r\delta_r\delta\theta \tag{4}$$

Let $k = r^2, \quad \delta_k = 2r\delta_r$

$$
\begin{aligned}
&= \int_0^{\left(\frac{\pi}{2}\right)} \int_0^{\infty} e^{\frac{-k^2}{2}} \delta_k \delta\theta \\
&= \int_0^{\left(\frac{\pi}{2}\right)} \left[-\frac{1}{2}e^{-k}\right]_0^{\infty} \delta\theta \\
&= \int_0^{\left(\frac{\pi}{2}\right)} \left(\frac{1}{2}\right) \delta\theta \\
&= \left[\frac{\theta}{2}\right]_0^{\left(\frac{\pi}{2}\right)} \\
&= \frac{\pi}{4} \\
\therefore I &= \frac{\sqrt{\pi}}{2} \qquad (5)
\end{aligned}
$$

Before the diversion into the polar coordinates, we had established equation 1 that $\Gamma\left(1/2\right) = 2\int_0^{\infty} e^{-u^2}\delta_u$

Then substitute equation 5 into equation 1, $\Gamma\left(1/2\right) = 2 \times \frac{1}{2}\sqrt{\pi}$

$$\Gamma\left(1/2\right) = \sqrt{\pi} \qquad (6)$$

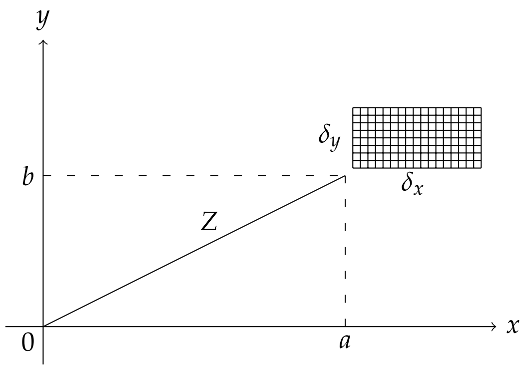


**Figure 3.1:** *The $(X - Y)$ Plane*

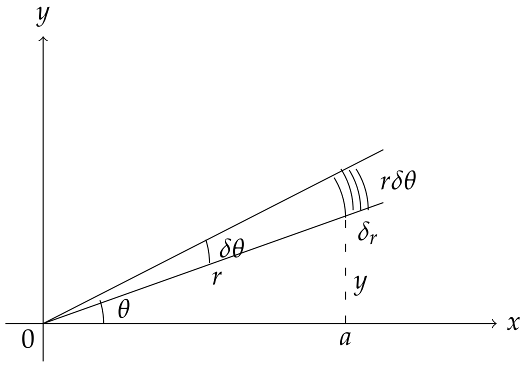


**Figure 3.2:** *The Complex Plane*

Figure 3.2 provides a graphical representation of the complex plane with polar coordinates $(r, \theta)$ used to represent a complex number. The figure demonstrates the conversion of an element of area from Cartesian coordinates $(\delta_x, \delta_y)$ to polar coordinates $(\delta_a = r\delta_\theta\delta_r)$. This conversion is essential in the proof of the gamma function of half $(?(1/2))$ using polar coordinates. The figure shows how the radial distance 'r' and the angle $\theta$ are used to define the position of a point in the complex plane. The element of area in polar coordinates, represented by the shaded sector, is used to integrate the function $e^( - r^2/2)$ over the complex plane, which is a crucial step in the derivation of the standard normal distribution.

This is the proof of the gamma function of half using the polar coordinates system of advanced calculus. This result opens the way for the proof of the standard normal distribution. ■

## 3.2. The Standard Normal Distribution

**Theorem 3.2.** $\int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \delta_z = 1$ **Proof.**

$$\int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \delta_z = 2\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{0} e^{-\frac{1}{2}z^2} \delta_z \tag{7}$$

Fortunately, we can now apply polar coordinates in the summation of the integral in equation 7. Dividing equation 3 by 2, we have $\frac{r^2}{2} = \frac{x^2+y^2}{2}$

$$e^{-\left(\frac{x^2+y^2}{2}\right)} = e^{-\frac{r^2}{2}}$$

With the same limits as of figure 3.1 we can easily see that

$$I^2 = \int\limits_{0}^{\left(\frac{\pi}{2}\right)} \int\limits_{0}^{\infty} e^{-\frac{1}{2}r^2} r\delta_r\delta\theta$$

Let $k = \frac{1}{2}r^2$, $\delta_r = \frac{\delta_k}{r}$

$$\begin{aligned}
&= \int_0^{\left(\frac{\pi}{2}\right)} \int_0^{\infty} e^{-k}\delta_k\delta\theta \\
&= \int_0^{\left(\frac{\pi}{2}\right)} \left[-e^{-k}\right]_0^{\infty} \delta\theta \\
&= \int_0^{\left(\frac{\pi}{2}\right)} \delta\theta \\
&= [\theta]_0^{\left(\frac{\pi}{2}\right)} \\
&= \frac{\pi}{4} \\
I &= \sqrt{\frac{\pi}{2}} \\
\therefore I &= \frac{\sqrt{2\pi}}{2} \qquad (8)
\end{aligned}$$

Substituting equation 8 into equation 7, we have,

$$\frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}z^2}\delta_z = \frac{2}{\sqrt{2\pi}} \times \frac{\sqrt{2\pi}}{2}$$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\delta_z = 1 \qquad (9)$$

■

## 3.3. Derivation of the Normal Curve

Now, back to [1], the derivation of the normal curve from the asymmetric binomial distribution was given in theorem 2.3. From equation 2.9 we had that, $Y = Ke^{-\frac{1}{2}Z^2}$ From proposition (2.0) it is obvious that:

$$y = k \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\delta_z = 1 \qquad (3.10)$$

From theorem 3.2, we had that

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\delta_z = \sqrt{2\pi} \qquad (3.11)$$

Substituting equation 3.11 into equation 3.10 we have that $k = \frac{1}{\sqrt{2\pi}}$

$$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \qquad (3.12)$$

Equation 3.12 is called the ”normal curve”.

### 3.4. The Normal Equation

The equation of the normal distribution was given by equation 1.1 as

If $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$$\int_{-\infty}^{\infty} f(x)\delta_x = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \delta_x$$
$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \delta_x$$

Let $z = \frac{x-\mu}{\sigma}$ and $x = z\sigma + \mu$ and $\delta_x = \sigma\delta_z$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp^{-\frac{1}{2}z^2} \cancel{\sigma}\delta_z$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp^{-\frac{1}{2}z^2} \delta_z$$

From theorem 3.2, we had that $\int_{-\infty}^{\infty} exp^{-\frac{1}{2}z^2}\delta_z = \sqrt{2\pi}$

$$= \frac{\sqrt{2\pi}}{\sqrt{2\pi}}$$
$$\therefore \int_{-\infty}^{\infty} f(x)\delta_x = 1$$

## 4. Discussion of Results

Ordinarily, all integral functions are difficult to integrate. They are not well behaved in regard to integration. Hence the integral functions

$$\Gamma(\tfrac{1}{2}) = \int_{0}^{\infty} t^{x-1}e^{-t}\delta_t$$
$$f(x) = \int_{0}^{\infty} t^{(1/2)}e^{-t}\delta_t$$

cannot easily be determined by normal means. This may be the root cause why [16] did not make any attempt to prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. [5] used the substitution method to prove the gamma function of half and the standard normal distribution. She knew that the direct integration method will leads to complex analysis. First, the so called "Assumption 1" which is the gamma function of half has being proved to be equal to $\sqrt{\pi}$ on theorem 4.0 with the aid of the polar coordinates system. To the student of statistics this should no longer be an assumption. Secondly the "Assumption 2" which is known as the standard normal distribution was proved on theorem 4.1. Also this is made possible by the aid of the polar coordinates system. The derivation of the normal distribution is also an area where many Authorities shy away from. [3] made an attempt to derive it, but he leaves the integral part of the function untouched.

## 5. Conclusion

I have not seen the direct integration method in the literature of the normal distribution but substitution method, before now. This work have used the integration method through the help of polar coordinate to derive the summation of the Gaussian Distribution.

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